

$$6 = (3 + 1) + 2$$

The Superconformal Index of Class \mathcal{S}

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Based on work with A. Gadde, D. Gaiotto, S. Razamat and W. Yan

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A golden age for exact results in susy QFT

New observables: Partition functions on (squashed) spheres, in the presence of supersymmetric defects.
Exactly computable by localization.

$$6 = 4+2$$

$$6 = 3+3$$

Beautiful and unexpected connections between susy theories in 4d (3d) and non-susy theories in 2d (3d), heuristically derived from the 6d (2,0) theory.

A vast new landscape of superconformal field theories (SCFTs).

Today we will describe another surprising 4d/2d relation:

Superconformal Index \mathcal{I} of $T[\mathcal{C}]$ \leftrightarrow **topological field theory correlator** on \mathcal{C}

$T[\mathcal{C}] \equiv$ 4d SCFT obtained from (2,0) theory on punctured Riemann surface \mathcal{C}

Index \equiv twisted $S^3 \times S^1$ partition function

Contrast with AGT:

S^4 partition function of $T[\mathcal{C}] \leftrightarrow$ **Liouville/Toda CFT** on \mathcal{C}

$S^4 \rightarrow S^3 \times S^1$, CFT \rightarrow TQFT

Motivations:

- practical: \mathcal{I} natural and useful observable, which we would like to compute;
- conceptual: perhaps easier entry point to the 4d/2d correspondence.

We will completely characterize the 2d TQFT and find a uniform answer for the index of any $T[\mathcal{C}]$, even when no weakly-coupled 4d Lagrangian is available.

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The superconformal index

Kinney-Maldacena-Minwalla-Raju, Romelsberger

It is a refined Witten index of the radially-quantized SCFT,

$$\mathcal{I}(\mu_i) = \text{Tr}_{S^3} (-1)^F \mu_i^{T_i} e^{-\beta \delta}, \quad \delta = \{Q, Q^\dagger\},$$

$\{T_i\}$ a complete set of commuting Q -closed generators.

- Independent of β since states with $\delta \neq 0$ cancel pairwise.
- Independent of continuous moduli.
- $\mathcal{I}[\text{Long multiplet}] = 0$ so \mathcal{I} encodes all recombination rules.
Most **sophisticated counting of protected spectrum** that can be obtained from representation theory alone.
- In path-integral language, partition function on $S^3 \times S^1$ with boundary conditions around S^1 twisted by $\mu_i^{T_i}$.

The $\mathcal{N} = 2$ index

Three superconformal fugacities (q, p, t) ,
several fugacities a_i associated to flavor symmetries,

$$\mathcal{I}(q, p, t; a_i) = \text{Tr}(-1)^F p^{j_{12}-r} q^{j_{34}-r} t^{R+r} \prod_i a_i^{f_i} .$$

R, r : Cartans of $SU(2)_R \times U(1)_r$ R-symmetry algebra

j_{12}, j_{34} : rotations of S^3 in two orthogonal planes

- Aside: \mathcal{I} can be defined for any $\mathcal{N} = 1$ theory with non-anomalous R-symmetry; it is an RG invariant. (Romelsberger, Festuccia-Seiberg)
Remarkable checks of Seiberg dualities and of AdS/CFT.
(Dolan-Osborn, Spiridonov-Vartanov, Gaiotto-Razamat-LR-Yan, Eager-Schmude-Tachikawa, ...)
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For a theory with a Lagrangian description, compute the index in the **free** limit, by counting gauge-invariant operators in terms of a **matrix integral**.
 (Unlike the S^4 partition function, which is sensitive to non-perturbative physics.)

- Assign to each elementary field, transforming in representation \mathcal{R} of the color \times flavor group, a “single-letter” index

$$f(p, q, t) \chi_{\mathcal{R}}(\mathbf{C}, \mathbf{F})$$

where \mathbf{C} and \mathbf{F} are color and flavor fugacities.

