What is the next number in the sequence?

(or, Sublattice counting and orbifolds)

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Outline

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Motivation

- Brane tilings have met with a lot of interest in the past few years.
- A brane tiling is the dual description of a quiver gauge theory,
- D3 branes probing a toric Calabi–Yau-three singularity
- M2 branes probing a toric Calabi–Yau-four singularity
- We would like to have a classification of all possible tilings.

Today's talk

- For the moment a complete classification seems too ambitious
- We consider a simpler problem: counting Abelian orbifolds of a given theory
- An orbifold is described by a repetition of the fundamental domain
- This is the definition of a sublattice of the initial lattice
- We map our orbifold counting problem to a sublattice enumeration. We can use methods from crystallography
- We find a number theoretical description for the generating functions. Asymptotically

$$f(n) \sim \frac{\sigma(n)}{|G|}$$

How do you do that?

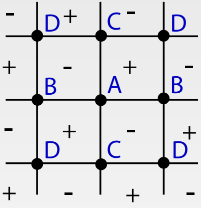
- Find the symmetries of a given lattice (cycle index)
- Decompose the counting function on the symmetries (Burnside's lemma)
- Count the sublattices for each symmetry (Hermite normal form)
- Study the structure (Dirichlet convolution)
- Write in a compact form (Dirichlet series)
- Analysis (Asymptotic behaviour)

Outline

String theory setup

- Consider a Calabi-Yau three-fold X and let $S \subset X$ be a surface shrinking to a point
- Placing D-branes on S we expect:
    - An enhanced gauge symmetry because some open strings will shrink to zero length
    - The branes are marginally stable against the decay into fractional branes
- These fractional branes are rigid branes generating the BPS states in the theory as bound states.
- The description of the brane as a manifold breaks down
- Geometric description: language of categories.
- For our purposes we can use quiver gauge theories.

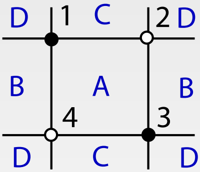
D-branes at singularities



quiver on the torus

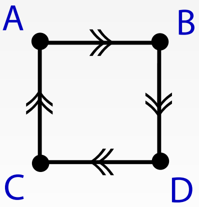
↑

graph dual



brane tiling

↓



quiver gauge theory

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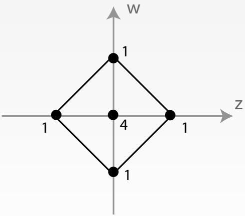


toric diagram

Orbifolds: algebraic construction

- Let us consider $\mathbb{Z}_n$ orbifolds of $\mathbb{C}^3$
- Let us denote the coordinates of $\mathbb{C}^3$ by $\{z_1, z_2, z_3\}$, and the orbifold action by (a_1, a_2, a_3):

$$\{z_1, z_2, z_3\} \sim \{\omega^{a_1} z_1, \omega^{a_2} z_2, \omega^{a_3} z_3\}$$

with $\omega^n = 1$ and $a_1 + a_2 + a_3 = 0 \mod n$.

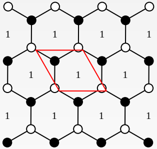
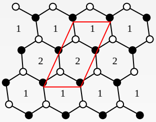
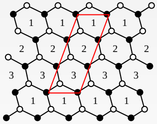
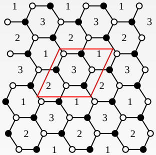
- In this notation, the problem is to find all triples (a_1, a_2, a_3) that give inequivalent orbifolds of $\mathbb{C}^3$.

