S^2 partition functions: Coulomb vs Higgs localization and vortices

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Introduction

- S^d partition functions:

Euclidean SUSY theory on S^d (not twisted as in [Witten 88; Vafa, Witten 94])

Compute path-integral: $$Z_{S^d}(t) = \int_{S^d} \mathcal{D}\Phi \; e^{-S[\Phi,t]}$$

Parameters t: from flat space Lagrangian or S^d

With enough SUSY, *exactly* computable with localization techniques.
One can compute VEVs of SUSY operators (*e.g.* line operators) as well.

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- Examples:

4d with $\mathcal{N}=2$ SUSY [Pestun 07]
3d with $\mathcal{N}=2$ SUSY [Kupustin, Willett, Yaakov 09; Jafferis 10; Hama, Hosomichi, Lee 11]
5d with $\mathcal{N}=1$ SUSY [Hosomici, Seong, Terashima 12; Kallen, Qiu, Zabzine 12]
2d with $\mathcal{N}=(2,2)$ SUSY [FB, Cremonesi 12; Doroud, Gomis, Le Floch, Lee 12]

Introduction

$Z_{S^d}(t)$ is an interesting function:

- Information about the theory that can be computed exactly (and non-perturbatively) at strong coupling
- Information about the IR fixed point
- Dualities correspond to interesting (and often not-yet-proven) mathematical identities
- In 3d it provides a "central charge" [Jafferis 10; Jafferis, Klabanov, Pufu, Safdi 11] that decreases from fixed point to fixed point along RG flows
- Very interesting mathematical structures [Alday, Gaiotto, Tachikawa 09; Dimofte, Gaiotto, Gukov 11; Cecotti, Cordova, Vafa 11]

Introduction

- Consider two-dimensional theories with $\mathcal{N} = (2, 2)$ SUSY and vector-like R-symmetry $U(1)_R$

Vector multiplets $+$ chiral multiplets

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and vector-like R-symmetry $U(1)_R$

Vector multiplets + chiral multiplets

Summary of results:

- Z_{S^2} computed with localization techniques

$\rightarrow$ integral over "Coulomb branch", sum over flux sectors:

$$Z_{S^2} = \sum_{\mathfrak{m}} \int d\sigma \; e^{-S_{\text{class}}} \; Z^{\text{1-loop}}_{\text{vector}} \; Z^{\text{1-loop}}_{\text{chiral}} \; .$$

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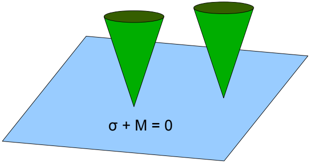
Surprise: localization can be performed in a *different* way

$\rightarrow$ sum over discrete "Higgs branch":

$$Z_{S^2} = \sum_{\text{Higgs vacua}} e^{-S_{\text{class}}} Z_{\text{1-loop}} Z_{\text{vortex}} Z_{\text{anti-vortex}}$$



Z_{vortex}: partition function of vortices on $\mathbb{R}^2_\varepsilon$ in Ω-background [Shadchin 06; Nekrasov 02]

Vortices at north pole, antivortices at south pole

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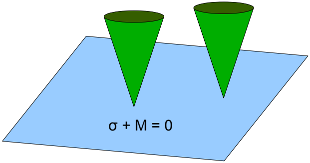

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Vortices at north pole, antivortices at south pole

- Different localizations because Euclidean fields are complexified, and path-integral is along a contour
- Higgs branch expression reminiscent of Pestun's S^4 result in terms of Z_{inst} [Nekrasov 02]
- Factorization as observed on S^3 [Pasquetti 11]
- $Z^{\text{Coulomb}}_{S^2} = Z^{\text{Higgs}}_{S^2}$ can be used to compute Z_{vortex}

Rigid supersymmetry on S^2

- Two-dimensional $\mathcal{N} = (2,2)$ theories with a vector-like $U(1)_R$ R-symmetry can be placed supersymmetrically on S^2 (2 complex supercharges):

$$\mathfrak{osp}^*(2|2) \cong \mathfrak{su}(2|1) \supset \mathfrak{su}(2) \times \mathfrak{u}(1)_R$$

Contained in global Euclidean superconformal algebra

$$\mathfrak{osp}(2|2,\mathbb{C}) \supset \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{u}(1)^2$$

Algebra:

$$[\delta_\epsilon, \delta_{\bar\epsilon}] = \mathcal{L}^A_\xi + \frac{i}{2r}\alpha R \qquad \xi^\mu = i\bar\epsilon\gamma^\mu\epsilon$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 0 \qquad \alpha = i\bar\epsilon\epsilon$$

$$D_\mu\epsilon = \frac{i}{2r}\gamma_\mu\epsilon$$

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- Vector multiplet: $(A_\mu, \lambda, \bar\lambda, \sigma, \eta, D)$
  Chiral multiplet: $(\phi, \bar\phi, \psi, \bar\psi, F, \bar F)$.