- The total index is given by enumerating the gauge-invariant words,

$$\mathcal{I}(p, q, t; \mathbf{F}) = \int [d\mathbf{C}] \text{PE} \left[\sum_j f^j(p, q, t) \chi_j(\mathbf{C}, \mathbf{F}) \right]$$

$$\text{PE}[f(x)] \equiv \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} f(x^n) \right)$$

Bosonic letter $\text{PE}[x] \equiv 1 + x + x^2 + \dots = 1/(1-x)$

Fermionic letter $\text{PE}[-x] \equiv 1 - x$

Basic example: **index of a chiral superfield**.

Contributing letters ($\delta = 0$) are q and $\tilde{\psi}_+$, and all of their ∂_{12} and ∂_{34} derivatives.

- Bosonic letters:

$$f[\partial_{12}^m \partial_{34}^n q] = t^{\frac{1}{2}} p^m q^n, \quad \text{PE}[f] = \frac{1}{1 - t^{\frac{1}{2}} p^m q^n}.$$

- Fermionic letters:

$$f[\partial_{12}^m \partial_{34}^n \tilde{\psi}_+] = -t^{-\frac{1}{2}} p^{m+1} q^{n+1}, \quad \text{PE}[f] = 1 - t^{-\frac{1}{2}} p^{m+1} q^{n+1}.$$

All in all, the index of a χ sf is an **elliptic gamma function** (Dolan-Osborn),

$$\mathcal{I} = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1} q^{n+1} t^{-\frac{1}{2}}}{1 - p^m q^n t^{\frac{1}{2}}} \equiv \Gamma(t^{\frac{1}{2}}; p, q).$$

For a hypermultiplet in the fundamental of $SU(N)$,

$$\prod_{i=1}^n \Gamma\left(t^{\frac{1}{2}} a_i^{\pm 1}; p, q\right),$$

where a_i are $SU(N)$ fugacities, and \pm means the product over both signs.

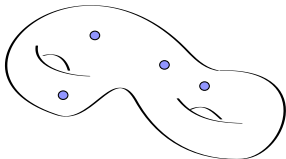
$\mathcal{N} = 2$ SCFTs of class $\mathcal{S}(ix)$ (Gaiotto, Gaiotto-Moore-Neitzke)

$T[\mathcal{C}]$ defined as the IR limit of the $A_{N-1}(2,0)$ theory on $\mathbb{R}^4 \times \mathcal{C}$,
 \mathcal{C} a Riemann surface with appropriate punctures (codimension-two defects).

$4d$ theory $T[\mathcal{C}]$	$2d$ theory on \mathcal{C}
Marginal gauge couplings	Complex moduli of \mathcal{C}
<i>n.a.</i> (irrelevant in IR) holographic RG check	Conformal factor of metric on \mathcal{C} (Anderson-Beem-Bobev-LR)
$SU(N)$ gauge group with coupling τ	cylinder with sewing parameter $q = \exp(2\pi i\tau)$
Flavor-symmetry factor $G \subset SU(N)$	Puncture labelled by $SU(2) \rightarrow SU(N)$ with commutant G
Weakly-coupled frame	Pair-of-pant decomposition of \mathcal{C}
Generalized S -duality	Moore-Seiberg groupoid of \mathcal{C}

$4d$ theory $T[\mathcal{C}]$	$2d$ theory on \mathcal{C}
Free bifundamental hypermultiplet $G_F = SU(N) \times SU(N) \times U(1)$	Three-punctured sphere with two maximal and one minimal puncture
T_N theory $G_F = SU(N) \times SU(N) \times SU(N)$	Three-punctured sphere with three maximal punctures
$Z[S^4]$	Liouville/Toda correlator (AGT)
$Z[S^3 \times S^1]$ (index)	TQFT correlator

Index $\mathcal{I}(q, p, t; \mathbf{a}_1, \dots, \mathbf{a}_k)$:
function of the flavor fugacities $\{\mathbf{a}_i\}$
but independent of gauge-couplings.
 $\mathcal{I} \equiv k$ -point TQFT correlator on \mathcal{C} .



