From Weak to Strong Coupling in ABJM Theory

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Based on arXiv:1007.3837: ND, M. Mariño, P. Putrov

and many other papers...

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Introduction and motivation

- The AdS/CFT correspondence is a powerful tool of modern theoretical physics.
- Allows to calculate gauge theory quantities at strong coupling (for large N).
- In the other direction, allows to calculate string theory quantities at large α'.
- In very simple situations the results agree, indicating that there is a non-renormalization principle at work.

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- In very simple situations the results agree, indicating that there is a non-renormalization principle at work.
- In other very specific cases one can derive (or guess) non-trivial functions that interpolate from weak to strong coupling:
  - BPS observables like Wilson loops, surface operators (topological subsectors).
  - Integrability: Cusp anomalous dimension. Konishi? Scattering amplitudes?
- This has been achieved so far only for the simplest example of exact AdS/CFT duality:

$$\mathcal{N} = 4 \text{ SYM} \qquad \Longleftrightarrow \qquad \text{Type IIB on } AdS_5 \times S^5$$

$\mathcal{N}=6$ super Chern-Simons-matter theory

[Aharony, Bergman Jafferis, Maldacena]

- Spring 2008: Building on Bagger-Lambert-Gustavsson, a new proposal for a highly supersymmetric AdS/CFT duality

$$d=3, \quad \mathcal{N}=6 \text{ super Chern-Simons} \qquad \Longleftrightarrow \qquad \begin{cases} \text{M-theory on } AdS_4\times S^7/\mathbb{Z}_k \\ \text{Type IIA on } AdS_4\times\mathbb{CP}^3 \end{cases}$$

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$$E(p)=\sqrt{J^2+4h^2(\lambda)\sin^2\tfrac{p}{2}}-J$$

  – Form obeyed at weak and strong coupling (constrained by symmetry).
  – In $\mathcal{N}=4$ SYM same structure with $h^2(\lambda)=\lambda/4\pi^2$.
  – In ABJM: Unknown function

$$h^2(\lambda)=\begin{cases} \lambda^2-4\lambda^4(4-\zeta(2))+\cdots & \text{small } \lambda \\ \frac{1}{2}\lambda+\cdots & \text{large } \lambda \end{cases}$$

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- Can BPS protected quantities be calculated exactly?

Outline

- Introduction and motivation.
- ABJM theory.
- Wilson loops.
- Localization to a super matrix model.
- Review: $1/2$ BPS Wilson loop in 4d.
- Solving the matrix model:
  - Weak coupling.
  - Strong coupling.
  - Near CS limit.
- Summary.

Lightning review of ABJ(M) theory

- $U(N_1) \times U(N_2)$ gauge symmetry.
- Chern-Simons terms at levels k and $-k$.
- Kinetic terms for scalars and fermions.
- Very specific sextic scalar potential and $(C)^2(\psi)^2$ terms.
- Normally 3d super Chern-Simons has $\mathcal{N} = 2$ or $\mathcal{N} = 3$ SUSY.

Field content		dim	rep	
A_μ	gauge field	1	adj	1
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C_I	scalar	1/2	N_1	$\overline{N}_2$
$\bar{C}^I$	scalar	1/2	$\overline{N}_1$	N_2
ψ_I	fermion	1	$\overline{N}_1$	N_2
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- This special quiver construction allows for $\mathcal{N} = 6$ SUSY.
- For $k = 1, 2$ should be enhanced to $\mathcal{N} = 8$ SUSY.
- Is the low energy theory of N_1 M2-branes on a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold (with $N_2 - N_1$ fractional branes).
- Gravity dual: M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$.
- For $k^5 \gg N$ a better description is IIA on $AdS_4 \times \mathbb{CP}^3$.
- Analogs of 't Hooft coupling: $\lambda_1 = N_1/k$, $\lambda_2 = N_2/k$.

