5-dim Superconformal Index with Enhanced E_n Global Symmetry

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Based on H.-C.Kim, S.-S. Kim and K. Lee, arXiv:1206.6781

References

- 5d superconformal field theory
  [Seiberg '96], [Intriligator, Seiberg, Morrison '97], …
- Superconformal algebra & index
  [Minwalla '97], [Kinney, Maldacena, Minwalla, Raju '05], …
- Localization
  [Pestun '07], [Gomis, Okuda, Pestun '11], [Hama, Hosomichi '12] …
- Instanton counting
  [Nekrasov '02], [Nekrasov, Shadchin '04], …

- $N^{5/2}$ degrees of freedom of 5d CFT [Jafferis, Pufu '12]
- Topological string amplitudes & 5d index [Vafa '12]
- AdS6/CFT5 for 5d quiver theories [Bergman, Rodriguez-Gomez '12]
- ….

5d gauge theory

- In general, non-renormalizable.

$$\frac{1}{g_{YM}^2}\int d^5x\, \mathrm{tr} F_{\mu\nu}F^{\mu\nu} \Longrightarrow [\, g_{YM}^2 \,] = L$$

- However, some SUSY gauge theories have UV completion !!
  - UV completion of maximally supersymmetric Yang-Mills

    = S^1 compactification of $D = 6$ (2,0) theory $\boxed{g_{YM}^2 = R_6}$

    (Lambert, Papageorgakis, Schmidt-Sommerfeld '10), (Douglas '10)

  - $\mathcal{N} = 1$ theories (Seiberg '96), (Intriligator, Seiberg, Morrison '97)
    - Particular gauge group and matter content give rise to smooth metric on the Coulomb branch.
    - Scale invariant theory lives at the UV fixed point !!
    - Here, we focus on these 5d CFTs at the conformal fixed point.

- Backgrounds
  - $SU(2)$ gauge theories with $N_f < 8$ fundamental flavors have non-trivial conformal fixed points in UV.
  - Global symmetry of the CFTs is enhanced to E_{N_f+1} symmetry.
  - Superconformal index (SCI) of the 5d CFTs

    Spectrum of BPS operators = partition function on $S^1 \times S^4$.

- Goal
  - Compute the superconformal index using localization.
  - Check the E_{N_f+1} global symmetry enhancement via the SCI.

5d Superconformal Field Theories

$\mathcal{N}=1$ SUSY gauge theories in 5d

- Minimum SUSY in 5d is $\mathcal{N}=1$
  - 8 supercharges $\bar{Q}_A = (Q^T)^B \epsilon_{BA}\Omega$
- $SU(2)_R$ R-symmetry. ($A=1,2$: $SU(2)_R$ doublet)
- Matter content : vector multiplet $A_\mu,\ \lambda^A,\ \phi$ and hypermultiplets $q^A,\ \psi$.
- Classical action

$$S = \frac{1}{g_0^2}\int d^5x\, \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{24\pi^2}\int \mathrm{tr}\, A\wedge F\wedge F + \cdots + S_{hyper}$$

g_0 : bare coupling
κ : Chern-Simons level

- Real scalar ϕ parametrizes the Coulomb branch moduli.

- Lagrangian on the Coulomb branch is determined by the prepotential $\mathcal{F}(\phi)$ (like 4d $\mathcal{N}=2$ theories).

$$\frac{1}{g_{eff}^2} \sim \frac{\partial^2 \mathcal{F}}{\partial\phi\partial\phi}\,, \quad \kappa_{eff} \sim \frac{\partial^3 \mathcal{F}}{\partial\phi\partial\phi\partial\phi}$$

- Since the C-S coefficient κ is quantized, the prepotential $\mathcal{F}(\phi)$ is at most cubic polynomial.
  - The quantum correction is one-loop exact !. (Witten '96)
  - Exact quantum prepotential is

m_i : hyper mass

$$\mathcal{F} = \frac{1}{2g_0^2}\mathrm{tr}\,\phi^2 + \frac{\kappa}{6}\mathrm{tr}\,\phi^3 + \frac{1}{12}\Bigg(\underbrace{\sum_{R\in root}|R\cdot\phi|^3 - \sum_i^{N_f}\sum_{\mathbf{w}\in\mathbf{W}_i}|\mathbf{w}\cdot\phi + m_i|^3}_{\text{one-loop correction}}\Bigg)$$

- Instantons

Mass : $m_I \sim \frac{k}{g^2}$ where $k = \frac{1}{8\pi^2}\int \mathrm{tr}F\wedge F$ is $U(1)_I$ instanton charge.