TQFT structure

- Parametrize index $\mathcal{I}[T_N]$ of three-punctured sphere:

$$\mathcal{I}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \sum_{\alpha, \beta, \gamma} C_{\alpha\beta\gamma} f^\alpha(\mathbf{a}) f^\beta(\mathbf{b}) f^\gamma(\mathbf{c}),$$

$C_{\alpha\beta\gamma}$ are a priori unknown.
(NB: labels α run over an infinite set.)

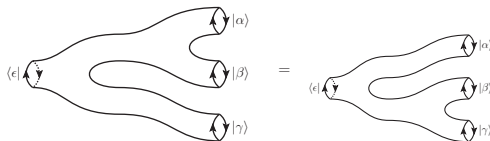
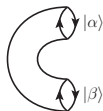
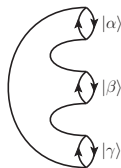
- “Propagators” known explicitly,

$$\eta(\mathbf{a}, \mathbf{b}) = \Delta(\mathbf{a}) \mathcal{I}_V(\mathbf{a}) \delta(\mathbf{a}, \mathbf{b}^{-1}) \rightarrow \eta^{\alpha\beta}$$

$\Delta(\mathbf{a}) = \text{Haar measure}$, $\mathcal{I}_V(\mathbf{a}) = \text{vector multiplet index}$.

S-duality translates into **associativity** of the TQFT structure constants,

$$C_{\alpha\beta}^\delta C_{\gamma\delta}^\epsilon = C_{\gamma\beta}^{\delta'} C_{\alpha\delta'}^\epsilon,$$



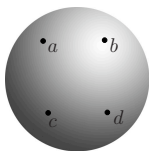
A_1 theories

All A_1 theories have Lagrangian descriptions:
generalized $SU(2)$ quivers with trivalent vertices.

Three-punctured sphere: half-hypermultiplet with
 $G_F \equiv SU(2) \times SU(2) \times SU(2)$.

Basic example: four-punctured sphere $\equiv SU(2)$ gauge theory with $N_f = 4$,

$$\mathcal{I} = \kappa \Gamma\left(\frac{pq}{t}\right) \oint \frac{dz}{2\pi i z} \frac{\Gamma\left(\frac{pq}{t} z^{\pm 2}\right)}{\Gamma(z^{\pm 2})} \Gamma\left(t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} z^{\pm 1}\right) \Gamma\left(t^{\frac{1}{2}} c^{\pm 1} d^{\pm 1} z^{\pm 1}\right).$$



S-duality demands invariance a and c . Rigorous proof by [van de Bult \(2009\)](#).
This checks associativity of the A_1 TQFT.

- Goal: compute index for the general $T[\mathcal{C}]$ of type A_{N-1} .
- Strategy: solve for all theories at once – “bootstrap” of the 2d TQFT.

In fact, very useful to further enlarge the space of theories to

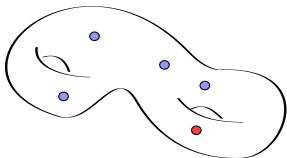
$T[\mathcal{C}, \mathfrak{S}] \equiv T[\mathcal{C}]$ probed by \mathfrak{S} , a BPS surface defect (codim. 4 in 6d).

$\mathfrak{S}_{(r,0)}$ located at $S^1 \subset S^3$ fixed by j_{12}

$\mathfrak{S}_{(0,s)}$ located at $S^1 \subset S^3$ fixed by j_{34}

(both wrapping the temporal S^1)

Surface defects correspond to special punctures on \mathcal{C}



As in Liouville CFT, the fusion of the special (defect) punctures with ordinary (flavor) punctures will be key to the bootstrap.

Basic claim:

To introduce surface defect, act on the index with a difference operator

$$\mathcal{I}[\mathcal{C}, \mathfrak{S}_{r,s}] = \mathfrak{S}_{r,s} \cdot \mathcal{I}[\mathcal{C}]$$

The difference operator shifts *one* of the flavor fugacities, e.g.

$$\mathfrak{S}_{(1,0)}(\mathbf{a}) \cdot \mathcal{I}(\mathbf{a}, \mathbf{b}, \dots) = \frac{\theta(t; p)}{\theta(q^{-1}; p)} \sum_{i=1}^N \prod_{j \neq i} \frac{\theta(\frac{t}{q} a_i / a_j; p)}{\theta(a_j / a_i; p)} \mathcal{I}_{\mathcal{C}}(a_i \rightarrow q^{\frac{1-N}{N}} a_i, a_{k \neq i} \rightarrow q^{\frac{1}{N}} a_k, \mathbf{b}, \dots).$$

Up to similarity, this is the basic Hamiltonian of the **elliptic RS model**.

