Fermi gas approach to Chern-Simons-matter matrix models

1110.4066 ($\mathcal{N} \geq 3$), 1206.6346 ($\mathcal{N} \geq 2$) with M. Mariño
1207.5066 (ABJM&TBA) with M. Yamazaki

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Introduction

Path integral for the partition function of $\mathcal{N} \geq 2$ Chern–Simons–matter theory on S^3 localizes to a matrix integral ([Kapustin, Willett, Yaakov], generalized by [Jafferis, Hama, Hosomichi, Lee]). This matrix model can be used to provide various explicit checks of $\text{AdS}_4/\text{CFT}_3$ duality as well as different properties of 3d gauge theories.

There are different methods to study large N limit of CSM matrix models ($F(N) \equiv \log Z(N)$):

- In the 't Hooft limit: $\lambda = N/k$ fixed (IIA limit $g_s \sim 1/k$). The standard machinery of matrix models: spectral curve, genus expansion $F(N) = \sum_{g \geq 0} k^{2-2g} F_g(\lambda)$, topological recursion, etc. [Drukker, Mariño, PP, ...]
- Directly in the $N \to \infty$ limit (M-theory limit). Variational principle w.r.t. the limiting distribution of eigenvalues: $F(N) = -N^{3/2} \min\limits_{\rho, y_a} \mathcal{F}[\rho, y_a]$ [Klebanov, Herzog, Pufu, Tesileanu,...]
- Fermi gas approach (M-theory limit).

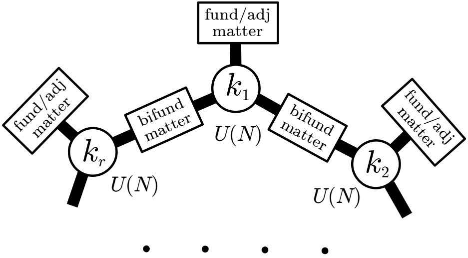
Features of the Fermi gas approach

- Exact equivalence to a Fermi gas system.
- Very simple "physical" derivation of the $N^{3/2}$ behavior for large N.
- Allows systematic study of $1/N$ corrections.
- Suggest the following conjecture:

$$F(N) \sim N^{3/2},\ N \to \infty \ \Leftrightarrow\ Z^{(\mathsf{pert})}(N) = e^{A}\, \mathrm{Ai}[C^{-1/3}(N - B)]$$

(First obtained by [Fuji-Hirano-Moriyama] for ABJM using holomorphic anomaly equations)

$\mathcal{N} = 2$ Chern-Simons-Matter theories



$$Z(N) = \int \prod_{a=1}^{r} \frac{1}{N!} \prod_{i=1}^{N} d\lambda_i^{(a)} \prod_{i<j} \left(\sinh \frac{\lambda_i^{(a)} - \lambda_j^{(a)}}{2} \right)^2 K_{a,a+1}^{(\text{matter})}(\lambda^{(a)}, \lambda^{(a+1)})\, e^{-S_{\text{cl}}(\lambda^{(a)})}$$

Fermi gas interpretation

$$Z(N) = \int \prod_{a=1}^{r} \frac{1}{N!} \prod_{i=1}^{N} d\lambda_i^{(a)} \prod_{i<j} \sinh\frac{\lambda_i^{(a)} - \lambda_j^{(a)}}{2} K_{a,a+1}^{(\mathsf{matter+cl})}(\lambda^{(a)}, \lambda^{(a+1)}) \prod_{i<j} \sinh\frac{\lambda_i^{(a+1)} - \lambda_j^{(a+1)}}{2} =$$

$$= \int \prod_{a=1}^{r} \prod_{i=1}^{N} d\lambda_i^{(a)} \left\{ \lambda_1^{(a)} \dots \lambda_N^{(a)} \right| K_{a,a+1}^{(N)} \left| \lambda_1^{(a+1)} \dots \lambda_N^{(a+1)} \right\}$$

where

$$|\lambda_1 \dots \lambda_N\} \equiv \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \left| \lambda_{\sigma(1)} \dots \lambda_{\sigma(N)} \right\rangle \in \mathcal{H}_N$$

is the standard basis in the Hilbert space $\mathcal{H}_N$ of N fermions in 1d.

$$K_{a,a+1}^{(N)} : \mathcal{H}_N \to \mathcal{H}_N$$

then

$$Z(N) = \mathrm{Tr}_{\mathcal{H}_N}\, \rho_{(N)}$$

$$\rho_{(N)} \equiv K_{1,2}^{(N)} \dots K_{r,1}^{(N)} \equiv e^{-H_N}$$

where H_N is a N-fermion Hamiltonian

Thermodynamic limit

Grand canonical partition function:

$$\Xi(\mu) \equiv \sum_{N \geq 0} Z(N)\, e^{N\mu}$$

Grand canonical potential:

$$J(\mu) \equiv \log \Xi(\mu)$$

$$Z(N) = \frac{1}{2\pi i} \int d\mu e^{J(\mu) - \mu N}$$

$$J(\mu) = \frac{C}{3}\, \mu^3 + B\mu + A + O(e^{-c\mu}),\ \mu \to \infty$$
$$\Leftrightarrow\ Z(N) \propto \mathrm{Ai}[C^{-1/3}(N-B)] \cdot \left(1 + O(e^{-c\sqrt{N}})\right),\ N \to \infty$$

Fact: The thermodynamic limit of a Fermi gas can be analyzed semi-classically. There is systematic way to calculate corrections.