Orbifolds: toric diagrams

- An alternative way of formulating the problem is by looking at the toric diagrams of these orbifolds.
- The toric diagram of $\mathbb{C}^3$ is a triangle of unit area
- the toric diagram of an orbifold of $\mathbb{C}^3$ by an Abelian group of order n is again a triangle but with an area which is n times larger.
- The problem of counting all inequivalent orbifolds of $\mathbb{C}^3$ is therefore equivalent to the problem of finding all triangles with vertices on integral points and area n.
- Since these are toric diagrams, two triangles which are related by a $GL(2,\mathbb{Z})$ are equivalent

Orbifolds: brane tilings

- Think of the brane tiling as forming a bipartite hexagonal lattice
- the problem of finding inequivalent toric diagrams is mapped to the problem of finding its sublattices

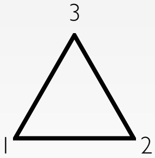
n	1	2	3	3
brane tiling				
geometry	$\mathbb{C}^3$	$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$	$\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$	$\mathbb{C}^3/\mathbb{Z}_3$

Outline

Cycle notation

We need a way to capture the symmetries of a given lattice.

- Label the vertices of the fundamental cell by the numbers $\{1, \ldots, m\}$.
- We want to describe the group of permutations G of the set $X = \{1, \ldots, m\}$ which result in the same fundamental cell.



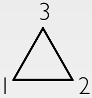
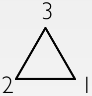
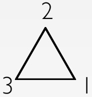
- identity
- 3 reflections
- 2 rotations by $2\pi/3$

Cycle notation

- Cycles of $g \in G$ are the orbits of the elements $\varepsilon \in X$ under g.
- For each group element g we start with $\varepsilon_1 \in X$ and write down its orbit in parentheses, $(\varepsilon_1\ g(\varepsilon_1)\ g^2(\varepsilon_1)\ \ldots\ g^{k-1}(\varepsilon_1))$, where $g^k(\varepsilon_1) = \varepsilon_1$.
- We continue with the next element that has not yet appeared in an orbit until we have exhausted all the elements of X.
- Each $g \in G$ can be expressed in terms of a_k disjoint cycles of length k
- The type of g is given by the partition of m $[1^{a_1} 2^{a_2} \ldots l^{a_l}]$, where $m = a_1 + 2a_2 + \ldots + l a_l$.
- The partition is represented by the expression

$$\zeta_g(x_1, \ldots, x_l) = x_1^{a_1} x_2^{a_2} \ldots x_l^{a_l}\,.$$

Cycle index

	g	a_1	a_2	a_3	$c(a_i)$	ζ
	1	3	-	-	1	x_1^3
	(12)	1	1	-	3	$x_1 x_2$
	(123)	-	-	1	2	x_3

The cycle index of G

$$Z_G(x_1,\dots,x_l) = \frac{1}{|G|}\sum_{g\in G}\zeta_g(x_1,\dots,x_l) = \frac{1}{|G|}\sum_{a} c(a_1,\dots,a_l)\, x_1^{a_1}\cdots x_l^{a_l}$$

Cycle index, examples

- *Cyclic group* C_m: symmetries of a circular object without reflections:

$$Z(C_m) = \frac{1}{m} \sum_{d|m} \varphi(d) x_d^{m/d},$$

where $\varphi(d)$ is the *totient function*.

- *Dihedral group* D_m: symmetries of a circular object:

$$Z(D_m) = \frac{1}{2} Z(C_m) + \begin{cases} \frac{1}{2} x_1 x_2^{(m-1)/2}, & \text{if } m \text{ is odd,} \\ \frac{1}{4} \left(x_1^2 x_2^{(m-2)/2} + x_2^{m/2} \right), & \text{if } m \text{ is even.} \end{cases}$$

- The *symmetric group* S_m is the group of all permutations:

$$Z(S_m) = \sum_{\alpha_1 + 2\alpha_2 + \cdots + kj_k = m} \frac{1}{\prod_{k=1}^m k^{\alpha_k} \alpha_k!} \prod_{k=1}^m x_k^{\alpha_k}.$$

Central point

Burnside's lemma

Let G be a group of permutations of the set X. The number $N(G)$ of *orbits* of G is given by the average over G of the sizes of the fixed sets:

$$N(G) = \frac{1}{|G|} \sum_{g \in G} |F_g| \; ; \qquad F_g = \{ x \in X \mid g(x) = x \} \; .$$