Wilson loops

1/6 BPS

$\begin{bmatrix}\text{ND, Plefka}\\ \text{Young}\end{bmatrix}\begin{bmatrix}\text{Chen}\\ \text{Wu}\end{bmatrix}\begin{bmatrix}\text{Rey, Suyama}\\ \text{Yamaguchi}\end{bmatrix}$

- Borrowing from the 4d theory, to make a BPS straight Wilson loop we can add a scalar piece to the connection

$$A_\mu \dot{x}^\mu \qquad \to \qquad \mathcal{A} = A_\mu \dot{x}^\mu + \frac{2\pi}{k}|\dot{x}| M_J^I C_I \bar{C}^J$$

- It's a bilinear on dimensional grounds and so it's in adjoint of $U(N)$.
- Checking SUSY gives a unique solution

$$\delta_{\text{SUSY}}\mathcal{A} = 0 \qquad \Longrightarrow \qquad M_J^I = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Preserves two Poincaré supercharges and two superconformal ones $\Rightarrow$ $1/6$ BPS.
- Was calculated perturbatively to order λ^2

$$\langle W \rangle = 1 + \frac{5\pi^2}{6}\lambda^2 + \cdots$$

- There was no simple guess on how to extend to all orders.
- More generally: Loop in arbitrary representation (R_1, R_2) of $U(N_1) \times U(N_2)$.

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- There was no simple guess on how to extend to all orders.
- More generally: Loop in arbitrary representation (R_1, R_2) of $U(N_1) \times U(N_2)$.
- Such Wilson loops exist in any $\mathcal{N} = 2$ super Chern-Simons theory. [Gaiotto Yin]

  SUSY is not enhanced in going from $\mathcal{N} = 2$ to $\mathcal{N} = 6$.

1/2 BPS

$\left[\text{ND, D. Trancanelli}\right]\left(\begin{array}{l}\text{Initiated in discussions with}\\ \text{V. Niarchos, G. Michalogiorgakis}\end{array}\right)\left[\begin{array}{l}\text{Lee}\\ \text{Lee}\end{array}\right]$

- A Wilson loop in both gauge groups can be written in terms of an $(N_1 + N_2) \times (N_1 + N_2)$ connection

$$L = \begin{pmatrix} A_\mu \dot{x}^\mu + \frac{2\pi}{k}|\dot{x}|M_J^I C_I \bar{C}^J & 0 \\ 0 & \widehat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k}|\dot{x}|\widehat{M}_J^I \bar{C}^J C_I \end{pmatrix}$$

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- Generalization: Write an $(N_1 + N_2) \times (N_1 + N_2)$ superconnection

$$L = \begin{pmatrix} A_\mu \dot{x}^\mu + \frac{2\pi}{k}|\dot{x}|M_J^I C_I \bar{C}^J & \sqrt{\frac{2\pi}{k}}\,|\dot{x}|\,\eta_I^\alpha\,\bar{\psi}_\alpha^I \\ \sqrt{\frac{2\pi}{k}}\,|\dot{x}|\,\psi_I^\alpha\,\bar{\eta}_\alpha^I & \widehat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k}|\dot{x}|\widehat{M}_J^I \bar{C}^J C_I \end{pmatrix}$$

The natural Wilson loop is then

$$W_\mathcal{R} \equiv \mathrm{Tr}_\mathcal{R} \mathcal{P} \exp\left(i \int L\, d\tau\right)$$

$\mathcal{R}$ is a representation of the supergroup $U(N_1|N_2)$.

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- To make a long story short, with the right choice of M_J^I and of η_I^α, this loop preserves 12 supercharges.

Localization to a super matrix model

[Kapustin, Willett, Yaakov]

- Consider any $\mathcal{N} = 2$ super Chern-Simons matter theory on S^3.
- Take a Wilson loop of that theory on the equator invariant under a supercharge Q.
- Add to the action a Q-exact term of the form $t\,Q(\Psi Q\Psi)$.
- VEV of Q-invariant observables is unmodified by this insertion.
- Take t large and look at the saddle points of $(Q\Psi)^2$.
- Get the VEV of the Wilson loop from the classical value at the saddle point and the one loop determinant around that point.