$Sp(1) = SU(2)$ gauge theories

- $SU(2)$ gauge group with N_f fundamental hypermultiplets.
  - Coulomb branch = V.E.V of scalar in vector multiplet.

$$\begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}, \quad \phi \in R^+$$

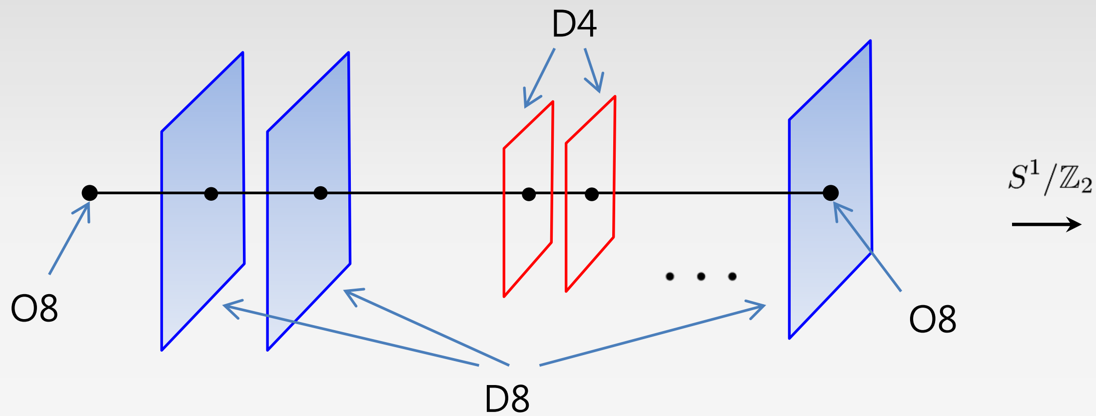
  - Effective coupling is

$$\frac{1}{g_{eff}^2} = \frac{1}{g_0^2} + 16\phi - \sum_i^{N_f} |\phi - m_i| - \sum_i^{N_f} |\phi + m_i|$$

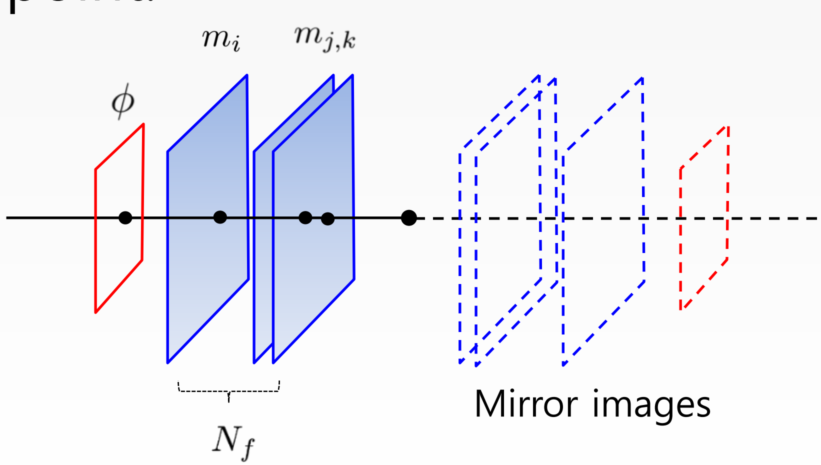
(Witten '96)

$Sp(1)$ gauge theories : Brane picture

- D4-branes with 16 D8-branes in Type I′ (Type IIA on $S^1/\mathbb{Z}_2$).



- The theory on a D4-brane with N_f D8-branes near the orientifold fixed point.



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- In general, the effective coupling g_{eff}^2 diverges at finite $\phi \sim \frac{1}{g_0^2}$
  - 5d theory is defined with UV cut-off $1/g_0^2$.

    ⇒ Non-renormalizable.

  - We need new theory to describe the short distance physics.