- The number $f(n)$ of sublattices of index n is the number of orbits of the symmetry group G when acting on the set X_n of sublattices of index n.
- This can be written as the average of the number of elements in X_n that are left invariant by the action of $g \in G$:

$$f(G) = \frac{1}{|G|} \sum_{g \in G} f_g(n) \; ; \qquad f_g(n) = |\{ x \in X_n \mid g(x) = x \}|$$

Decomposition of the counting function

- Using the cycle decomposition we can rewrite this expression as a sum over the types of the elements g, indexed by partitions α:

$$f(n) = \frac{1}{|G|} \sum_{\alpha} c(\alpha) f_{x^{\alpha}}(n) \,.$$

- there is a subsequence for each monomial in the cycle index Z_G.

$$Z_G(x_1, \dots, x_l) = \frac{1}{|G|} \sum_{\alpha} c(\alpha)\, x_1^{\alpha_1} \cdots x_l^{\alpha_l}$$

- we have decomposed the counting problem into subproblems fixed by the symmetry groups.

Recipe

For a given lattice, find the symmetry group G, write the cycle index Z_G and identify the counting functions for each of the terms.

The hexagonal lattice

- Consider the bipartite hexagonal lattice corresponding to the geometry of $\mathbb{C}^3$.
- Because of the bipartiteness, the symmetry group is S_3 (equilateral triangle)
- From the cycle decomposition above:

$$Z_{S_3} = \frac{1}{6}\left(x_1^3 + 3\,x_1 x_2 + 2\,x_3\right) .$$

- Using Burnside's lemma, the number of sublattices of index n can be decomposed as:

$$\boxed{f^{\triangle}(n) = \frac{1}{6}\left(f^{\triangle}_{x_1^3}(n) + 3\,f^{\triangle}_{x_1 x_2}(n) + 2\,f^{\triangle}_{x_3}(n)\right)}$$

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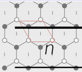
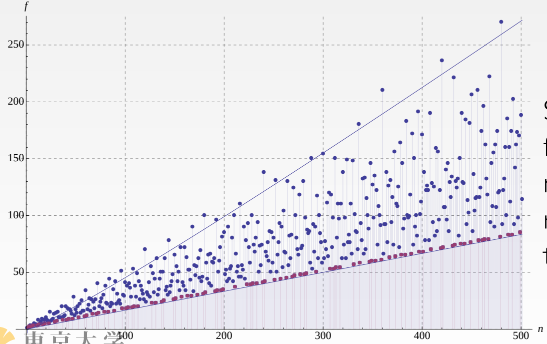
The hexagonal lattice



n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f^{\triangle}_{x_1^3}$	1	3	4	7	6	12	8	15	13	18	12	28	14	24	24	31
$f^{\triangle}_{x_1 x_2}$	1	1	2	3	2	2	2	5	3	2	2	6	2	2	4	7
$f^{\triangle}_{x_3}$	1	0	1	1	0	0	2	0	1	0	0	1	2	0	0	1
$f^{\triangle}$	1	1	2	3	2	3	3	5	4	4	3	8	4	5	6	9



Scatter plot of the sequence $f^{\triangle}$ for a hexagonal lattice. Prime numbers are emphasized in red. The two lines correspond to $n/6$ and $e^{\gamma} n \log \log n/6$.

Hermite normal form

- The problem of counting sublattices with a given symmetry can be mapped to the problem of counting a set of matrices
- We learn what is the general structure
- We obtain an algorithm to count the sublattices
- Consider a lattice L_d generated by the d vectors $\langle y_1, \dots, y_d \rangle$. Any sublattice L' of L_d is generated by d vectors $\langle x_1, \dots, x_d \rangle$

$$\begin{cases} x_1 = a_{11} y_1 \\ x_2 = a_{21} y_1 + a_{22} y_2 \\ \dots \\ x_d = a_{d1} y_1 + a_{d2} y_2 + \dots + a_{dd} y_d \,, \end{cases}$$

- The integer coefficients a_{ij} satisfy the conditions

$$0 \le a_{ij} < a_{ii} \qquad \forall j < i \qquad\qquad n = \prod_{i=1}^{d} a_{ii} \,.$$