On S^2 freedom to choose R-charges q of chiral multiplets $\quad\rightarrow\quad$ couplings

Supersymmetric actions on S^2

Action constructed order by order in $\frac{1}{r}$ or by coupling to SUGRA [Festuccia, Seiberg 11]

- Yang-Mills action for vector multiplets:

$$\mathcal{L}_{YM} = \frac{1}{g^2}\,\mathrm{Tr}\left\{\frac{1}{2}\Big(F_{12}-\frac{\eta}{r}\Big)^2+\frac{1}{2}\Big(D+\frac{\sigma}{r}\Big)^2+\frac{1}{2}(D_\mu\sigma)^2+\frac{1}{2}(D_\mu\eta)^2-\frac{1}{2}[\sigma,\eta]^2 + \frac{i}{2}\bar{\lambda}\not{D}\lambda+\frac{i}{2}\bar{\lambda}[\sigma,\lambda]+\frac{1}{2}\bar{\lambda}\gamma_3[\eta,\lambda]\right\}$$

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- Twisted superpotential $\widetilde{W}(\Sigma)$

$$\mathcal{L}_{\widetilde{W}} = i\widetilde{W}' \left(D - iF_{12} + \frac{\sigma + i\eta}{r} \right) - \frac{i}{2}\widetilde{W}''\, \bar{\lambda}(1 + \gamma_3)\lambda - \frac{i}{r}\widetilde{W}$$

and its anti-chiral counterpart $\widetilde{W}^*(\Sigma)$. We will take complex conjugate.

Twisted chiral superfield: $\Sigma = (\sigma + i\eta,\ \lambda,\ D - iF_{12})$

Special case: complexified Fayet-Iliopoulos term: $\widetilde{W}(z) = \frac{1}{2}\big(- \xi + \frac{i\theta}{2\pi}\big) z$

$$\mathcal{L}_{FI} = -i\xi\, D + i\frac{\theta}{2\pi}\, F_{12}$$

Supersymmetric actions on S^2

- Matter kinetic action for chiral multiplets (of R-charge q):

$$\mathcal{L}_{\text{mat}} = |D_\mu \phi|^2 + \bar{\phi}\Big(\sigma^2 + \eta^2 + iD + \frac{iq}{r}\sigma + \frac{q(2-q)}{4r^2}\Big)\phi + \bar{F}F$$
$$+ \bar{\psi}\Big(- i\not{D} + i\sigma - \gamma_3\eta - \frac{q}{2r}\Big)\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi$$

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- Superpotential ($R[W]=2$):

$$\mathcal{L}_W = \sum_j \frac{\partial W}{\partial \phi_j} F_j - \frac{1}{2}\sum_{j,k} \frac{\partial^2 W}{\partial \phi_j \partial \phi_k}\psi_j \psi_k$$

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- Couple global flavor symmetries to external vector multiplets, give VEV to $\sigma^{\text{ext}} = -rD^{\text{ext}}$, $\eta^{\text{ext}} = rF_{12}^{\text{ext}}$.

σ^{ext} $\quad\rightarrow\quad$ real (or twisted) masses M

$\sigma^{\text{ext}} + \frac{iq}{2r}$ form a holomorphic pair.

Localization

- Supersymmetric action S and operators $\mathcal{O}$ w.r.t. supercharge $\mathcal{Q}$:

$$[\mathcal{Q}, S] = [\mathcal{Q}, \mathcal{O}] = 0$$

$\mathcal{Q}$-exact terms do not affect the path-integral:

$$\frac{\partial}{\partial t} \int \mathcal{D}\Phi \; \mathcal{O} \; e^{-S - t\{\mathcal{Q}, \mathcal{P}\}} = 0$$

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- Choose exact deformation action with positive definite bosonic part:

$$S_{\mathsf{loc}} = t \sum_{\text{fermions } \chi} \mathcal{Q}\left((\overline{\mathcal{Q}\chi})\, \chi \right) \qquad\qquad S_{\mathsf{loc}}\big|_{\mathsf{bos}} = t \sum_{\chi} \left|\mathcal{Q}\chi\right|^2$$

In $t \to \infty$ limit, only BPS configurations $\mathcal{Q}\chi = 0$ contribute:

$$Z = \sum_{\Phi_0 \,|\, \mathcal{Q}\chi = 0} e^{-S[\Phi_0]} \; Z_{\text{1-loop}}[\Phi_0]$$

Localization

- Choose "equivariant" supercharge:

$$\mathcal{Q}^2 = J + \frac{R}{2} + i\Lambda(\sigma, \eta)$$

Form a superalgebra $\mathfrak{su}(1|1)$.