UV fixed point

- The short distance theories for $N_f < 8$ are known.

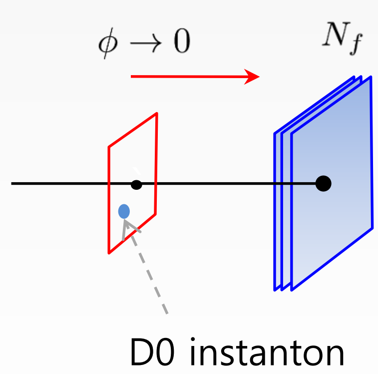
- Note : g_{eff}^2 is always positive when $N_f < 8$

$$\frac{1}{g_{eff}^2} = \frac{1}{g_0^2} + 2(8 - N_f)\phi \geq 0 \ , \qquad (m_i = 0)$$

  - Sensible theories on the entire Coulomb branch.

- In UV ($g_0 \to \infty$), new scale invariant theory emerges ! (Seiberg '96)
  - Non-trivial 5d CFT lives at the center of Coulomb branch $\phi \to 0$.
  - Strongly interacting theory , $g_{eff} \to \infty$

  ⇒ Non-perturbative (instanton) effects become important



5d superconformal field theory

- UV compeletion of the SU(2) gauge theory with $N_f < 8$.

- Strongly interacting $\mathcal{N}=1$ SCFT preserving 8 Poincare supercharge Q and 8 conformal supercharge S.

- Superalgebra is the superconformal F(4)

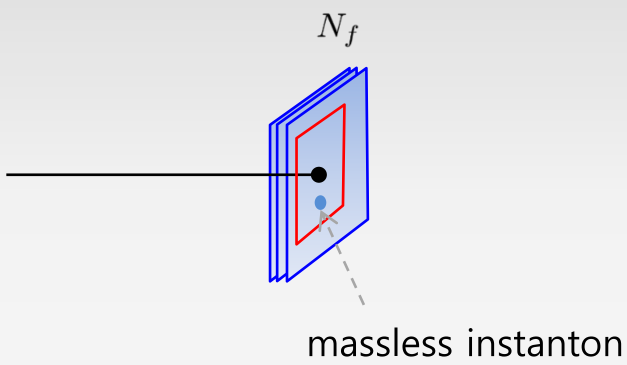
  Bosonic $SO(2,5) \times SU(2)_R$ and $\{Q^A_m, S^n_B\} = \delta^n_m \, \delta^A_B \, D + 2 \, \delta^A_B \, M_m{}^n - 3 \, \delta^n_m \, R_B{}^A$

- Global symmetry ~ $SO(2N_f)$ flavor × $U(1)_I$ instanton ?

- In the CFT, instantons become massless !! $m_I \sim \dfrac{1}{g^2_{eff}} \to 0$

Global symmetry enhancement

- Extra massless excitations (instantons) at the fixed point.



Symmetry enhancement ?

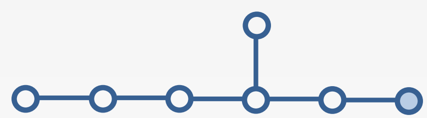
Global symmetry enhancement

- Extra massless excitations (instantons) at the fixed point.

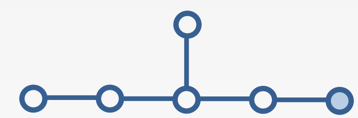
- Global symmetry $SO(2N_f) \times U(1)_I$ is enhanced to E_{N_f+1}.
  - Instanton charge provides the extra Cartan of E_{N_f+1}. (Seiberg '96)
  - Massless instantons are the missing charged states.

