Ground States of S-duality Twisted $N = 4$ SYM

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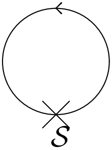
Based on 1007.3749 (with O. J. Ganor & H. S. Tan), 0812.1213 (with O. J. Ganor).

Outline

Compactification with Duality Twist

Consider the $N = 4$ U(n) SYM, compactified on a circle, with S-duality twist:



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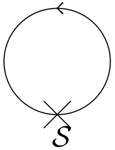

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The Hamiltonian Formalism

- Consider the theory on $X^4 = \mathbb{R} \times M^3$.
- Work in the temporal gauge $A_0 = 0$, so that

$$A = A_i dx^i, \quad i = 1, 2, 3\,.$$

- States are described by gauge-invariant wavefunctions $\Psi = \Psi[A]$.
- Electric and magnetic fields are represented by operators acting on wavefunctions:

$$B^i \Psi[A] = (\epsilon^{ijk} \partial_j A_k) \Psi[A] \qquad \text{"Position operator"}$$

$$E^i \Psi[A] = -2\pi i \frac{\delta}{\delta A_i} \Psi[A] \qquad \text{"Momentum operator"}$$

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S-duality exchanges electric and magnetic fields:

$$E \rightarrow B\,, \quad B \rightarrow -E\,.$$

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Is there a functional integral transform that realizes the S-duality?

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Answer

Define $\mathcal{S} : \Psi[A] \mapsto \tilde{\Psi}[A]$ by

$$\begin{aligned}\tilde{\Psi}[A] &= \int \mathcal{D}A' \mathcal{S}(A, A') \Psi[A'] \\ &\equiv \int \mathcal{D}A' e^{-\frac{i}{2\pi} \int A \wedge dA'} \Psi[A'] \,.\end{aligned}$$

Checking the Formula

To check the formula, suppose

$$\Psi[A] = \delta[A - a]$$

which is an eigenstate of the magnetic field operator:

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Then

$$\tilde{\Psi}[A] = \int \mathcal{D}A' e^{-\frac{i}{2\pi}\int A\wedge dA'}\Psi[A'] = e^{-\frac{i}{2\pi}\int A\wedge da}\,,$$

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Full Duality Group Action

In general, the full duality group is SL(2,$\mathbb{Z}$); an element

$$\mathbf{s} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \quad \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{Z}, \quad \mathbf{ad} - \mathbf{bc} = 1$$

of this group acts on the electric and magnetic fields as

$$E \longrightarrow \mathbf{a}E + \mathbf{b}B, \quad B \longrightarrow \mathbf{c}E + \mathbf{d}B\,.$$

Full Duality Group Action

This is realized by [Lozano; Witten]

Action of Full Duality Group

$$\tilde{\Psi}[A] = \int \mathcal{D}A' \mathcal{S}(A, A') \Psi[A'] ,$$

where

$$\mathcal{S}(A, A') = \exp \left[\frac{i}{4\pi \mathbf{c}} \int (\mathbf{a} A \wedge dA - 2A \wedge dA' + \mathbf{d} A' \wedge dA') \right] .$$

Can check that they form a representation of SL(2,$\mathbb{Z}$).

Compactification with S-duality Twist: Abelian Case

In the $R \to 0$ limit,

$$Z = \operatorname{tr} \mathcal{S} = \int \mathcal{D}A\, \mathcal{S}(A, A)\,,$$

where

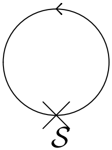
$$\mathcal{S}(A, A) = \exp\left[\frac{i(\mathbf{a} + \mathbf{d} - 2)}{4\pi \mathbf{c}} \int A \wedge dA\right].$$

So the 3d theory is the abelian Chern–Simons theory at level

$$k = \frac{\mathbf{a} + \mathbf{d} - 2}{\mathbf{c}}\,.$$

Question

What happens in the non-abelian case?



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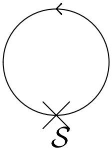

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Self-dual Coupling Constants

- The S-duality also acts on the gauge coupling:

$$\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \to \tilde{\tau} = \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}\,.$$

Therefore, $\mathcal{S}$ should be regarded as a unitary map

$$\mathcal{S} : \mathcal{H}(\tau) \to \mathcal{H}(\tilde{\tau})\,.$$

- For the S-twist to be well-defined, we need $\tau = \tilde{\tau}$.