North and south pole: fixed points of J

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North and south pole: fixed points of J

- All actions constructed before are $\mathcal{Q}$-exact, except the *twisted superpotential*

Z_{S^2} depends on $\widetilde{W}$, (complexified) real masses M and external fluxes $\mathfrak{n}$

Coulomb branch localization

- Regard A_μ, σ, η, D real, and $(\lambda, \bar{\lambda})$, $(\psi, \bar{\psi})$, $(\phi, \bar{\phi})$, $(F, \bar{F})$ complex conjugates

$$\mathcal{L}_{YM} = \mathrm{Tr}\, \mathcal{Q}\big[(\overline{\mathcal{Q}\lambda})\lambda + \lambda^\dagger(\overline{Q\lambda^\dagger})\big] \qquad \mathcal{L}_\psi = \mathrm{Tr}\, \mathcal{Q}\big[(\overline{\mathcal{Q}\psi})\psi + \psi^\dagger(\overline{Q\psi^\dagger})\big]$$

Solve BPS equations:

$$0 = \mathcal{Q}\lambda = \mathcal{Q}\lambda^\dagger \qquad 0 = \mathcal{Q}\psi = \mathcal{Q}\psi^\dagger$$

Simple BPS configurations:

$$\sigma = -r\, D = \text{constant} \qquad F_{12} = \frac{\eta}{r} \equiv \frac{\mathfrak{m}}{2r^2} \qquad [\sigma, \mathfrak{m}] = 0$$

$$\phi = F = 0$$

This is a “Coulomb branch” (very similar to S^3 case)

Coulomb branch localization

The S^2 partition function is:

$$Z_{S^2} = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}} \int \Big(\prod_j \frac{d\sigma_j}{2\pi} \Big) \; Z_{\text{class}}(\sigma, \mathfrak{m}) \; Z_{\text{gauge}}(\sigma, \mathfrak{m}) \; \prod_{\Phi} Z_{\Phi}(\sigma, \mathfrak{m}; M, \mathfrak{n})$$

The one-loop determinants are:

$$Z_{\text{gauge}} = \prod_{\alpha \in G \, , \, \alpha > 0} \Big(\frac{\alpha(\mathfrak{m})^2}{4} + \alpha(\sigma)^2 \Big)$$

$$Z_{\Phi} = \prod_{\rho \in R_\Phi} \frac{\Gamma\Big(\frac{R[\Phi]}{2} - i\rho(\sigma) - i f^a[\Phi] M_a - \frac{\rho(\mathfrak{m}) + f^a[\Phi] n_a}{2} \Big)}{\Gamma\Big(1 - \frac{R[\Phi]}{2} + i\rho(\sigma) + i f^a[\Phi] M_a - \frac{\rho(\mathfrak{m}) + f^a[\Phi] n_a}{2} \Big)}$$

The classical action is:

$$Z_{\text{class}} = e^{-4\pi i \xi \, \mathrm{Tr}\, \sigma - i\theta \, \mathrm{Tr}\, \mathfrak{m}} \exp\Big\{ 8\pi i r \, \mathbb{R}\mathrm{e} \, \widetilde{W}\big(\tfrac{\sigma}{r} + i \tfrac{\mathfrak{m}}{2r} \big) \Big\}$$

We isolated the linear piece in $\widetilde{W}$ (Fayet-Iliopoulos term)

Some checks

- Give large twisted mass to a chiral multiplet: $w = \rho(\sigma) + f^a M_a \to \pm\infty$

$$Z_\Phi \;\to\; e^{8\pi i r \, \mathbb{R}\mathrm{e}\, \widetilde{W}_{\text{eff}}}$$

$$\widetilde{W}_{\text{eff}}(\Sigma) = -\frac{1}{4\pi}\Sigma_s\big[\log(-ir\Sigma_s) - 1\big] \qquad\qquad \Sigma_s = \rho(\Sigma) + f^a M_a$$

reproduces the correct one-loop running of FI term

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- $U(1)$ with 1 fundamental of charge Q:

$$Z_{S^2} = \frac{1}{Q^2}\sum_{n=0}^{|Q|-1} \exp\left[2ie^{-2\pi\xi/Q}\sin\left(\frac{\theta - 2\pi n}{Q}\right)\right]$$

Mirror symmetry [Hori, Vafa 00] : twisted chiral Σ, Y with

$$\widetilde{W} = \frac{1}{4\pi}\big[\Sigma\big(QY - \tau(\mu)\big) + i\mu e^{-Y}\big]$$

The on-shell action evaluated at critical points precisely reproduces Z_{S^2}.