$SO(14) \times U(1) \subset E_8$

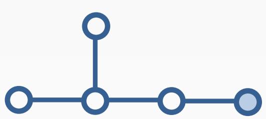
$SO(12) \times U(1) \subset E_7$



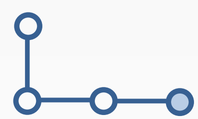
$SO(10) \times U(1) \subset E_6$



$SO(8) \times U(1) \subset SO(10) = E_5$



$SO(6) \times U(1) \subset SU(5) = E_4$



$E_3 = SU(3) \times SU(2)$

= $U(1)_I$ instanton charge

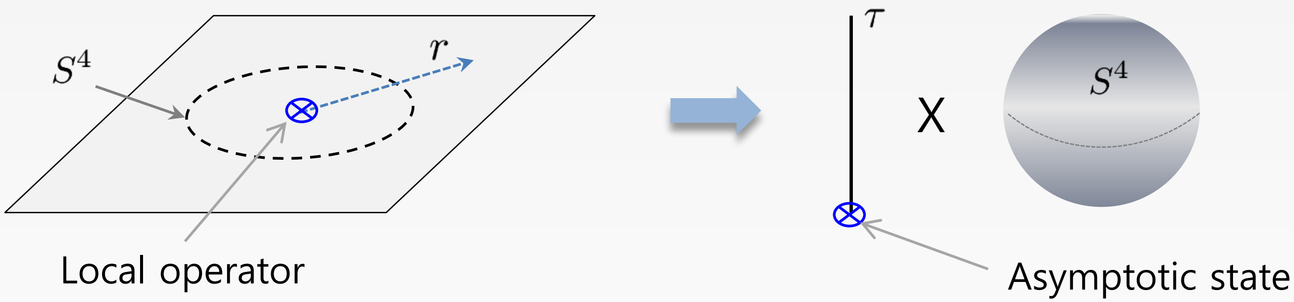
Superconformal Index

Superconformal index

- Spectrum of the BPS operators is exactly calculable.
  Protected by supersymmetry.

- Alternatively, we count the spectrum of BPS states on $R \times S^4$

  * Radial quantization (Operator-State map)



Superconformal index

- Witten index counting BPS states(operators) in the CFT

$$I(x,y,m_i,q) = \mathrm{tr}\left[(-1)^F e^{-\beta\Delta} x^{2(J_1+R)} y^{2J_2} e^{-iH_i m_i} q^k\right]$$

$$\Delta = \{Q,S\} = D - 2J_1 - 3R$$

- D is Dilatation and J_1, J_2 are Cartans of the Lorentz rotation $SO(5)$.
- H_i are generators of flavor symmetry and k is instanton number.
- $x,\ y,\ m_i,\ q$ are chemical potentials for the corresponding rotations.

- Only short multiplets contribute due to boson & fermion cancellation.

- Independent of β and any continuous parameters.

- $S^1 \times S^4$ partition function with twisted boundary conditions along S^1 .

- We evaluate it using localization technique.

Localization

- Partition ftn. is independent of Q-exact deformation of action

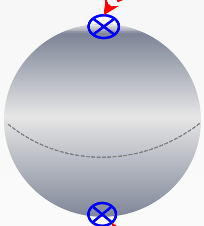
$$S \to S + t\{Q, V\} \quad \text{with} \quad Q^2 V = 0$$

- Taking $t \to \infty$ limit, path integral localizes around the classical solutions to the saddle point equation

$$\{Q, V\} \sim F_{0i}F^{0i} + \cos^2\frac{\theta_1}{2}(F_{ij}^-)^2 + \sin^2\frac{\theta_1}{2}(F_{ij}^+)^2 + \cdots = 0$$

- Solution :

$A_0 = \alpha$ ~ S^1 holonomy &



- Localizes to two fixed points of $Q^2 \sim \mathcal{L}_\tau + \gamma_1(J_1 + R) + \gamma_2 J_2$.
- Analogous to S^4 partition function. (Pestun '07), (Hama, Hosomichi '12)

Localization

- Integrals over the small fluctuations around the saddle points yield the exact result.

- Perturbative part ($F^{\pm}=0$) + Instanton part ($F^{+}\neq 0$ or $F^{-}\neq 0$)

- Perturbative part can be computed using equivariant index theorem.

$$I^{pert}=\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f_{vec}(\cdot^{n})+\frac{1}{n}f_{mat}(\cdot^{n})\right]$$

(Pestun '07),
(Gomis,Okuda,Pestun '11)