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Self-dual Coupling Constants

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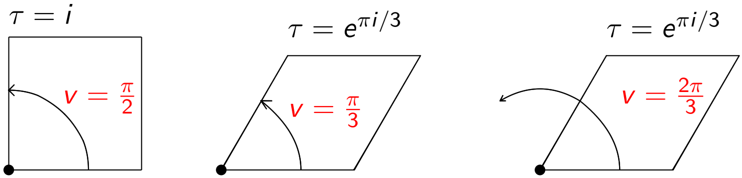
- $\tau = e^{\pi i/3}$ and $\mathbf{s} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$;
- $\tau = i$ and $\mathbf{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;
- $\tau = e^{\pi i/3}$ and $\mathbf{s} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

S-duality as Rotation of a Torus

For self-dual values of τ and corresponding $\mathbf{s} \in \mathrm{SL}(2,\mathbb{Z})$, one can show that

$$\mathbf{c}\tau + \mathbf{d} = e^{iv} ,$$

and the action of $\mathbf{s}$ can be regarded as a rotation of a torus with complex structure τ by an angle v:



Broken Supersymmetries

- Under the S-duality action of $\mathbf{s} \in \mathrm{SL}(2,\mathbb{Z})$ listed above, supercharges transform as [Kapustin & Witten]

$$Q_{a\alpha} \to e^{-iv/2} Q_{a\alpha}\,, \quad a = 1, \ldots, 4, \quad \alpha = 1, 2\,.$$

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R-symmetry twist

Consider an element of the R-symmetry group Spin(6) $= SU(4)$

$$\gamma = \begin{pmatrix} e^{i\varphi_1} & & & \\ & e^{i\varphi_2} & & \\ & & e^{i\varphi_3} & \\ & & & e^{i\varphi_4} \end{pmatrix} \in SU(4)\,.$$

Transformations of Scalars and Fermions

- Fermions ψ^a_α are in the **4** of $SU(4)$:

$$\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\\ \psi^4\end{pmatrix} \to \begin{pmatrix}e^{i\varphi_1} & & & \\ & e^{i\varphi_2} & & \\ & & e^{i\varphi_3} & \\ & & & e^{i\varphi_4}\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\\ \psi^4\end{pmatrix}.$$

- Scalars Φ^I are in the **6** of $SO(6)$:

$$\begin{pmatrix}Z^1\\ Z^2\\ Z^3\end{pmatrix} \to \begin{pmatrix}e^{i(\varphi_1+\varphi_4)} & & \\ & e^{i(\varphi_2+\varphi_4)} & \\ & & e^{i(\varphi_3+\varphi_4)}\end{pmatrix}\begin{pmatrix}Z^1\\ Z^2\\ Z^3\end{pmatrix},$$

where $Z^I = \Phi^{2I-1} + i\Phi^{2I}$.

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Supersymmetry Restored

Under the combined S-duality and R-symmetry twists, supercharges transform as

$$Q_{a\alpha} \to e^{i(\varphi_a - v/2)} Q_{a\alpha} \,.$$

By choosing

$$\gamma = \begin{pmatrix} e^{iv/2} & & & \\ & e^{iv/2} & & \\ & & e^{iv/2} & \\ & & & e^{-3iv/2} \end{pmatrix} ,$$

we can preserve 12 supercharges, or $N = 6$ supersymmetry in 3D.

The Low Energy Limit?

Question

What is the low energy limit of the $N = 4$ SYM with gauge group of $U(n)$ compactified on S^1 with S-duality and R-symmetry twists?

- For $n = 1$, it was a topological theory.
- For large enough n, it cannot be a topological theory—there are low energy propagating degrees of freedom.

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Zero Modes

Consider the gauge-invariant operator

$$O_p = \frac{1}{g^p} \mathrm{tr}\, Z^p \,, \quad Z = \Phi^1 + i\Phi^2 \,.$$

Under the combined $\mathcal{S}\gamma$-twist, it transforms as [Intrilligator]

$$O_p \to e^{-ipv} O_p \,.$$

So non-zero $\langle O_p \rangle$ is compatible with the twist only if

$$p \in \begin{cases} 6\mathbb{Z} & \text{if } \tau = e^{\pi i/3},\ v = \pi/3 \,, \\ 4\mathbb{Z} & \text{if } \tau = i,\ v = \pi/2 \,, \\ 3\mathbb{Z} & \text{if } \tau = e^{\pi i/3},\ v = 2\pi/3 \,. \end{cases}$$

Zero Modes

- For example, in the U(3) theory at $\tau = i$, O_1, O_2, O_3 have zero expectation values, and so do all O_p.
- On the other hand, in the U(4) theory, one can explicitly construct a zero mode.