Hermite normal form

- For a two dimensional lattice the coeficients are $\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$.
- From the condition $a_{11} a_{22} = n$ we choose $a_{22} = m$ and $a_{11} = n/m$, where m is a divisor of n.
- To count the number of sublattices invariant under the symmetry x^{a} we enumerate the possible values of a_{21}.
- The constraint $a_{21} < a_{22}$ introduces a dependence of the number of possible values of a_{21}, $\#\{a_{21}\} = g_{x^a}(a_{22})$, on a_{22}
- The total number of sublattices $f_{x^a}(n)$ is given by summing $g_{x^a}(m)$ over all the divisors of n:

$$f_{x^a}(n) = \sum_{m|n} g_{x^a}(m) .$$

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Multiplicative functions

- Prime numbers play a special role.
- This is one of the clues that point to the sequences being multiplicative.

Multiplicative sequence

A sequence f is multiplicative if

$$f(nm) = f(n)f(m)\,, \quad \text{when } (n,m) = 1\,,$$

where (n,m) is the *greatest common divisor* between n and m.

- f is completely determined by its values for primes and their powers
- for any $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, the sequence decomposes as

$$f(n) = f(p_1^{a_1}) f(p_2^{a_2}) \dots f(p_r^{a_r})$$

Dirichlet convolution

Dirichlet convolution

The *Dirichlet convolution* of two sequences g and h is

$$f(n) = (g * h)(n) = \sum_{m|n} g(m)\, h(\frac{n}{m}) \,,$$

where the sum runs over all the divisors m of n.

- Commutative, $f * g = g * f$,
- Associative, $f * (g * h) = (f * g) * h$
- Has an identity $f * \mathrm{Id} = f$ defined by

$$\mathrm{Id}(n) = \{\, 1, 0, 0, \ldots \,\}$$

- to each sequence f one can associate its inverse f^{-1} satisfying

$$f * f^{-1} = f^{-1} * f = \mathrm{Id} \,.$$

Möbius function

- We have seen that the counting functions have the structure

$$f_{x^a}(n) = \sum_{m|n} g_{x^a}(m) \,.$$

- In terms of Dirichlet convolution: $f_{x^a} = g_{x^a} * \mathrm{u}$ where:

$$\mathrm{u}(n) = \{1, 1, 1, \dots\} \,.$$

- its inverse is the Möbius function defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is square-free,} \\ 0 & \text{otherwise.} \end{cases}$$

  where k is the number of distinct prime factors of n.
- It follows that if $f = g * \mathrm{u}$, then $g = \mu * f$.

The hexagonal lattice

- The sequence $f^{\triangle}_{x_1^3} = \{\, 1, 3, 4, 7, 6, 12, 8, 15, \dots \}$ corresponds to the identity permutation x_1^3

$$f^{\triangle}_{x_1^3} = \mathrm{u} * \mathrm{N} \,,$$

where

$$\mathrm{N}(n) = \{\, 1, 2, 3, \dots \} \,.$$

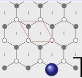
- The sequence $f^{\triangle}_{x_1 x_2} = \{\, 1, 1, 2, 3, 2, 2, 2, 5, \dots \}$ can be written as the convolution of a periodic sequence of period 4 and the unit:

$$f^{\triangle}_{x_1 x_2} = \{\, 1, 0, 1, 2, 1, 0, 1, 2, 1, \dots \} * \mathrm{u} \,.$$

$g^{\triangle}_{x_1 x_2}$ is in turn the convolution of a finite sequence and u:

$$f^{\triangle}_{x_1 x_2} = \{\, 1, 0, 1, 2, 1, 0, 1, 2, 1, \dots \} * \mathrm{u} = \{\, 1, -1, 0, 2 \} * \mathrm{u} * \mathrm{u} \,.$$

The hexagonal lattice



- The last sequence $f^{\triangle}_{x_3} = \{ 1, 0, 1, 1, 0, 0, 2, 0, \ldots \}$ also has the form of the convolution of the unity with a periodic sequence of period 3:

$$f^{\triangle}_{x_3} = \{ 1, -1, 0, 1, -1, 0, 1, -1, 0, \ldots \} * \mathrm{u} \, .$$

