Finite-size effect for four-loop Konishi of the β-deformed SYM

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- Based on
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Contents

1. Review on integrability in AdS/CFT

2. Recent results on $(\text{AdS/CFT})_\beta$

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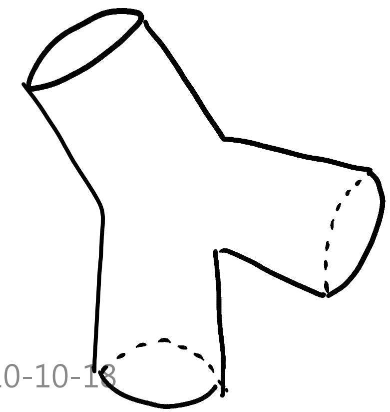
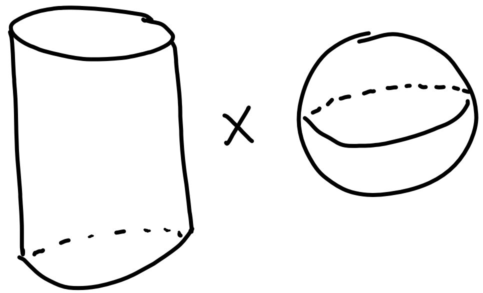
AdS/CFT duality

Type IIB on $AdS_5 \times S^5 \equiv \mathcal{N}=4$ $SU(N)$ SYM

$$g_s = \frac{4\pi\lambda}{N} \qquad \frac{R^2}{\alpha'} = \sqrt{\lambda} \qquad \lambda = N g_{YM}^2$$





Planar limit $N \rightarrow \infty$

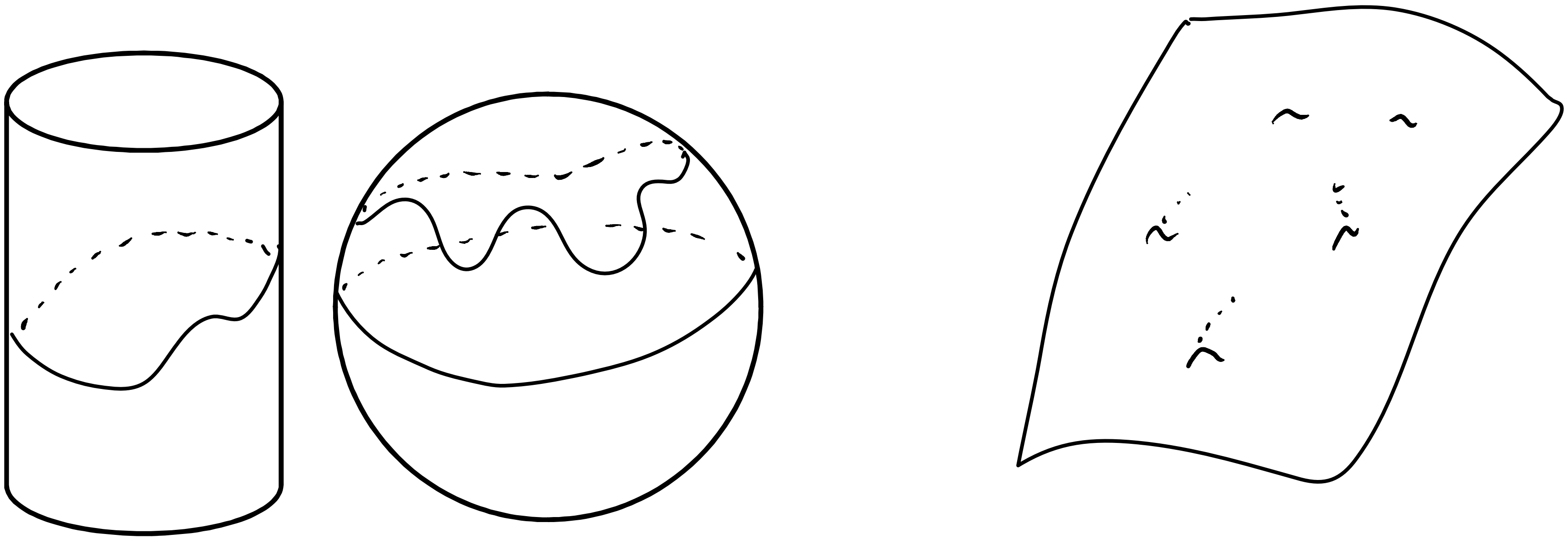
- Non-interacting string as $g_s \to 0$

- Integrability
  - Exact results for any value of λ
  - Truly non-perturbative !