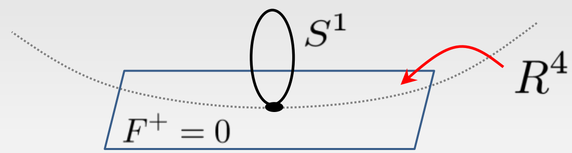
$$f_{vec}=-\frac{x(y+1/y)}{(1-xy)(1-x/y)}\left[\sum_{i<j}^{N}(e^{-i\alpha_i-i\alpha_j}+e^{-i\alpha_i+i\alpha_j}+e^{i\alpha_i-i\alpha_j}+e^{i\alpha_i+i\alpha_j})+\sum_{i=1}^{N}(e^{-2i\alpha_i}+e^{2i\alpha_i})+N\right],$$

$$f_{mat}=\frac{x}{(1-xy)(1-x/y)}\sum_{i=1}^{N}\sum_{l=1}^{N_f}(e^{-i\alpha_i-im_l}+e^{i\alpha_i-im_l}+e^{-i\alpha_i+im_l}+e^{i\alpha_i+im_l})$$

Instanton part

- Point (anti-)instanton is localized at the (north)south pole.

- Focus on the south pole



- Instanton contribution
  - Nekrasov's instanton partition function on $S^1 \times R^4$ for $Sp(N)$ gauge theory.
  - We can identify our chemical potentials with Nekrasov's parameters

$$\alpha = a \ , \quad -i\epsilon_1 = \gamma_1 + \gamma_2 \ , \quad -i\epsilon_2 = \gamma_1 - \gamma_2$$

Coulomb VEV

Omega parameters

$$e^{-\gamma_1} = x, \ e^{-\gamma_2} = y$$

  - Gauge group on the instanton moduli space is $O(k)$ not $SO(k)$.
  - $O(k)$ group consists of $O(k)_+ = SO(k) \ , \ O(k)_-$.

(Nekrasov, Shadchin '03)

- Contour integral formula of the instanton index
  - Integral over $O(k)$ algebra elements $\mathrm{diag}\left(e^{i\sigma_2\phi_1}, e^{i\sigma_2\phi_2}, \cdots\right)$.

$$I_+^k \sim \oint [d\phi] \left[\frac{\prod_{l=1}^{N_f} \sin\frac{m_l}{2}}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\prod_{i=1}^{N}\sin\frac{i\gamma_1\pm\alpha_i}{2}}\prod_{I=1}^{n}\frac{\sin(\frac{\phi_I\pm 2i\gamma_1}{2})}{\sin\frac{\phi_I\pm i\gamma_1\pm i\gamma_2}{2}}\right]^\chi$$

$$\times\prod_{I=1}^{n}\left[\frac{\sinh\gamma_1}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\sin\frac{2\phi_I\pm i\gamma_1\pm i\gamma_2}{2}}\frac{\prod_{l=1}^{N_f}\sin\frac{m_l\pm\phi_I}{2}}{\prod_{i=1}^{N}\sin\frac{\phi_I\pm\alpha_i\pm i\gamma_1}{2}}\right]\prod_{I<J}^{n}\left[\frac{\sin\frac{\phi_I\pm\phi_J\pm 2i\gamma_1}{2}}{\sin\frac{\phi_I\pm\phi_J\pm i\gamma_1\pm i\gamma_2}{2}}\right]$$

$$I_-^k \sim \cdots$$

$k = 2n+\chi$ ($\chi = 0$ or 1)

I_+^k is for $O(k)_+$, I_-^k is for $O(k)_-$

$\sin(a\pm b) \equiv \sin(a+b)\sin(a-b)$

- For $O(k=1) \equiv Z_2$,

N_f Fermion zero modes

$$I^{k=1} = \frac{1}{2}\left(I_+^{k=1} + I_-^{k=1}\right), \qquad I_+^{k=1} \sim \frac{\prod_{l=1}^{N_f}\sin\frac{m_l}{2}}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\sinh\frac{\gamma_1\pm i\alpha}{2}}$$

Bosonic zero modes :
R^4 position + internal

- Note : irrelevant poles at $z_I(\equiv e^{i\phi_I}) = 0$ when $N_f > 5$.

(No resolution yet.)