Higgs branch localization

- In the Euclidean theory fields are complexified and we can choose a contour.

Allow σ, D to be complex in BPS eqns $\rightarrow$ Higgs branches and vortex solutions

Motivated by [Pasquetti 11] we might hope to be able to perform localization in such a way that vortices (and *not* Coulomb branch) contribute.

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Motivated by [Pasquetti 11] we might hope to be able to perform localization in such a way that vortices (and *not* Coulomb branch) contribute.

- Trick: introduce *exact* FI term ζ and impose D-term equation:

$$\mathcal{L}_H = i\Big(D + \frac{\sigma}{r}\Big)(\phi\phi^\dagger - \zeta\,\mathbb{1}) + \dots$$

D appears quadratically in localizing action $\mathcal{L}_{\text{loc}} = t\big(\mathcal{L}_{YM} + \mathcal{L}_H + \mathcal{L}_\psi\big)$

Gaussian path-integral $\quad \rightarrow \quad D + \dfrac{\sigma}{r} + i(\phi\phi^\dagger - \zeta\mathbb{1}) = 0$

A posteriori: $D \notin \mathbb{R}$.

Higgs branch localization

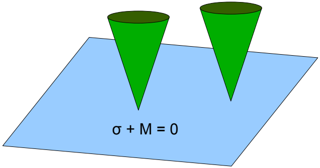
When gauge group gets completely broken, and with generic real masses M:

- Higgs branch: $\phi\phi^\dagger = \zeta\,\mathbb{1} \qquad \big(\sigma + M\big)\phi = 0 \qquad F_{12} = \eta = 0$

  $\rightarrow$ vacua where N chirals get VEV, at fixed positions on Coulomb branch

$$\sigma_a = -M_{l_a} \qquad\qquad a = 1, \ldots, N$$

E.g.: $U(N)$ with (N_f, N_a) flavors: $\vec{l} \in C(N, N_f)$



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- Vortices at north pole, antivortices at south pole

  Limit $\zeta \to \infty$:
  $$\text{NP:} \quad D_{\bar z}\phi = 0 \qquad F_{12} = -\big(|\phi|^2 - \zeta\mathbb{1}\big)$$
  $$\text{SP:} \quad D_z\phi = 0 \qquad F_{12} = |\phi|^2 - \zeta\mathbb{1}$$

Close to poles: $\mathbb{R}^2_\Omega$ [Shadchin 06] $\qquad \mathcal{Q}^2 = J + \frac{R}{2} + i\sigma \quad \rightarrow \quad \varepsilon = \frac{1}{r},\; a = -iM_{\text{eff}}$

Partition function is equivariant integral on the vortex moduli space

$$Z_{\text{vortex}}(z, \varepsilon, a) = \sum_{k=0}^{\infty} z^k Z_k(\varepsilon, a) \qquad\qquad z = e^{-2\pi\xi - i\theta}$$

Higgs branch localization

Result:

$$\boxed{Z_{S^2} = \sum_{\text{vacua}} e^{-4\pi i \xi \sum_{j=1}^{N} \sigma_j} Z'_{\text{1-loop}} \, Z_{\text{v}} \, Z_{\text{av}}}$$

with

$$Z_{\text{v}} = Z_{\text{vortex}}\big((-1)^N z \, , \, \tfrac{1}{r} \, , \, -iM_{\text{eff}}\big)$$
$$Z_{\text{av}} = Z_{\text{vortex}}\big((-1)^N \bar{z} \, , \, -\tfrac{1}{r} \, , \, iM_{\text{eff}}\big)$$

$Z'_{\text{1-loop}}$ does not include the N non-vanishing chiral multiplets.