- Only a few physical quantities are exact
  - Anomalous dimensions
  - S-matrix

Strong coupling limit $\lambda \gg 1$

- string fluctuations on world-sheet
  - Energies of string states
  - Excitations and their scattering on world-sheet



$\mathcal{N}$=4 SU(N) SYM

- Action

$$S = \frac{2}{g_{\rm YM}^2}\int d^4x\left\{\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2}(D_\mu\Phi_i)^2-\frac{1}{4}([\Phi_i,\Phi_j])^2+\frac{1}{2}\bar{\chi}\not{D}\chi-\frac{i}{2}\bar{\chi}\Gamma_i[\Phi_i,\chi]\right\}$$

- "R-symmetry": $\mathcal{N}$=4 SO(6)≡SU(4)

- Scalar fields : $\Phi_i,\ i=1,\ldots,6$ $\quad\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$

- Gauginos : fundamental in SU(4) □

Composite operators

$$\mathcal{O}(x) = \mathrm{Tr}\left[XYZF_{\mu\nu}\chi^{\alpha}(D_\mu Y)\ldots\right]$$

- Color singlets (all in adjoint rep.)

$$X \equiv \Phi_1 + i\Phi_2, \quad Y \equiv \Phi_3 + i\Phi_4, \quad Z \equiv \Phi_5 + i\Phi_6$$

- Conformal dimensions

$$\langle \mathcal{O}_n(x)\mathcal{O}_m(0)\rangle = \frac{\delta_{mn}}{x^{2\Delta_n}}$$

- How to compute them ?

Perturbative way

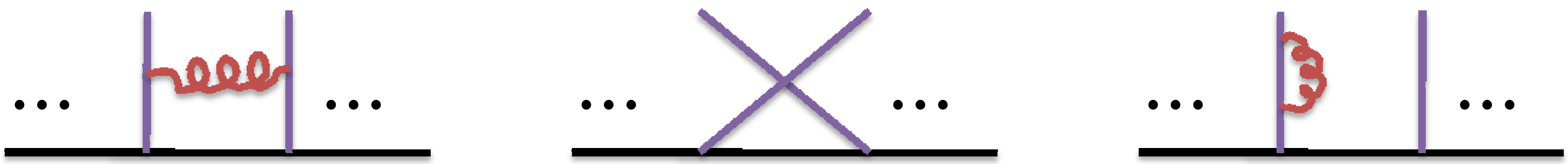
- (ex) su(2) sector

$$\left\{\mathrm{Tr}\left[Z^N\right], \mathrm{Tr}\left[Z^{N-1}X\right], \mathrm{Tr}\left[Z^{N-n-1}XZ^{n-1}X\right], \ldots, \mathrm{Tr}\left[X^N\right]\right\}$$

- Operator mixing

$$\left\{\mathrm{Tr}\left[ZZZZXX\right], \mathrm{Tr}\left[ZZZXZX\right], \mathrm{Tr}\left[ZZXZZX\right], \mathrm{Tr}\left[ZXZZZX\right]\right\}$$

- One-loop



Integrable spin-chains

- Need eigenstates of operator mixings
- Mapped to solving some spin-chain model
- (ex) su(2) $$\Gamma = \frac{\lambda}{4\pi^2}\sum_{i=1}^{J}\left[\frac{\mathbf{1}}{4} - \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}\right]$$
  - Map: $|\Uparrow\rangle \equiv |Z\rangle, \qquad |\Downarrow\rangle \equiv |X\rangle$
  - BPS vacuum: $|0\rangle \equiv \mathrm{Tr}\left[Z^N\right]$
  - Excited states:

$$|\Uparrow\Downarrow\Uparrow\Downarrow\Uparrow\Uparrow\ldots\rangle + \ldots \equiv \mathrm{Tr}\,[ZXZXZZ + \ldots] + \ldots$$