$\mathcal{N} \geq 3$ case

$$\rho_{(N)} = \left(\rho_{(1)}\right)^{\otimes N}$$

Fermi gas is free. The system is described in terms of 1-particle Hamiltonian H_1 ($\rho_{(1)} = e^{-H_1}$).

$$J(\mu) = \mathrm{Tr} \log \left(1 + e^{\mu - H_1}\right) = \int dn(E) \, \log(1 + e^{\mu - E})$$

$$n(E) \equiv \mathrm{Tr}\, \theta(E - \hat{H}_1) = \sum_n \theta(E - E_n)$$

The thermodynamic (i.e. large N) limit of $F(N) \equiv \log Z(N)$ for an ideal Fermi gas is determined by the large E limit of $n(E)$. Particularly,

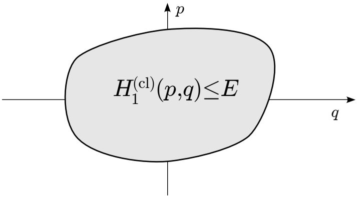
$$n(E) \approx C E^s \Longrightarrow F(N) \approx -\frac{s}{s+1} C^{-1/s} N^{\frac{s+1}{s}}$$

corrections to $n(E) \longrightarrow$ corrections to $F(N)$

Semi-classics & Wigner transform

In the large energy limit this 1-particle quantum system can be treated semi-classically:

$$n(E) \approx \frac{\mathrm{Vol}(\{H_1^{(\mathsf{cl})}(p,q) \leq E\})}{2\pi\hbar}$$



The Wigner transform (inverse to the Weyl quantization map):

$$\text{operators} \longrightarrow \text{functions on the phase space}$$

$$\hat{K} \longmapsto K_{\mathrm{W}}(p,q) = \int dx\, \mathrm{e}^{-ipx/\hbar}\, \langle q + x/2 | \hat{K} | q - x/2 \rangle$$

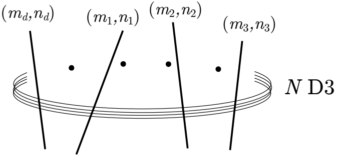
Properties:

- $\hat{A}\hat{B} \longmapsto A_{\mathrm{W}} \star B_{\mathrm{W}}$ where $\star = \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial_q}\,\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\,\overrightarrow{\partial_q}\right)\right]$ is the Moyal product.
- $\mathrm{Tr}\,\hat{K} = \int \frac{\mathrm{d}p\mathrm{d}q}{2\pi\hbar}\, K_{\mathrm{W}}(p,q)$

(In what follows $\hbar = 2\pi$)

$\mathcal{N}=3$ examples

Type IIB picture:

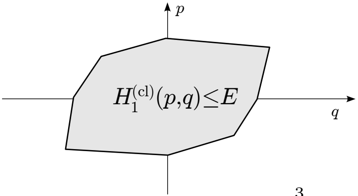


$U(N)$ with CS level k $\leftrightarrow$ $(1,n)$-$(1,n')$ branes $(n-n'=k)$

fundamental matter $\leftrightarrow$ $(0,1)$ brane

$$\left[e^{-H_1}\right]_{\mathrm{W}} = \frac{1}{2\cosh\frac{m_1 p + n_1 q}{2}} \star \cdots \star \frac{1}{2\cosh\frac{m_d p + n_d q}{2}}$$

$$H_1^{(\mathrm{cl})}(p,q) \approx \frac{1}{2}\sum_{i=1}^{d} |m_i p + n_i q|$$



$$n(E) \approx \frac{\mathrm{Vol}\left\{H_1^{(\mathrm{cl})}(p,q) \le E\right\}}{2\pi\hbar} = CE^2 \Longrightarrow F(N) \approx -\frac{2N^{\frac{3}{2}}}{3C^{1/2}}$$

$C \propto \mathrm{Vol}(X_7)$ s.t. the target space of the dual M-theory is $\mathrm{AdS}_4 \times X_7$.

(cf. [Gulotta, Herzog, Pufu])

$\mathcal{N} = 3$ examples, cont.

The full perturbative result is given by just the first quantum correction to

$$n(E) \equiv \mathrm{Tr}\, \theta(E - \hat{H}) = C E^2 + n_0 + O(\mathrm{e}^{-cE})$$

$$\Rightarrow \; J^{\text{(pert)}}(\mu) = \frac{C}{3}\, \mu^3 + B\mu + A$$

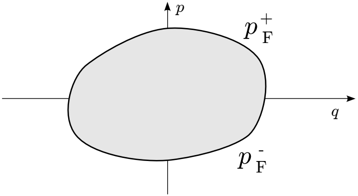
$$\Rightarrow \; Z^{\text{(pert)}}(N) \propto \mathrm{Ai}[C^{-1/3}(N - B)]$$

for the considered class of $\mathcal{N} \geq 3$ theories.