Vortex partition function of SQCD

$U(N)$ with (N_f, N_a) flavors (assume $N_f \geq N_a$):
k-vortex moduli space in a given vacuum $\vec{l}$ is a symplectic quotient

ADHM-like: Higgs branch of an $\mathcal{N} = 2$ quantum mechanics [Hanany, Tong 03; Eto, Isozumi, Nitta, Ohashi, Sakai 05] , dimensional reduction of a 2d $\mathcal{N} = (0, 2)$ $U(k)$ gauge theory

Z_k is the equivariant volume, getting contribution from fixed points of unbroken symmetry (color-flavor locked phase):

$$U(1)_\varepsilon \times S\big[U(N) \times U(N_f - N)\big] \times U(1) \times SU(N_a)$$

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The result can be written as a contour integral [Nekrasov, Shadchin 04; Dimofte, Gukov, Hollands 10]

$$Z_k = \oint \Big[\prod_{j=1}^{k} \frac{d\varphi_j}{2\pi i} \Big] \, \mathcal{Z}_{\text{vec}}(\varphi, \varepsilon) \; \mathcal{Z}_{\text{fund}}(\varphi, \varepsilon, a) \; \mathcal{Z}_{\text{antifund}}(\varphi, \varepsilon, \tilde{a})$$

Contour encircles multi-poles parametrized by $\vec{k} \in \mathbb{Z}^N_{\geq 0}$ with $\sum k_i = k$:

$$\{\varphi_j\} = \{a_r + (l_r - 1)\varepsilon \mid r \in \vec{l}, \; l_r = 1, \ldots, k_r\}$$

Vortex partition function of SQCD

- Sum over residues at the poles:

$$Z_k = \varepsilon^{(N_a - N_f)k} \sum_{\substack{\vec{k} \in \mathbb{Z}^N_{\geq 0} \\ |\vec{k}|=k}} \prod_{r \in \vec{l}} \frac{\prod_{f=1}^{N_a} \left(\frac{\tilde{a}_f + a_r}{\varepsilon} \right)_{k_r}}{k_r! \, \prod_{\substack{s \in \vec{l} \\ s \neq r}} \left(\frac{a_s - a_r}{\varepsilon} - k_r \right)_{k_s} \prod_{j \notin \vec{l}} \left(\frac{a_j - a_r}{\varepsilon} - k_r \right)_{k_r}}$$

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- Explicitly verify that this Z_k plugged into the Higgs branch localization formula agrees with the Coulomb branch expression

To evaluate Coulomb branch integral, close the contour of integration and sum over residues

Dualities

Equality of Z_{S^2} for pair of theories ($\rightarrow$ conjecture duality):

$$U(N) \text{ with } (N_f, 0) \quad \leftrightarrow \quad U(N_f - N) \text{ with } (N_f, 0) \qquad N_f > 1$$

$$SU(N) \text{ with } (N_f, 0) \quad \leftrightarrow \quad SU(N_f - N) \text{ with } (N_f, 0)$$

[Hori,Tong 06]

$$U(N) \text{ with } (N_f, N_a) \quad \leftrightarrow \quad U(N_f - N) \text{ with } (N_f, N_a) \qquad N_f > N_a + 1$$

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$$U(N) \text{ with } (N_f,N_a) \quad \leftrightarrow \quad U(N_f-N) \text{ with } (N_f,N_a) \qquad N_f>N_a+1$$

$$N_f N_a \text{ singlets} + W = \tilde{q} M q$$

- Unitary: use Higgs branch expression
  - 1-1 correspondence of vacua $\vec{l} \in C(N,N_f)$
  - Classical action + 1-loop determinants easily coincide
  - To prove coincidence of Z_k $\forall k$, use contour integral expression

- Special unitary: perform Fourier transform

$$Z^{(N_F,0)}_{SU(N)}(b;a_j) = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_{-\infty}^{+\infty} 4\pi\, d\xi\, e^{4\pi i \xi}\; Z^{(N_f,0)}_{U(N)}(\xi,\theta;a_j)$$

S^2 partition function and Gromow-Witten invariants

[Jockers, Kumar, Lapan, Morrison, Romo 12] have recently observed that when the 2d GLSM theory flows to a conformal non-linear σ-model on a compact CY,
Z_{S^2} computes the full quantum genus-zero Kḧaler potential on the Kähler moduli space of the CY:

$$Z_{S^2} = e^K$$

Does not need to know what the mirror is (and do the computation in the mirror)

K computes Gromow-Witten invariants.

Conclusions

We have computed the p.f. of a 2d $\mathcal{N} = (2,2)$ theory on S^2. Generalizations:

- include twisted chiral superfields (mirror symmetry)
- squash S^2
- higher genus Riemann surfaces?
- $\mathcal{N} = (0,2)$ supersymmetry?

Explore the connection with non-linear σ-models and Gromov-Witten invariants

Alternative localization allowing (some) complex fields

- compute Z_{vortex} in absence of ADHM-like construction
- does it work in higher dimensions?