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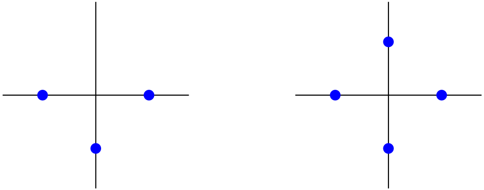

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Compactifying on a Torus

We further compactify the theory on a torus and study the space of ground states:

$$X = S^1 \times M^3 = S^1 \times \mathbb{R} \times T^2 \,.$$

Embedding into String Theory

To embed the setting into string theory,

- Start with type IIB theory on $\mathbb{R}^{10}$ with string coupling τ_{IIB}.
- Put n D3-branes along x^1, x^2, x^3 directions and at $x^4 = \cdots = x^9 = 0$.
- Compactify x^3 with $\mathcal{S}$-twist and by a rotation $\gamma \in \text{Spin}(6)$ in the transverse direction.
- Compactify x^1, x^2 directions on circles of radii L_1, L_2.

Duality Maps

Type	0	1	2	3	4,. . .,9	10	
D3	–	–	–	–	·	×	IIB on T^2
D2	–	·	–	–	·	×	IIA on T^2
M2	–	·	–	–	·	·	M on T^3
F1	–	·	×	–	·	·	IIA on T^2

- The torus in the x^1x^{10}-directions have complex structure $\tau = \tau_{\text{IIB}}$ and area $A/\alpha'_{\text{IIA}} = L_2/L_1$.
- Take the limit

$$g_{\text{IIA}} \sim \frac{L_1^{1/2} L_2^{3/2}}{\alpha'_{\text{IIB}}} \to 0 \qquad \text{and} \qquad \frac{R}{\alpha'^{1/2}_{\text{IIA}}} = \frac{R(L_1L_2)^{1/2}}{\alpha'_{\text{IIB}}} \gg 1\,.$$

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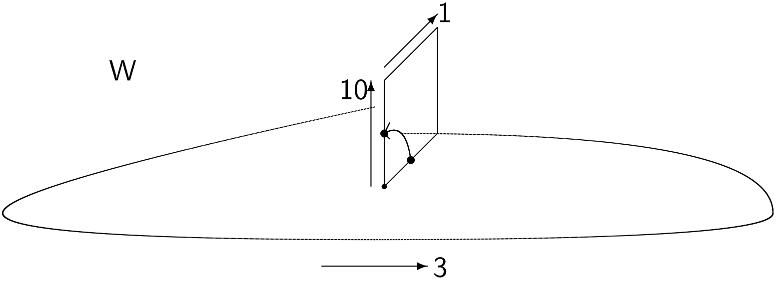
Transverse Configuration

- Rotation by $\gamma \in \mathrm{Spin}(6)$ fixes the branes at $x^4 = \cdots = x^9 = 0$ for low enough n.
- On the T^2 in the $x^1 x^{10}$ directions, $\mathcal{S}$ rotates the torus by an angle v, so the background forms a torus bundle over a circle.



Ground States of the System

Conclusion

- Given n, ground states of the system correspond to string configurations of minimal lengths with total winding number n.
- Strings of minimal lengths start and end at the fixed points of the T^2.

Counting Single String States

Focus on $\tau = i$ ($v = \frac{\pi}{2}$) case.

- For $n = 1$, there are two fixed points $z = 0$ and $z = \frac{1}{2} + \frac{1}{2}\tau$:

- For $n = 2$, there are three states:

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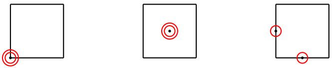