Bethe ansatz Equation

- Ansatz:

$$|p_1, p_2, \cdots\rangle = \sum_{n_1, n_2, \cdots = 1}^{N} e^{i(n_1 p_1 + n_2 p_2 + \cdots)} |\cdots \Uparrow \underset{n_1}{\Downarrow} \Uparrow \cdots \Uparrow \underset{n_2}{\Downarrow} \Uparrow \cdots\rangle$$

- Condition for eigenstate : BAE

$$e^{ip_j N} = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{\cot\frac{p_j}{2} - \cot\frac{p_k}{2} + 2i}{\cot\frac{p_j}{2} - \cot\frac{p_k}{2} - 2i}, \qquad j = 1, \ldots, M$$

- Eigenvalues : conformal dimensions

$$\gamma = \frac{\lambda}{2\pi^2} \sum_{j=1}^{M} \sin^2 \frac{p_j}{2}$$

Non-perturbative way

- All-loop conjecture : (ex) su(2)
  - No known all-loop spin chain Hamiltonian
  - Still educated guess leads to

$$\left(\frac{x^+(u_j)}{x^-(u_j)}\right)^N = \prod_{\substack{k=1\\k\neq j}}^{M}\left[\sigma^2(x_j,x_k)\frac{u_j-u_k+i}{u_j-u_k-i}\right]$$

$$x_j^\pm = e^{\pm i\frac{p_j}{2}}\left[\frac{1+\sqrt{1+16g^2\sin^2\frac{p_j}{2}}}{4g\sin\frac{p_j}{2}}\right], \qquad x_j^\pm + \frac{1}{x_j^\pm} = u_j \pm \frac{i}{2g}$$

$$\triangle = \sum_{j=1}^{M}\left(1+\frac{2ig}{x_j^+}-\frac{2ig}{x_j^-}\right) = \sum_{j=1}^{M}\sqrt{1+16g^2\sin\frac{p_j}{2}}$$

$$g \equiv \frac{\sqrt{\lambda}}{4\pi}$$

Full sector & all-loop

$$\mathcal{O}(x) = \mathrm{Tr}\Big[\ldots ZX \ldots ZY \ldots ZF_{\mu\nu} \ldots Z\chi^{\alpha} \ldots ZD_{\mu}Y \ldots\Big]$$

$$1 = \prod_{k=1}^{M_2} \frac{u_{1j} - u_{2k} + \frac{i}{2}}{u_{1j} - u_{2k} - \frac{i}{2}} \prod_{k=1}^{M_4} \frac{1 - g^2/2x_{1j}x_{4k}^+}{1 - g^2/2x_{1j}x_{4k}^-}$$

$$1 = \prod_{k=1}^{M_2} \frac{u_{2j} - u_{2k} - i}{u_{2j} - u_{2k} + i} \prod_{k=1}^{M_3} \frac{u_{2j} - u_{3k} + \frac{i}{2}}{u_{2j} - u_{3k} - \frac{i}{2}} \prod_{k=1}^{M_1} \frac{u_{2j} - u_{1k} + \frac{i}{2}}{u_{2j} - u_{1k} - \frac{i}{2}}$$

$$1 = \prod_{k=1}^{M_2} \frac{u_{3j} - u_{2k} + \frac{i}{2}}{u_{3j} - u_{2k} - \frac{i}{2}} \prod_{k=1}^{M_4} \frac{x_{3j} - x_{4k}^+}{x_{3j} - x_{4k}^-}$$