- For a theory with one $U(N)$ vector multiplet

$$Z = \int \prod_{a=1}^{N} d\mu_a \, e^{ik\pi\mu_a^2} \prod_{a<b} \sinh^2(\pi(\mu_a - \mu_b))$$

- Wilson loop in the fundamental: Insert into the integral

$$\sum_{a=1}^{N} e^{2\pi\mu_a}$$

- This is the matrix model for regular $U(N)$ topological Chern-Simons on S^3.

[Mariño] [Aganagic, Klemm Mariño, C. Vafa] [Halmagyi Yasnov]

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- Applying this to ABJ(M) theory one finds the partition function

$$Z = \int \prod_{a=1}^{N_1} d\mu_a \, e^{ik\pi\mu_a^2} \prod_{\hat{a}=1}^{N_2} d\nu_{\hat{a}} \, e^{-ik\pi\nu_{\hat{a}}^2} \frac{\prod_{a<b} \sinh^2(\pi(\mu_a - \mu_b)) \prod_{\hat{a}<\hat{b}} \sinh^2(\pi(\nu_{\hat{a}} - \nu_{\hat{b}}))}{\prod_{a,\hat{a}} \cosh^2(\pi(\mu_a - \nu_{\hat{a}}))}$$

- $1/6$ BPS Wilson loop in the fundamental of first group is same insertion as above.

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What about the $1/2$ BPS loop?

What is this matrix model?

How do we solve it?

Review: 1/2 BPS Wilson loop in 4d

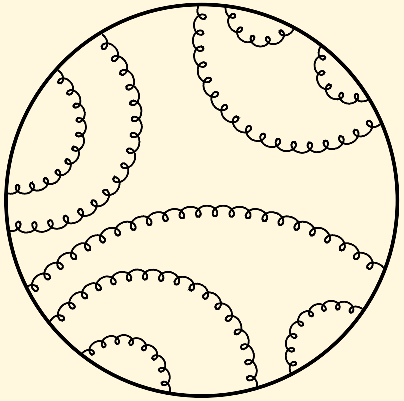
[Erickson, Semenoff, Zarembo][ND, Gross][Pestun]

- In $\mathcal{N}=4$ SYM can take the circular Wilson loop

$$W=\frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\left[i\int(A_\mu\dot{x}^\mu+i\Phi|\dot{x}|)dt\right]$$

- Sum over ladder graphs given by a Gaussian matrix model!

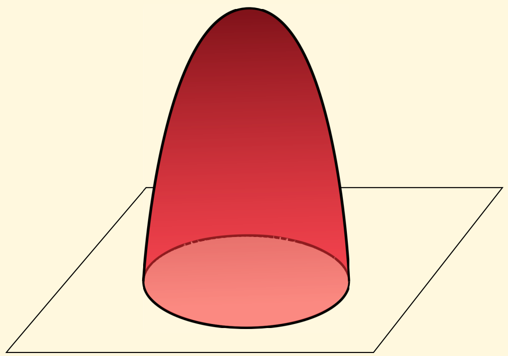
$$\langle W\rangle=\frac{1}{Z}\int\mathcal{D}M\,\frac{1}{N}\mathrm{Tr}\,e^M\,e^{-\frac{2}{g^2}\mathrm{Tr}\,M^2}$$



- Proven to be exact.
- In the planar limit the eigenvalues condense to a cut

$$\langle W\rangle=\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}}d\mu\,\rho_0(\mu)e^\mu=\frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda})\underset{\lambda\to\infty}{\longrightarrow}e^{\sqrt{\lambda}}\qquad\rho_0(\mu)=\frac{2}{\pi\lambda}\sqrt{\lambda-\mu^2}$$

- Exactly matches the action for the corresponding classical string in $AdS_5\times S^5$.
- Generalized to theories with $\mathcal{N}=2$ supersymmetry.
- By using D3 or D5 brane can also match $1/N$ terms.