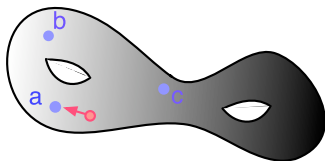


Figure: Fusion of **special (defect) puncture** with **ordinary (flavor) puncture**.

To justify the claim, we will give an RG construction of the surface defects.

A key step will use a more general idea:
the physical interpretation of the singularities of the index as a function of the complex fugacities a_i .

General principle: a pole in \mathcal{I} as $a \rightarrow a_*$ arises from a bosonic flat direction, parametrized by a vev $\langle \mathcal{O} \rangle$.

$$\text{Res}_{a \rightarrow a_*} \mathcal{I}[T_{UV}] \sim \mathcal{I}[T_{IR}],$$

where T_{IR} is the 4d theory at the end of the RG flow triggered by $\langle \mathcal{O} \rangle$.

RG construction of $T[\mathcal{C}, \mathfrak{S}_{(r,s)}]$

Embed $T[\mathcal{C}]$ into larger theory $T[\mathcal{C}']$, by “gluing” an extra bi-fundamental hyper.

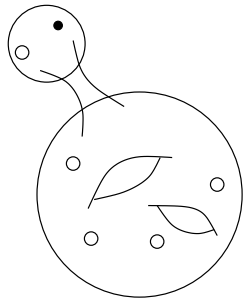
- By turning on a constant baryonic vev, $T[\mathcal{C}'] \rightarrow T[\mathcal{C}]$ in the IR.
- Space-dependent vev $B(z, w) \sim z^r w^s$: $T[\mathcal{C}'] \rightarrow T[\mathcal{C}, \mathfrak{S}_{(r,s)}]$ in the IR.

$$\mathcal{I}_{\mathcal{C}'}(\mathbf{a}, \mathbf{c}, \dots) = \oint [d\mathbf{b}] \mathcal{I}_{hyp}(\mathbf{b}, \mathbf{c}; \mathbf{a}) \mathcal{I}_{\mathcal{C}}(\mathbf{b}, \dots)$$

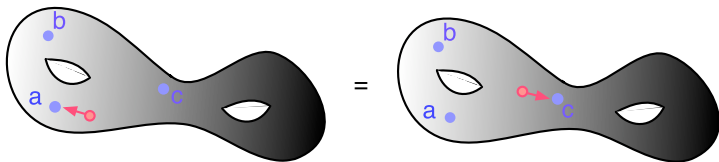
$T[\mathcal{C}']$ has an extra $U(1)_f$ flavor symmetry, parametrized by fugacity \mathbf{a} .

$\mathcal{I}[\mathcal{C}'](\mathbf{a}, \dots)$ has poles at $\mathbf{a} = t^{\frac{1}{2}} q^{\frac{r}{N}} p^{\frac{s}{N}}$.

The residues correspond to $\mathcal{I}[\mathcal{C}, \mathfrak{S}_{(r,s)}]$.



Extracting the residues gives the RS difference operators $\mathfrak{S}_{(r,s)}$ acting on $\mathcal{I}[\mathcal{C}]$.



$$\mathfrak{G}_{(r,s)}(\mathbf{a}) \mathcal{I}_C(\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots) = \mathfrak{G}_{(r,s)}(\mathbf{c}) \mathcal{I}_C(\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots)$$