$$\left(\frac{x_{4j}^+}{x_{4j}^-}\right)^L = \prod_{k=1}^{M_4} \sigma^2(x_{4j}, x_{4k}) \frac{x_{4j}^+ - x_{4k}^-}{x_{4j}^- - x_{4k}^+} \frac{1 - g^2/2x_{4j}^+x_{4k}^-}{1 - g^2/2x_{4j}^-x_{4k}^+}$$

$$\times \prod_{k=1}^{M_1} \frac{1 - g^2/2x_{4j}^-x_{1k}}{1 - g^2/2x_{4j}^+x_{1k}} \prod_{k=1}^{M_3} \frac{x_{4j}^- - x_{3k}}{x_{4j}^+ - x_{3k}} \prod_{k=1}^{M_5} \frac{x_{4j}^- - x_{5k}}{x_{4j}^+ - x_{5k}} \prod_{k=1}^{M_7} \frac{1 - g^2/2x_{4j}^-x_{7k}}{1 - g^2/2x_{4j}^+x_{7k}}$$

$$1 = \prod_{k=1}^{M_6} \frac{u_{5j} - u_{6k} + \frac{i}{2}}{u_{5j} - u_{6k} - \frac{i}{2}} \prod_{k=1}^{M_4} \frac{x_{5j} - x_{4k}^+}{x_{5j} - x_{4k}^-}$$

$$1 = \prod_{k=1}^{M_6} \frac{u_{6j} - u_{6k} - i}{u_{6j} - u_{6k} + i} \prod_{k=1}^{M_5} \frac{u_{6j} - u_{5k} + \frac{i}{2}}{u_{6j} - u_{5k} - \frac{i}{2}} \prod_{k=1}^{M_7} \frac{u_{6j} - u_{7k} + \frac{i}{2}}{u_{6j} - u_{7k} - \frac{i}{2}}$$

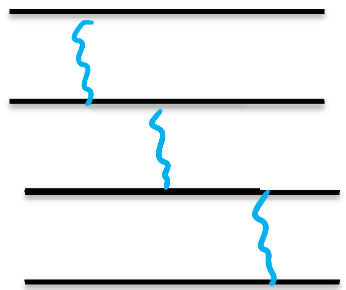
$$1 = \prod_{k=1}^{M_6} \frac{u_{7j} - u_{6k} + \frac{i}{2}}{u_{7j} - u_{6k} - \frac{i}{2}} \prod_{k=1}^{M_4} \frac{1 - g^2/2x_{7j}x_{4k}^+}{1 - g^2/2x_{7j}x_{4k}^-}$$

Not enough

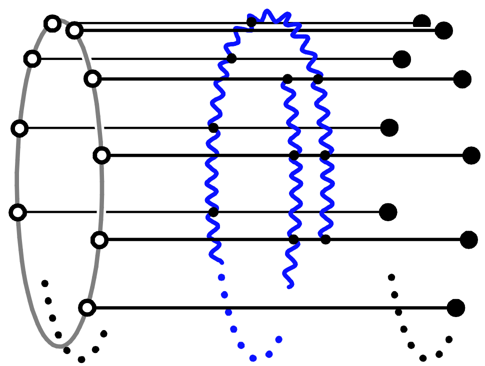
- Works very well !
- But not enough because
  - E.V.s for unknown Hamiltonians
  - BAE works for only infinitely long operators
- If apply to "short" operators, wrapping problem arises