Localization for $1/2$ BPS Wilson loop

- Can use the same localization term for the $1/2$ BPS loop (they share the supercharges).
- Take a $1/6$ BPS Wilson loop of the form ($\mathcal{R}$ is a rep of $U(N_1|N_2)$)

$$W_{\mathcal{R}}^{(1/6)} \equiv \operatorname{Tr}_{\mathcal{R}} \mathcal{P} \exp\left(i \int L^{(1/6)}\, d\tau\right), \qquad L^{(1/6)} = \begin{pmatrix} \mathcal{A}^{(1/6)} & 0 \\ 0 & \widehat{\mathcal{A}}^{(1/6)} \end{pmatrix}$$

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> The difference between the $1/2$ BPS Wilson loop and this specific $1/6$ BPS loop is Q-exact.

- The Wilson loop in the fundamental of $U(N_1|N_2)$ inserts into the matrix model

$$W = \sum_{a=1}^{N_1} e^{2\pi\mu_a} + \sum_{\hat{a}=1}^{N_2} e^{2\pi\nu_{\hat{a}}}$$

- For a general representation the insertion is

$$W_{\mathcal{R}} = \mathrm{Tr}_{\mathcal{R}} \begin{pmatrix} \mathrm{diag}(e^{2\pi\mu_a}) & 0 \\ 0 & \mathrm{diag}(e^{2\pi\nu_{\hat{a}}}) \end{pmatrix} = \mathrm{sTr}_{\mathcal{R}} \begin{pmatrix} \mathrm{diag}(e^{2\pi\mu_a}) & 0 \\ 0 & -\mathrm{diag}(e^{2\pi\nu_{\hat{a}}}) \end{pmatrix}$$

- These are the natural observables in this super matrix model!

What is this matrix model

- From localization $U(N)$ Chern Simons theory on S^3 is captured by the matrix model

$$Z = \int \prod_{a=1}^{N} d\mu_a \, e^{ik\pi\mu_a^2} \prod_{a<b} \sinh^2(\pi(\mu_a - \mu_b))$$

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- On $S^3/\mathbb{Z}_2$ there are different saddle points to expand around, with different Wilson loops around the non-contractible cycle. This breaks $U(N) \to U(N_1) \times U(N_2)$. The relevant matrix model is

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- Replacing $U(N_1 + N_2) \to U(N_1) \times U(N_2)$ by $U(N_1|N_2) \to U(N_1) \times U(N_2)$ gives the cosh in the denominator and the ABJM matrix model.

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  Same result can be found by doing a semiclassical expansion.

- On $S^3/\mathbb{Z}_2$ there are different saddle points to expand around, with different Wilson loops around the non-contractible cycle. This breaks $U(N)\to U(N_1)\times U(N_2)$. The relevant matrix model is