The periodic sequence is the (non-principal) *Dirichlet character* of modulus three:

$$g^{\triangle}_{x_3} = \chi_{3,2}(n) = \{ 1, -1, 0, 1, -1, 0, \ldots \} \, .$$

Putting all together we find that the sequence $f^{\triangle}$ can be written as

$$\boxed{f^{\triangle} = \frac{1}{6}\left(f^{\triangle}_{x_1^3} + 3\, f^{\triangle}_{x_1 x_2} + 2\, f^{\triangle}_{x_3} \right) = \frac{1}{6}\left(\mathrm{N} + 3 \{ 1, 0, -1, 2 \} * \mathrm{u} + 2\, \chi_{3,2} \right) * \mathrm{u}}$$

Outline

Generating functions

Now that we have the numbers we need a good way to encode them in a compact form.

- the formal power series (partition function)

$$F(t) = \sum_{n=1}^{\infty} f(n) t^n ;$$

- the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} .$$

- The corresponding inverse transformations are given by

$$f(n) = \frac{1}{2\pi\imath} \oint \frac{F(t)}{t^{n+1}} \, \mathrm{d}t , \quad f(n) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(s) n^s |_{s=\sigma+\imath\tau} \, \mathrm{d}\tau .$$

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Generating functions and Dirichlet convolution

- Both types of generating functions have a simple behavior under Dirichlet convolution.
- Let f, g and h be such that
$$f = g * h\,.$$
- The power series for h reads:

$$F(t) = \sum_{n=1}^{\infty} f(n) t^n = \sum_{n=1}^{\infty} \sum_{m|n} g(m)\, h(\frac{n}{m})\, t^n = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} g(m)\, h(k)\, t^{mk}\,.$$

This can be expressed in two ways, using the generating function for g or for h:
$$F(t) = \sum_{m=1}^{\infty} g(m) H(t^m) = \sum_{k=1}^{\infty} h(k) G(t^k)\,.$$

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Generating functios and Dirichlet convolution

- All our sequences are sums over divisors (or equivalently as Dirichlet convolutions with the unit), we will always write

$$F(t) = \sum_{k=1}^{\infty} G(t^k) \,.$$

- It is also possible to write the power series for the inverse of the Dirichlet convolution as follows. Let

$$f(t) = \sum_{k,m=1}^{\infty} g(m)\, h(k)\, t^{mk} \,,$$

then

$$H(t) = \sum_{k=1}^{\infty} h(k)\, t^k = \sum_{m=1}^{\infty} \mu\,(k)\, g(k)\, F(t^k) \,,$$

where μ is the Möbius function.

Dirichlet series

The Dirichlet series is even more adapted to these structures.

- if f is multiplicative, the series can be expanded in terms of an infinite product over the primes, the Euler product:

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots \right) .$$

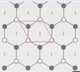
  remember that a multiplicative sequence is determined by the values taken for powers of prime numbers.

- the Dirichlet series of a convolution is decomposed as

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=1}^{\infty} \sum_{m|n} \frac{g(m)\, h(\frac{n}{m})}{n^s} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{g(m)\, h(k)}{m^s k^s} = G(s)H(s) .$$

The Dirichlet series is the Laplace transform of a discrete measure. It exchanges convolution and pointwise products.

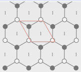
The hexagonal lattice



- The generating series are linear transformations. The decomposition in terms of symmetries remains the same.
- We can calculate explicitly the generating series for each term

symmetry	Dirichlet series $G(s)$	power series $G(t)$
x_1^3	$\zeta(s-1)$	$\frac{1+t^3}{(1-t)(1-t^2)} - 1$
$x_1 x_2$	$(1 - 2^{-s} + 2^{1-2s})\ \zeta(s)$	$\frac{1+t^3}{(1-t)(1+t^2)} - 1$
x_3	$L(s, \chi_{3,2})$	$\frac{(1+t)(1-t^2)}{1-t^3} - 1$