Wrapping problem

- High-order Feynman diagrams connect operators farther away

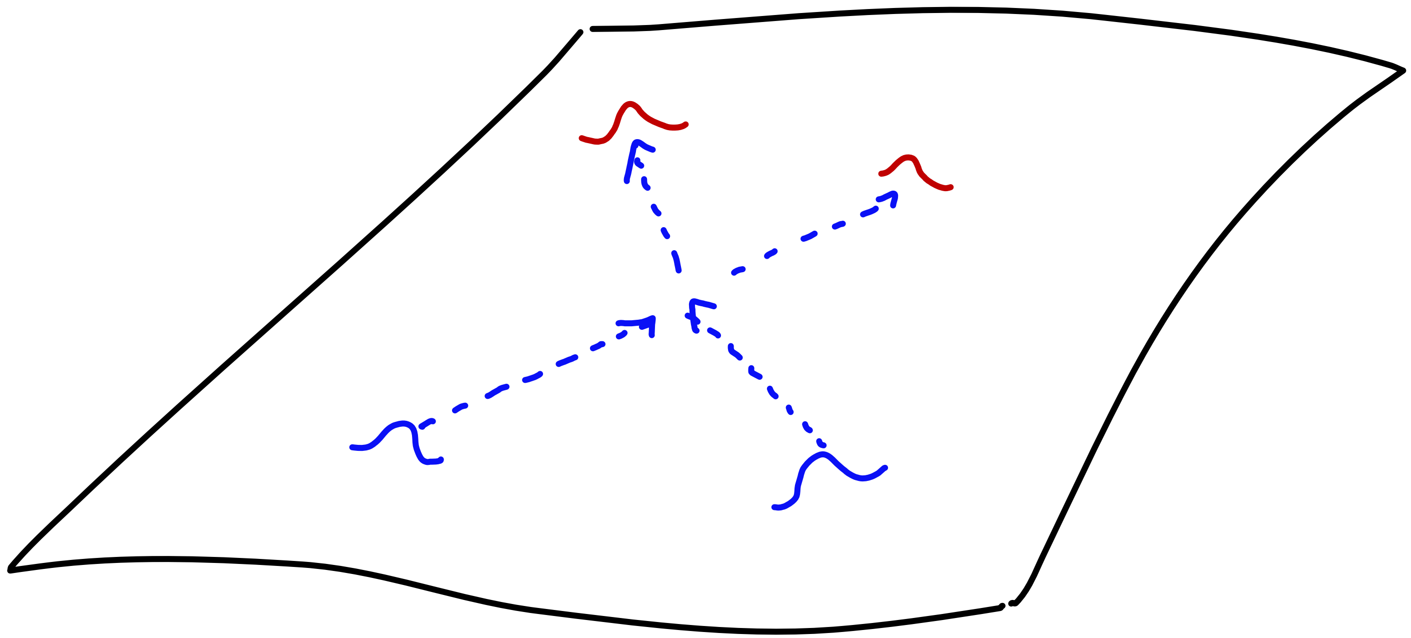


- For finite-size chain, the wrapping interations occur



New Way – Integrable program

S-matrix



S-matrix

- SYM Spin chain:

$$S_{AB}^{A',B'}(p_1,p_2) \quad : \quad \sum_{n_1,n_2=1}^{N} e^{i(n_1p_1+n_2p_2)}|\cdots \mathcal{Z}\underset{n_1}{\Phi_A}\mathcal{Z}\cdots\mathcal{Z}\underset{n_2}{\Phi_B}\mathcal{Z}\cdots\rangle$$

$$\rightarrow \quad \sum_{n_1,n_2=1}^{N} e^{i(n_1p_2+n_2p_1)}|\cdots \mathcal{Z}\underset{n_1}{\Phi_{B'}}\mathcal{Z}\cdots\mathcal{Z}\underset{n_2}{\Phi_{A'}}\mathcal{Z}\cdots\rangle$$

- String-side:
  - World-sheet σ-model in light-cone gauge

Symmetries and Spectrum

- SYM operators

$$(\boxslash;\boxslash) = \Phi_{a\dot{a}} \oplus \chi^{a}_{\dot{\alpha}} \oplus \chi^{\dot{a}}_{\alpha} \oplus D_{\alpha\dot{\alpha}}$$

- tensor product of two fundamental rep.