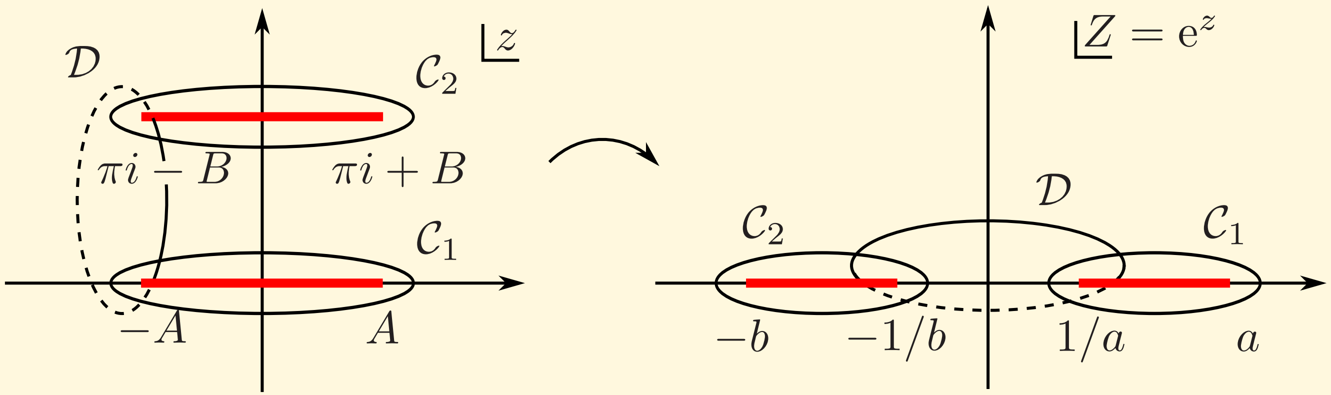
$$Z=\int\prod_{a=1}^{N_1}d\mu_a\,e^{ik\pi\mu_a^2}\prod_{\hat a=1}^{N_2}d\nu_{\hat a}\,e^{ik\pi\nu_{\hat a}^2}\prod_{a<b}\sinh^2(\pi(\mu_a-\mu_b))\prod_{\hat a<\hat b}\sinh^2(\pi(\nu_{\hat a}-\nu_{\hat b}))$$
$$\times\prod_{a,\hat a}\cosh^2(\pi(\mu_a-\nu_{\hat a}))$$

  The ν eigenvalues are effectively shifted by πi from the real line.

- Replacing $U(N_1+N_2)\to U(N_1)\times U(N_2)$ by $U(N_1|N_2)\to U(N_1)\times U(N_2)$ gives the cosh in the denominator and the ABJM matrix model.

- Same is achieved by doing the calculation for $U(N_1+N_2)$ and analytically continuing to negative N_2.

Solving the matrix model



- Using the matrix

$$U = \begin{pmatrix} e^{\mu_i} & 0 \\ 0 & -e^{\nu_j} \end{pmatrix},$$

we define the resolvent

$$\omega(z) = g_s \left\langle \mathrm{tr} \left(\frac{Z+U}{Z-U} \right) \right\rangle = g_s \left\langle \sum_{i=1}^{N_1} \coth\left(\frac{z-\mu_i}{2} \right) \right\rangle + g_s \left\langle \sum_{j=1}^{N_2} \tanh\left(\frac{z-\nu_j}{2} \right) \right\rangle$$

- It is possible to show that in the planar approximation the resolvent is [Halmagyi Yasnov]

$$\omega_0(z) = 2\log\left(\frac{1}{2\sqrt{\beta}} \left[\sqrt{(Z+b)(Z+1/b)} - \sqrt{(Z-a)(Z-1/a)} \right] \right)$$

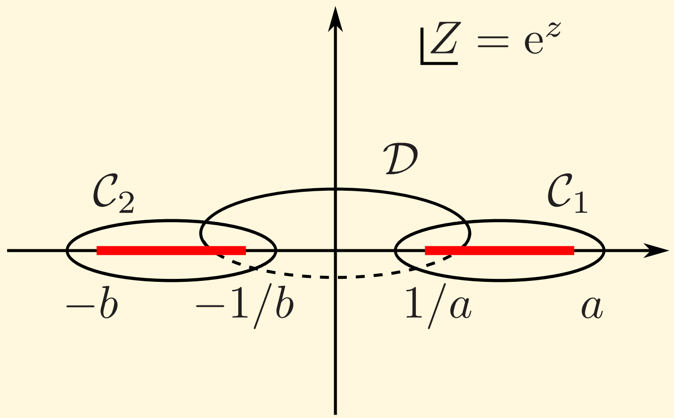
with

$$\zeta = \frac{1}{2}\left(a + \frac{1}{a} - b - \frac{1}{b} \right), \qquad \beta = \frac{1}{4}\left(a + \frac{1}{a} + b + \frac{1}{b} \right).$$