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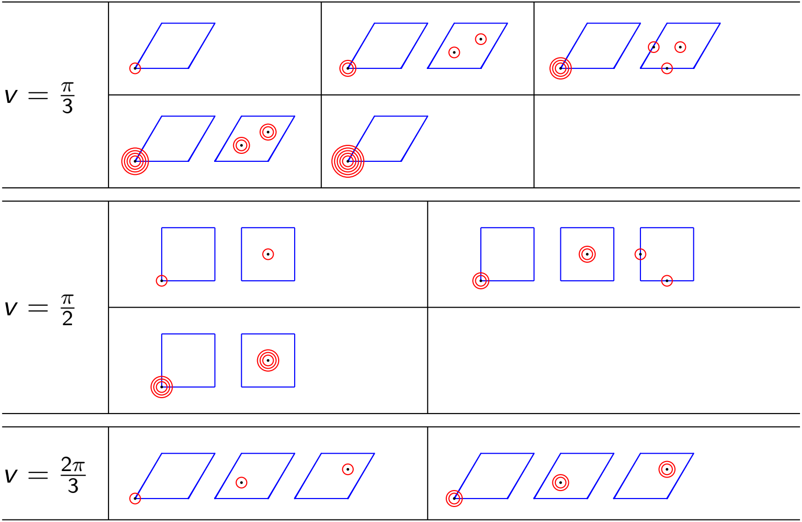
Single String Ground States



Multi-String Ground States

- Multi-string ground states are just (symmetric) tensor products of single string ground states.
- Given total winding number n, ground states are classified into sectors, each corresponding to a partition of n:

$$\sigma = (n_1, \ldots, n_s) \qquad \text{with} \qquad n_1 + \cdots + n_s = n\,.$$

Multi-String Ground States

Focus on $\tau = i$ ($v = \frac{\pi}{2}$) case.

Total Winding n	Partition of n	Dimension
1	(1)	2
2	(2)	3
	(1,1)	3
3	(3)	2
	(2,1)	6
	(1,1,1)	4

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Discrete Momenta and Winding Numbers

$\mathbb{Z}_k$-valued momenta and winding numbers

- The $\mathbb{Z}_k$-valued momentum is associated with the discrete isometry of the torus bundle W:

$$U : (x^3, z) \mapsto (x^3, z + \frac{1+\tau}{k}) .$$

- The $\mathbb{Z}_k$-valued winding number is given by the torsion part of the first homology group of the torus bundle.

$$H_1(W; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_k .$$

Discrete Momenta and Winding Numbers

For $\tau = i$ ($v = \frac{\pi}{2}$) and $n = 1$ case,

- States with momentum m have eigenvalues $U = e^{2\pi i m/k}$.

- States with winding number n have eigenvalues $V = e^{2\pi i n/k}$.

Modular Transformation from String Dualities

- After the embedding the setting into the string theory and following the chain of dualities, the complex structure ρ of x^1x^2 torus becomes the Kähler structure of the x^1x^{10} torus.
- Therefore, the modular transformation of type IIB torus becomes the T-duality of the type IIA theory.
- In particular, T-duality exchanges the momentum and winding states.

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Action of T-duality Group

- The T-duality group $SL(2,\mathbb{Z})$ is generated by $S : \rho \mapsto -\frac{1}{\rho}$ and $T : \rho \mapsto \rho + 1$, and we expect

$$S^{-1}US = V^{-1}\,, \qquad S^{-1}VS = U\,,$$
$$T^{-1}UT = e^{i\phi}UV^{-1}\,, \qquad T^{-1}VT = V\,.$$

- From known action of U and V, these commutation relations can be solved for S and T.

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- For $\tau = i$ $(v = \frac{\pi}{2})$ case, we get

$$S|{}_{\odot}\square\rangle = \frac{1}{\sqrt{2}}(|{}_{\odot}\square\rangle + |\boxed{\odot}\rangle)\,, \quad S|\boxed{\odot}\rangle = \frac{1}{\sqrt{2}}(|{}_{\odot}\square\rangle - |\boxed{\odot}\rangle)\,.$$

and

$$T|{}_{\odot}\square\rangle = |{}_{\odot}\square\rangle\,, \quad T|\boxed{\odot}\rangle = e^{\frac{\pi i}{2}}|\boxed{\odot}\rangle\,.$$

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The U(1) Chern–Simons Theory

- The Hilbert space of $U(1)$ Chern–Simons theory on $\mathbb{R} \times M^2$ can be obtained by quantizing the space of flat connections of line bundles on M^2.
- For a flat connection $A = A_1 dx^1 + A_2 dx^2$ on T^2, define

$$\mathfrak{a} = -\frac{i\rho_2}{\pi} A_{\bar{z}} \equiv \frac{1}{2\pi}(-\rho A_1 + A_2)\,,$$

which is valued on the ("dual") torus of complex structure ρ.
- Wavefunctions are holomorphic sections of the line bundle over the "dual" torus with first Chern class k:

$$\psi_p(\mathfrak{a}) = \theta(k\mathfrak{a}+p\rho; k\rho)e^{\frac{\pi k}{2\rho_2}\mathfrak{a}^2+\pi i\rho p^2/k+2\pi i p\mathfrak{a}}\,, \qquad p = 0, \ldots, k-1.$$