Superconformal index

- Full superconformal index is

$$\boxed{I(x,y,m_i,q) = Z_{S^1\times S^4} = \int d\alpha\; I^{pert}(x,y,\alpha,m_i)\times |I^{inst}(x,y,\alpha,m_i,q)|^2}$$

- Index for $Sp(1)$ gauge theory with N_f fundamental flavors

$$I = 1 + \left(1 + \chi_{adj}^{SO(2N_f)} + \chi_{s}^{SO(2N_f)} q + \chi_{\bar{s}}^{SO(2N_f)} q^{-1}\right) x^2 + \mathcal{O}(x^3)$$

Single (anti-)instanton states

$\chi_R^{SO(2N_f)}(e^{im_i})$: $SO(2N_f)$ character of R representation

Ex : $\chi_4^{SO(4)} = e^{im_1} + e^{im_2} + e^{-im_1} + e^{-im_2}$

Superconformal index

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$$I = 1 + \Big(\underbrace{1+\chi_{adj}^{SO(2N_f)} + \chi_s^{SO(2N_f)}\, q + \chi_{\bar{s}}^{SO(2N_f)}\, q^{-1}}_{=\;\chi_{adj}^{E_{N_f+1}}}\Big)\, x^2 + \mathcal{O}(x^3)$$

$$SO(2N_f)\times U(1)_I \subset E_{N_f+1}$$

Symmetry enhancement !!

- Global symmetry of CFT is enhanced to E_{N_f+1}.

⇨ Superconformal index has to be written by E_{N_f+1} characters $\chi_R^{E_{N_f+1}}$!!

- Example : $N_f = 5$ ➡ $E_6 \supset SO(10) \times U(1)_I$

$$
\begin{aligned}
I \;=\;& 1 + \chi^{E_6}_{\mathbf{78}} x^2 + \chi_2(y)[1+\chi^{E_6}_{\mathbf{78}}]x^3 + \Big(\chi_3(y)[1+\chi^{E_6}_{\mathbf{78}}] + 1 + \chi^{E_6}_{\mathbf{2430}}\Big)x^4 \\
&+\Big(\chi_4(y)[1+\chi^{E_6}_{\mathbf{78}}] + \chi_2(y)[1+\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{2430}} + \chi^{E_6}_{\mathbf{2925}}]\Big)x^5 \\
&+\Big(\chi_5(y)[1+\chi^{E_6}_{\mathbf{78}}] + \chi_3(y)[2+2\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{650}} + 2\chi^{E_6}_{\mathbf{2430}} + \chi^{E_6}_{\mathbf{2925}}] + 2\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{2925}} + \chi^{E_6}_{\mathbf{43758}}\Big)x^6 \\
&+\Big(\chi_6(y)[1+\chi^{E_6}_{\mathbf{78}}] + \chi_4(y)[2+4\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{650}} + 2\chi^{E_6}_{\mathbf{2430}} + 2\chi^{E_6}_{\mathbf{2925}}] \\
&+\chi_2(y)[2+4\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{650}} + 2\chi^{E_6}_{\mathbf{2430}} + \chi^{E_6}_{\mathbf{2925}} + \chi^{E_6}_{\mathbf{43758}} + \chi^{E_6}_{\mathbf{105600}}]\Big)x^7 \\
&+\Big(\chi_7(y)[1+\chi^{E_6}_{\mathbf{78}}] + \chi_5(y)[4+5\chi^{E_6}_{\mathbf{78}} + 2\chi^{E_6}_{\mathbf{650}} + 3\chi^{E_6}_{\mathbf{2430}} + 2\chi^{E_6}_{\mathbf{2925}}] \\
&+\chi_3(y)[3+8\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{650}} + 3\chi^{E_6}_{\mathbf{2430}} + 3\chi^{E_6}_{\mathbf{2925}} + \chi^{E_6}_{\mathbf{34749}} + 2\chi^{E_6}_{\mathbf{43758}} + 2\chi^{E_6}_{\mathbf{105600}}] \\
&+3 + 2\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{650}} + 3\chi^{E_6}_{\mathbf{2430}} + \chi^{E_6}_{\mathbf{2925}} + \chi^{E_6}_{\mathbf{70070}} + \chi^{E_6}_{\mathbf{105600}} + \chi^{E_6}_{\mathbf{537966}}\Big)x^8 + \mathcal{O}(x^9)
\end{aligned}
$$