- The $\mathcal{C}$ cycles give the 't Hooft couplings

$$t_i = \frac{1}{4\pi i}\oint_{\mathcal{C}_i} \omega_0(z)\, dz, \qquad i = 1, 2.$$

$$t_i = \pm g_s N_i = \pm 2\pi\lambda_i\,, \qquad g_s = \frac{2\pi i}{k}$$



- A simple quantity is $t = t_1 + t_2 = 2\pi i(\lambda_1 - \lambda_2)$.
- It can be found from the asymptotics of ω_0 at infinity

$$\omega_0 = \log\beta + \frac{\zeta}{Z} + \cdots$$

- Useful also to define

$$B = \lambda_1 - \lambda_2 + \tfrac{1}{2}\,, \qquad \kappa = e^{-\pi i B}\zeta\,.$$

- It is harder to calculate λ_1 and λ_2 independently.
- The $\mathcal{D}$-cycle integral gives the derivative of the planar free energy

$$\mathcal{I} \equiv \frac{\partial F_0}{\partial t_1} - \frac{\partial F_0}{\partial t_2} - \pi i t = -\frac{1}{2}\oint_{\mathcal{D}} \omega_0(z)\, dz, \qquad Z = \exp\left[g_s^{-2}F_0 + O(g_s^0)\right]$$

Wilson loops

- Wilson loops are also $\mathcal{C}$-cycle integrals

$$\left\langle W_{\square}^{1/6}\right\rangle = \oint_{\mathcal{C}_1} \frac{dz}{4\pi i}\,\omega(z)\,e^z = \oint_{\mathcal{C}_1} \frac{dZ}{4\pi i}\,\omega(Z).$$

$$\left\langle W_{\square}^{1/2}\right\rangle = \oint_{\mathcal{C}_1+\mathcal{C}_2} \frac{dz}{4\pi i}\,\omega(z)\,e^z = \oint_{\mathcal{C}_1+\mathcal{C}_2} \frac{dZ}{4\pi i}\,\omega(Z).$$

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- The $1/6$ BPS Wilson loop is complicated.
- The $1/2$ BPS Wilson loop is much simpler. Can be calculated from the asymptotics of ω_0 at infinity.

$$\left\langle W_{\Box}^{1/2}\right\rangle = \frac{\zeta}{2}.$$

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To evaluate the $1/2$ BPS Wilson loop we "just" need to relate ζ to λ_i...

Weak coupling

- Can expand around $\zeta = 0$ (or $a \sim b \sim 1$). And calculate the different quantities.
- Inverting the $\mathcal{C}$-cycle integrals we get

$$\begin{aligned}\kappa = & -2i(t_1 - t_2) - \frac{i}{12}\left(t_1^3 + 3t_1^2 t_2 - 3t_1 t_2^2 - t_2^3\right) \\ & - \frac{i}{960}\left(t_1^5 + 5t_1^4 t_2 - 10t_1^3 t_2^2 + 10t_1^2 t_2^3 - 5t_1 t_2^4 - t_2^5\right) + O(t^7).\end{aligned}$$

- κ is the same as ζ, up to a phase. So this is the VEV of the $1/2$ BPS Wilson loop.
- likewise for the $1/6$ BPS loop

$$\left\langle W_{\square}^{1/6}\right\rangle = e^{\pi i \lambda_1} 2\pi i \lambda_1 \left(1 - \frac{\pi^2}{6}\lambda_1(\lambda_1 - 6\lambda_2) - \frac{\pi^3 i}{2}\lambda_1 \lambda_2^2 + \frac{\pi^4}{120}\lambda_1\left(\lambda_1^3 - 10\lambda_1^2\lambda_2 - 20\lambda_2^3\right) + \cdots\right)$$

- The free energy is

$$F = \frac{N_1^2}{2}\log\left(\frac{2\pi i N_1}{k}\right) + \frac{N_2^2}{2}\log\left(-\frac{2\pi i N_2}{k}\right) - \frac{3}{4}(N_1^2 + N_2^2) - \log(4) N_1 N_2 + \cdots$$