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$$\mathfrak{a} = -\frac{i\rho_2}{\pi} A_{\bar{z}} \equiv \frac{1}{2\pi}(-\rho A_1 + A_2)\,,$$

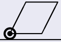
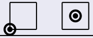
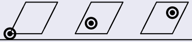
  which is valued on the ("dual") torus of complex structure ρ.
- Wavefunctions are holomorphic sections of the line bundle over the "dual" torus with first Chern class k:

$$\psi_p(\mathfrak{a}) = \theta(k\mathfrak{a}+p\rho; k\rho) e^{\frac{\pi k}{2\rho_2}\mathfrak{a}^2+\pi i\rho p^2/k+2\pi i p\mathfrak{a}}\,, \qquad p = 0, \ldots, k-1.$$

Identifying Ground States

The $U(1)$ Result

- Dimensions match: level k CS theory has k states.

k	Ground States
1	
2	
3	

- Furthermore, the action of modular transformations $S : \rho \mapsto -\frac{1}{\rho}$ and $T : \rho \mapsto \rho + 1$ agree with the action of T-duality group. (This can be used to identify the ground states on both sides.)

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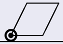
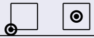
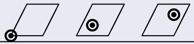
1	
2	
3	

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Non-abelian Case: the Easy Part

- To define $U(n)$ Chern–Simons theory, need to specify two levels:

$$U(n)_{k',k} = [U(1)_{k'} \times SU(n)_k]/\mathbb{Z}_n \,.$$

- The case with $k' = kn$ is especially nice [Eliztur, Moore, Schwimmer & Seiberg]:

$$\mathcal{H}[U(n)_{kn,k}] = \otimes^n \mathcal{H}[U(1)_k]/S_n \,.$$

- But the RHS is nothing but the space of multi-particle ground states corresponding to $\sigma = (1, \ldots, 1)$!
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Non-abelian Case: the Hard Part

Our strategy goes as follows. First, focus on single string sectors $\sigma = (n)$.

- Make an Ansatz that the type IIA Hilbert space equals that of $U(n)_{k',k}$ CS theory for k', k:

$$|\text{String State}\rangle = \sum_{p=0}^{k'-1} |\psi; p\rangle_{SU(n)} \otimes |p\rangle_{U(1)} \,.$$

- Perform T-duality/modular transformation on both sides, and read off the modular transformation property of $SU(n)$ part.
- Check if it gives the correct modular transformation for $SU(n)_k$ CS theory. The latter can be read off from the known result $U(n)_{kn,k} = [U(1)_{kn} \times SU(n)_k]/\mathbb{Z}_n = \otimes^n U(1)_k / S_n$.

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The Non-abelian Result

For $\tau = i$ ($v = \frac{\pi}{2}$) case,

Total Winding n	Partition of n	CS Counterpart
1	(1)	$U(1)_2$
2	(2)	$U(2)_{4,-2}$
	(1,1)	$U(2)_{4,2}$
3	(3)	$U(3)_{6,-1}$
	(2,1)	$U(2)_{4,-2} \times U(1)_2$
	(1,1,1)	$U(3)_{6,2}$

- For $\tau = e^{\pi i/3}$ and $v = \frac{\pi}{3}$ case, there are three sectors, all of which can be explicitly identified as CS Hilbert space.
- For $\tau = e^{\pi i/3}$ and $v = \frac{2\pi}{3}$ case, there are 16 sectors, all but one of which can be explicitly identified as CS Hilbert space.
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Summary

- Compactification of $N = 4$ SYM with duality twist seems to contain topological sectors in the low energy limit.
- Upon further compactification on T^2, the space of ground states can be obtained via string theory dualities.
- Their dimensions and modular transformation properties match those of Chern–Simons theory (in nearly all cases considered).

Outlook

- Calculation of Wilson–'t Hooft operators?
- For large enough rank of gauge group, is this theory ABJM?
- Can we work with $N = 2$ supersymmetry?
- What can we learn from 6-dimensional (2,0) theory perspective?
- What is the nature of the full S-duality kernel $\mathcal{S}(A, A')$?