The hexagonal lattice



Collecting all the terms we find:

- For the power series:

$$\boxed{F^{\triangle}(t) = \sum_{m=1}^{\infty} \sum_{\substack{n_1,n_2,n_3=0 \\ \neq(0,0,0)}}^{\infty} (-)^{n_2}\, t^{m(n_1+2n_2+3n_3)}} .$$

- For the Dirichlet series:

$$\boxed{F^{\triangle}(s) = \frac{\zeta(s)}{6}\left(\zeta(s-1) + 3\left(1 - \frac{1}{2^s} + \frac{2}{2^{2s}}\right)\zeta(s) + 2L(s,\chi_{3,2})\right)} .$$

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Asymptotic behaviour

The asymptotic behavior of a sequence can be derived by looking at the corresponding Dirichlet series.

Asymptotic behaviour

Let $F(s)$ be a Dirichlet series with non-negative coefficients that converges for $\Re(s) > \alpha > 0$. If $F(s)$ is holomorphic in all points of the line $\Re(s) = \alpha$, except for $s = \alpha$ and

$$F(s) \sim A(s) + \frac{B(s)}{(s - \alpha)^{m+1}},$$

where $m \in \mathbb{N}$, then the partial sum of the coefficients is asymptotic to:

$$\sum_{n=1}^{N} f(n) \sim \frac{B(\alpha)}{\alpha\, m!} N^{\alpha} \log^{m}(N).$$

Asymptotic behaviour

We just need to know the analytic properties of some functions:

- The Riemann zeta function $\zeta(s)$ is analytic everywhere, except for a simple pole at $s = 1$ with residue 1;
- The L-function $L(s, \chi)$ is analytic everywhere, except for a simple pole at $s = 1$ if χ is a principal character.

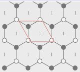
Also useful is:

Robin's inequality

$$\sigma(n) < e^{\gamma} n \log\log n \,, \qquad n \text{ large},$$

where γ is Euler's constant. This is true for large n, where large means $n \geq 5041$, and if and only if Riemann's hypothesis is true

Hexagonal lattice



- The rightmost pole of the Dirichlet series $F^{\triangle}(s)$ is found for $s = 2$
- The pole has order 1 and its residue is $\zeta(2)/6$.
- The partial sum of the terms in the sequence $f^{\triangle}$ behaves asymptotically as

$$\boxed{\sum_{n=1}^{N} f^{\triangle}(n) \sim \frac{\zeta(2)}{12} N^2 = \frac{\pi^2}{72} N^2},$$

- for large n, the leading term is $\zeta(s)\,\zeta(s-1)/6$, hence

$$\boxed{f^{\triangle}(n) < \frac{e^{\gamma}\, n \log\log n}{6}\,, \qquad n \text{ large.}}$$

Outline

東京大学 THE UNIVERSITY OF TOKYO

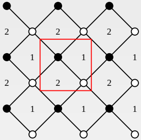
IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

How do you do that?

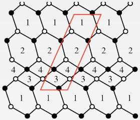
- Find the symmetries of a given lattice (cycle index)
- Decompose the counting function on the symmetries (Burnside's lemma)
- Count the sublattices for each symmetry (Hermite normal form)
- Study the structure (Dirichlet convolution)
- Write in a compact form (Dirichlet series)
- Analysis (Asymptotic behaviour)

Results

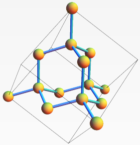
- We have seen the construction for the orbifolds of $\mathbb{C}^3$.
  - the conifold



- In the paper:
  - L^{aba}



  - $\mathbb{C}^4$



- These exhausts all the symmetries in the plane
- Classification of string vacua

Results

- Brane tilings describe quiver gauge theories obtained by placing branes at CY singularities
- We would like to obtain a complete classification
- We started by studying a subclass: Abelian orbifolds of a given geometry
- The problem is the same as counting sublattices of a given lattice
- Techniques from number theory
- Generating functions for any given geometry

The end

Thank you

for your attention