Strong coupling

- Differentiating with respect to ζ or β gives elliptic integrals

$$\frac{dt_{1,2}}{d\zeta} = -\frac{1}{4\pi i}\oint_{\mathcal{C}_{1,2}} \frac{dZ}{\sqrt{(Z^2-\zeta Z+1)^2-4\beta^2 Z^2}} = \pm\frac{\sqrt{ab}}{\pi(1+ab)}\,K(k), \qquad k^2 = 1-\left(\tfrac{a+b}{1+ab}\right)^2.$$

- Similar expressions exist for the free energy and to the $1/6$ BPS Wilson loop.
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- Useful fancy machinery: Picard-Fuchs equations.
- For $\beta = 1$ ($\lambda_1 = \lambda_2$) this is

$$\lambda(\kappa) = \frac{\kappa}{8\pi} \, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16}\right).$$

- The full solution for all ζ, β is known too. The large λ behavior is

$$\lambda_1(\kappa, B) = \frac{1}{2}\left(B^2 - \frac{1}{4}\right) + \frac{1}{24} + \frac{\log^2 \kappa}{2\pi^2} + \cdots$$

- The natural variable has a shift

$$\hat{\lambda} = \lambda_1 - \frac{1}{2}\left(B^2 - \frac{1}{4}\right) - \frac{1}{24} = \frac{1}{2}(\lambda_1 + \lambda_2) - \frac{1}{2}(\lambda_1 - \lambda_2)^2 - \frac{1}{24}.$$

  Same shift arises in calculating the D2-brane charge in the supergravity background!

- After the shift we have

$$\hat{\lambda}(\kappa, B) = \frac{\log^2 \kappa}{2\pi^2} + \cdots$$

- Inverting this we get

$$\kappa = e^{\pi\sqrt{2\hat{\lambda}}}$$

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- Including subleading terms we get instanton corrections

$$\kappa(\hat{\lambda}, B) = e^{\pi\sqrt{2\hat{\lambda}}} \left(1 + \sum_{\ell \geq 1} c_\ell \left(\frac{1}{\pi\sqrt{2\hat{\lambda}}}, \beta \right) e^{-2\ell\pi\sqrt{2\hat{\lambda}}} \right)$$

$$c_1(x, \beta) = -\left(\beta + \beta^{-1}\right)\left(1 - \frac{x}{2}\right),$$

- Instanton action agrees with a string wrapping a $\mathbb{CP}^1 \subset \mathbb{CP}^3$.
- Can also calculate the $1/6$ BPS loop.
- Also non-planar corrections.

Free energy

- Similarly can calculate the $\mathcal{D}$ period.

$$\partial_{\hat{\lambda}} F_0(\lambda_1, \lambda_2) = 2\pi^2 \log \kappa - 2\pi^3 i \left(B - \frac{1}{2} \right) + \cdots$$

- Using the expression fo κ in terms of $\hat{\lambda}$ and integrating gives

$$F_0(\hat{\lambda}, B) = \frac{4\pi^3\sqrt{2}}{3} \hat{\lambda}^{3/2} + \sum_{\ell \geq 1} e^{-2\pi\ell\sqrt{2\hat{\lambda}}} f_\ell \left(\frac{1}{\pi\sqrt{2\hat{\lambda}}}, \beta \right) - \pi^3 i (\lambda_1^2 - \lambda_2^2),$$

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- Subleading corrections are world-sheet instantons.
- Can calculate non-planar corrections efficiently to high-genus.
- Large N expansion seems to be Borel resummable.

Near CS limit

- Can consider the case of finite λ_1 infinitesimal λ_2.
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- Would be interesting to explore this for non-protected operators.

Summary

- The $1/2$ BPS is the natural dual of the fundamental string in AdS_4.
  - Has a very natural expression in the supergroup Chern-Simons matrix model.
- The BPS Wilson loop provide the first weak to strong coupling interpolating function in ABJM theory.
- $1/6$ BPS loop (without a $1/2$ BPS completion) can also be calculated exactly.
- The vacuum is another BPS protected state and we can calculate it
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The end