  - Branching rules ~

$$
\begin{aligned}
\mathbf{78} &= \mathbf{1}_0 + \mathbf{16}_1 + \overline{\mathbf{16}}_{-1} + \mathbf{45}_0 \\
\mathbf{2430} &= \mathbf{1}_0 + \mathbf{16}_1 + \overline{\mathbf{16}}_{-1} + \mathbf{45}_0 + \mathbf{126}_{-2} + \overline{\mathbf{126}}_2 + \mathbf{210}_0 + \mathbf{560}_1 + \overline{\mathbf{560}}_{-1} + \mathbf{770}_0 \\
\mathbf{43758} &= \mathbf{1}_0 + \mathbf{16}_1 + \overline{\mathbf{16}}_{-1} + \mathbf{45}_0 + \mathbf{126}_{-2} + \overline{\mathbf{126}}_2 + \mathbf{210}_0 + \mathbf{560}_1 + \overline{\mathbf{560}}_{-1} + \mathbf{770}_0 \\
&\quad + \mathbf{672}_{-3} + \overline{\mathbf{672}}_3 + \mathbf{1440}_1 + \overline{\mathbf{1440}}_{-1} + \mathbf{3696}'_{-2} + \overline{\mathbf{3696}}'_2 + \mathbf{5940}_0 + \mathbf{7644}_0 \\
&\quad + \mathbf{8064}_1 + \overline{\mathbf{8064}}_{-1} \\
&\cdots
\end{aligned}
$$

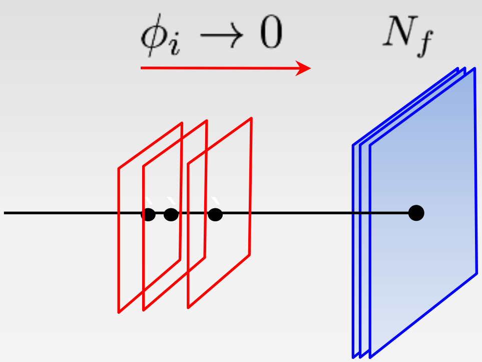
- We computed the index for $N_f = 1, 2, 3, 4, 5$ cases up to 4 instanton contributions.

All show E_{N_f+1} symmetry enhancement !

- Superconformal index provides strong evidence of the global symmetry enhancement of 5d CFT at the UV fixed point.

$Sp(N)$ Generalization

- N D4-branes near the orientifold point.



- $Sp(N)$ gauge theory with N_f fundamental hypermultiplets.
  - An additional hypermultiplet in anti-symmetric representation.
  - When $N_f < 8$, CFT arises at the UV fixed point.
  - E_{N_f+1} global symmetry enhancement. (Seiberg '96) (Intriligator,Seiberg,Morrison '97)

- For $N_f = 5$,

$\chi_n(e^{im})$: SU(2) character

$$I = 1 + \chi_2(e^{im})\,x + \Big(\chi_2(y)\chi_2(e^{im}) + 2\chi_3(e^{im}) + \chi^{E_6}_{\mathbf{78}}\Big)x^2$$
$$+\Big(\chi_3(y)\chi_2(e^{im}) + \chi_2(y)[2 + 2\chi_3(e^{im}) + \chi^{E_6}_{\mathbf{78}}] + 2\chi_4(e^{im}) + \chi_2(e^{im}) + 2\chi_2(e^{im})\chi^{E_6}_{\mathbf{78}}\Big)x^3 + \mathcal{O}(x^4)$$

Conclusions

- 5d gauge theories flowing to non-trivial CFT in UV have been studied.
  - $Sp(N)$ gauge theories with $N_f < 8$ fundamentals.
  - Massless instantons appear and E_{N_f+1} symmetry enhancement occurs.

- We have computed the superconformal index and checked that it is written in terms of E_{N_f+1} characters.

- Future directions
  - Resolution for 2 or more instantons when E_7, E_8 cases or $Sp(N>1)$.
  - Generalization to other gauge theories : SU(N), SO(N), quivers, ...
    (Intriligator, Seiberg, Morrison '97), (Aharony, Hanany '97)
  - AdS6/CFT5 : Gravity dual is warped $AdS_6 \times S^4$.
    (Ferrara, Kehagias, Partouche, Zaffaroni '98), (Bergman, Rodriguez-Gomez '12)
  - E_n Instanton moduli space.