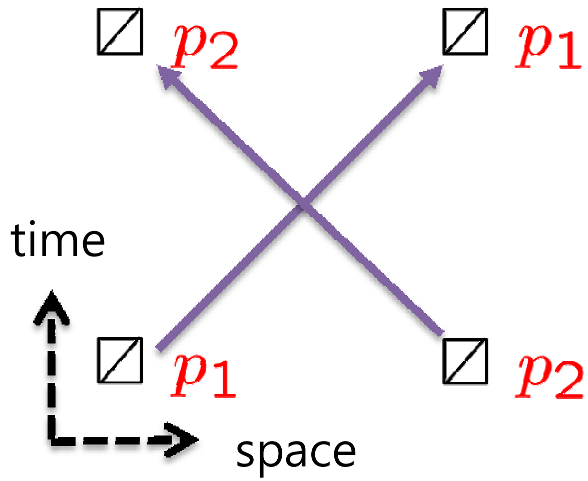
$$\boxslash = (\Box, \mathbf{1}) \oplus (\mathbf{1}, \Box) = (\phi^1, \phi^2 | \psi^1, \psi^2)$$

- Symmetries: SU(2|2) x SU(2|2)

$$\left(\begin{array}{c|c} \mathfrak{R}^a{}_b & \mathfrak{Q}^\alpha{}_b \\ \hline \mathfrak{S}^a{}_\beta & \mathfrak{L}^\alpha{}_\beta \end{array}\right) \left(\begin{array}{c|c} \mathfrak{R}^a{}_b & \mathfrak{Q}^\alpha{}_b \\ \hline \mathfrak{S}^a{}_\beta & \mathfrak{L}^\alpha{}_\beta \end{array}\right)$$

S-matrix from symmetry

- Two particle S-matrix



- Commutativity with SU(2|2)

$$\left[S(p_1, p_2), \left(\begin{array}{c|c} \mathfrak{R}^a{}_b & \mathfrak{Q}^\alpha{}_b \\ \hline \mathfrak{S}^a{}_\beta & \mathfrak{L}^\alpha{}_\beta \end{array} \right) \right] = 0$$

Beisert's SU(2|2) S-matrix

$$
\left(\begin{array}{cccc|cccc|cccccccc}
a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{10} & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_2 & 0 & 0 & -a_7 & 0 & 0 & a_7 & 0 & 0 & a_1+a_2 & 0 & 0 & 0 \\
\hline
0 & a_5 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_8 & 0 & 0 & -a_4 & 0 & 0 & a_3+a_4 & 0 & 0 & a_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\
\hline
0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_8 & 0 & 0 & a_3+a_4 & 0 & 0 & -a_4 & 0 & 0 & -a_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & a_5 & 0 \\
\hline
0 & 0 & 0 & a_1+a_2 & 0 & 0 & a_7 & 0 & 0 & -a_7 & 0 & 0 & -a_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & a_{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1
\end{array}\right)
$$

$$
a_1 = \frac{x_2^- - x_1^+}{x_2^+ - x_1^-},\ a_2 = \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}, \ldots
$$

S-matrix is good for

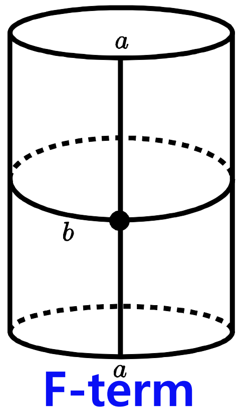
- Directly interpolates between SYM and string theory
- Derives asymptotic BAE
- Solves wrapping problem
- Leads to thermodynamic BAE

Luscher correction

- S-matrix approach to the finite-size effects

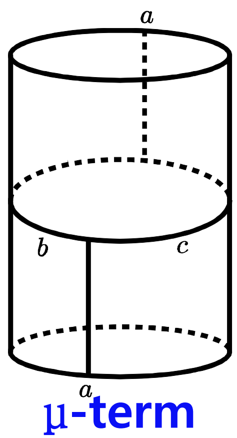
- (1+1)-dim with finite-size volume
  - F-term

$$\Delta m_F(L) = -\frac{m}{\cosh\theta_p}\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh(\theta-\theta_p)\, e^{-mL\cosh\theta}\sum_b\left(S_{ab}^{ab}\left(\theta+\frac{i\pi}{2}-\theta_p\right)-1\right)$$