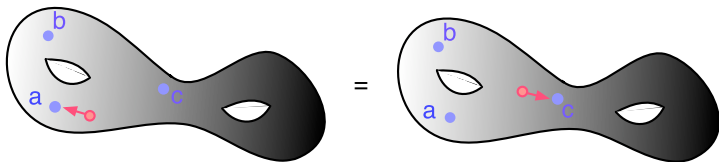
- The difference operators have common non-degenerate eigenfunctions ψ^α

$$\mathfrak{G}_{(r,s)} \cdot \psi^\alpha = E_{(r,s)}^\alpha \psi^\alpha,$$

S-duality then requires that the index is diagonal in the $\{\psi^\alpha\}$ basis,

$$\mathcal{I}_{0,3} = \sum_{\alpha} C_{\alpha\alpha\alpha} \psi^\alpha(\mathbf{a}) \psi^\alpha(\mathbf{b}) \psi^\alpha(\mathbf{c}).$$

- By comparing with weakly-coupled frames one can also completely fix the structure constants $C_{\alpha\alpha\alpha}$
- Precise algorithm to “partially-close” punctures (by giving vevs to moment maps), and obtain theories with reduced flavor symmetries.



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- This amounts to a **complete solution**. *E.g.*, immediate to evaluate index for genus g surface with s (maximal) punctures,

$$\mathcal{I}_{g,s}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s) = \sum_{\alpha} (C_{\alpha\alpha\alpha})^{2g-2+s} \prod_{i=1}^s \psi^{\alpha}(\mathbf{a}_i).$$

- Answer takes a close form in the limit $p \rightarrow 0$ (susy enhancement). $\psi^{\alpha}(\mathbf{a}|0, q, t)$ are proportional to **Macdonald polynomials** $P^{\alpha}(\mathbf{a}|q, t)$

$$\begin{aligned} C_{\alpha\alpha\alpha} &= \frac{\mathcal{A}(q, t)}{\dim_{q,t}(\alpha)} \\ \dim_{q,t}(\alpha) &= P^{\alpha}(t^{\frac{N-1}{2}}, \dots, t^{\frac{1-N}{2}} | q, t) \\ \mathcal{A}(q, t) &= \text{PE} \left[\frac{1}{2}(N-1) \frac{t-q}{1-q} \right] \prod_{j=2}^N (t^j; q). \end{aligned}$$

(Also a well-defined rule for reduced punctures by specializing the \mathbf{a}_i).

(Partial) identification of 2d theory:

- For $q = t$, Macdonald \rightarrow Schur.

The associated 2d TQFT is recognized as

q-deformed 2d Yang-Mills in the zero area limit

as defined by Aganagic Ooguri Saulina Vafa.

Recent extension to finite area (Tachikawa) if one keeps KK modes on \mathcal{C} .

- For $q \neq t$ (Macdonald index)

2d TQFT related to the “refinement” of Aganagic and Shakirov.

$\mathcal{I}(0, q, t)$ appears to coincide with $Z[MC_q \times S_t^1]$ as defined by

Cecotti-Neitzke-Vafa: \mathcal{Q} -exact deformation?

Some checks and applications:

- Argyres-Seiberg dualities, expected symmetry enhancements ($E_{6,7,8}$).
- Reduction to S^3 partition function: surface defects \rightarrow expected line defects.
- For $p = q = 0$, Hall-Littlewood index.
HL index of linear quivers \equiv **Hilbert series** of Higgs branch.

Nice application: Hilbert series of moduli space of two exceptional instantons (of E type). (Gaiotto-Razamat, Hanany-Mekareeya-Razamat, Keller-Song)

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Summary

- We have obtained explicit expressions for the superconformal index of **all** class \mathcal{S} theories (of type A), with or without surface defects.
- A “topological” 4d/2d relation.
Close parallel with the AGT correspondence:
ordinary punctures associated to **flavor**,
special punctures associated to **surface defects**.

Some questions

- More direct 2d interpretation of RS operators (ratios of θ functions...)?
- Recover our results by honest localization (for Lagrangian theories).
- Extension to $\mathcal{N} = 1$?
- Microscopic derivation of the 2d Lagrangian from the (2,0) theory?

Perhaps, two general lessons:

Conformal defects are useful.

Enlarging the view to the whole theory-space is useful.