  - μ-term: from the poles of S-matrix

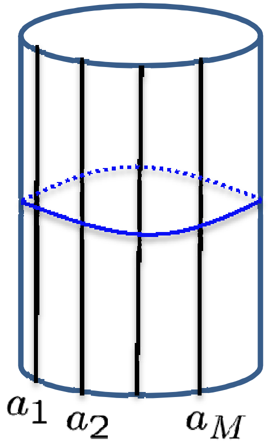
$$\Delta m_\mu(L) = m\cos\alpha\sum_{b,c} i f_{abc}\left[\mathrm{Res}S_{ab}^{ab}(\theta)\right]\cdot e^{-mL\cos\alpha}$$



AdS/CFT Luscher formula

- Wrapping corrections for Konishi

$$\delta\varepsilon^F_{a_1,\ldots,a_M} = -\sum_b (-1)^{F_b} \int_{-\infty}^{\infty} \frac{dq_E^0}{2\pi} \left(1 - \sum_{k=1}^{M} \alpha_k \frac{\epsilon'_{a_k}(p_k)}{\epsilon'_b(q)}\right) e^{-iqL} \left(\prod_{l=1}^{M} S^{ba_l}_{ba_l}(q, p_l) - 1\right)$$



β-deformed SYM

- superpotential

$$W = \mathrm{tr}(e^{i\pi\beta}\phi\psi Z - e^{-i\pi\beta}\phi Z\psi)$$

New AdS/CFT correspondence

- Dual to type IIB string theory on TsT transformed $AdS_5 \times S^5$ (Lunin-Maldacena)
- Classical string has giant magnon [Bykov-Frolov] and dyonic GM [Bozhilov-CA]
- all-loop integrable (?): Bethe-ansatz was conjectured [Beisert-Roiban]

Weak coupling of SYM$_\beta$

- Konishi operator

$$\mathrm{Tr}(\phi Z \phi Z), \qquad \mathrm{Tr}(\phi \phi Z Z)$$

Perturbative results

- Fiamberti-Santambrogio-Sieg-Zanon

$$
\begin{aligned}
\gamma^{(+)} &= 4 + g^2\gamma_1 + g^4\gamma_2 + g^6\gamma_3 + g^8\gamma_4 + \dots \\
\gamma_1 &= 6(1+\Delta) \\
\gamma_2 &= -\frac{3}{\Delta} - 15 - 21\Delta - 9\Delta^2 \\
\gamma_3 &= -\frac{3}{4\Delta^3} + \frac{153}{4\Delta} + 114 + \frac{495}{4}\Delta + 54\Delta^2 + \frac{27}{4}\Delta^3 \\
\gamma_4 &= -\frac{3}{8\Delta^5} + \frac{33}{2\Delta^3} - \frac{1701}{4\Delta} - 1230 \\
&- \frac{2427}{2}\Delta - 180\Delta^2 + 162\Delta^4 + \frac{2997}{8}\Delta^3 \\
&+ \left(-\frac{9}{\Delta} + 297 + 702\Delta + 234\Delta^2 - 405\Delta^3 - 243\Delta^4\right)\zeta(3) \\
&- 360\,(1+\Delta)^2\,\zeta(5)
\end{aligned}
$$

$$
\gamma^{(-)} = \gamma^{(+)}(\Delta \to -\Delta), \qquad \Delta \equiv \frac{\sqrt{5 + 4\cos(4\pi\beta)}}{3}
$$

Beisert-Roiban BAE

- for su(2)

$$e^{ip_1 L} = e^{2\pi i\beta L} \cdot e^{2i\theta(p_1,p_2)} \cdot \frac{u_1 - u_2 + i}{u_1 - u_2 - i}$$

$$e^{ip_2 L} = e^{2\pi i\beta L} \cdot e^{-2i\theta(p_1,p_2)} \cdot \frac{u_2 - u_1 + i}{u_2 - u_1 - i}$$

- eigenvalues

$$u_i = \frac{1}{2}\cot\frac{p_i}{2}\sqrt{1 + 16g^2 \sin^2\frac{p_i}{2}}$$

$$E(p_i) = \sqrt{1 + 16g^2 \sin^2\frac{p_i}{2}}$$