There is a straightforward procedure of computing C and B. For ABJM theory it gives $C = \frac{2}{\pi^2 k}$, $B = \frac{k}{24} + \frac{1}{3k}$ and confirms the result obtained by resumming the genus expansion obtained by solving the holomorphic anomaly equation [Fuji-Hirano-Moriyama].

$\mathcal{N} = 2$ case

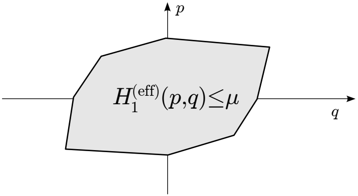
Fermi gas is not free. The thermodynamic limit is given by Thomas-Fermi approximation.



$$J(\mu) \approx \min_{p_{\mathsf{F}}^{\pm}} \left(E_{\mathsf{tot}}[p_{\mathsf{F}}^{\pm}] - \mu \mathrm{Vol}[p_{\mathsf{F}}^{\pm}]/(2\pi\hbar) \right)$$

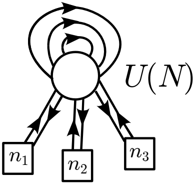
One can study corrections to the TF approximation systematically. There are indications that for theories with $N^{3/2}$ scaling

$$J(\mu) = \frac{C}{3}\mu^3 + B\mu + A + O(e^{-c\mu})$$



$\mathcal{N} = 2$ example

Simple example:



One can check explicitly that indeed

$$J^{(\mathsf{pert})}(\mu) = \frac{C}{3}\,\mu^3 + B\mu + A$$

$$\Rightarrow \; Z^{(\mathsf{pert})}(N) \propto \mathrm{Ai}[C^{-1/3}(N - B)]$$

ABJM and TBA

Consider ABJM theory

$$J(\mu) = \mathrm{Tr}\log(1 + e^{\mu}\rho), \quad \rho = e^{-H_1}$$

$$\Xi(\mu) = \det(1 + e^{\mu}\rho) = \sum_{N\geq 0} Z(N)e^{\mu N}$$

$$Z(N) = \sum_{\sum_\ell \ell m_\ell = N} \prod_\ell \frac{(-1)^{(\ell-1)m_\ell}\left(\mathrm{Tr}\,\rho^\ell\right)^{m_\ell}}{m_\ell!\ell^{m_\ell}}$$

$$\langle x|\rho|y\rangle = \frac{1}{\sqrt{2\cosh\frac{kx}{2}}}\frac{1}{2\cosh\frac{x-y}{2}}\frac{1}{\sqrt{2\cosh\frac{ky}{2}}} = \frac{e^{-u(x)-u(y)}}{\cosh\frac{x-y}{2}}$$

$$u(\xi) = \frac{1}{2}\log 4\cosh\frac{\xi}{2}$$

For general $u(\xi)$ the traces $\mathrm{Tr}\,\rho^\ell$ can be computed from a system of TBA-like equations [Zamolodchikov '94, Tracy-Widom '95]

ABJM and TBA, cont.

$$R_+(x) = \left\langle x \left| \frac{\rho}{1-\lambda^2\rho^2} \right| x \right\rangle, \qquad R_-(x) = \left\langle x \left| \frac{\lambda\rho^2}{1-\lambda^2\rho^2} \right| x \right\rangle$$

$$\int R_+(x)dx = \sum_n \lambda^{2n}\,\mathrm{Tr}\,\rho^{2n+1}, \quad \int R_-(x)dx = \sum_n \lambda^{2n+1}\,\mathrm{Tr}\,\rho^{2n+2}$$

Define $\epsilon(\theta)$, $\eta(\theta)$ by:

$$e^{-\epsilon(\theta)} = R_+(\theta)\ , \quad \eta(\theta) = 2\lambda \int_{-\infty}^{\infty} \frac{e^{-\epsilon(\theta')}}{\cosh(\theta-\theta')} d\theta'$$

Then they satisfy TBA-like equations:

$$\epsilon(\theta) = 2u(\theta) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\log\left(1+\eta^2(\theta')\right)}{\cosh(\theta-\theta')} d\theta'\ ,$$

$$R_-(\theta) = \frac{1}{\pi} R_+(\theta) \int_{-\infty}^{\infty} \frac{\arctan \eta(\theta')}{\cosh^2(\theta-\theta')} d\theta'\ .$$

The system can be solved order-by-order in λ recursively. At each step integrals can be evaluated using residue theorem.

ABJM and TBA, cont.

$(k=1)$

$$\mathrm{Tr}\,\rho = \frac{1}{4},\ \mathrm{Tr}\,\rho^2 = \frac{-2+\pi}{16\pi},\ \mathrm{Tr}\,\rho^3 = \frac{\pi-3}{16\pi},\ \mathrm{Tr}\,\rho^4 = \frac{-4-8\pi+3\pi^2}{128\pi^2},$$
$$\mathrm{Tr}\,\rho^5 = \frac{10-\pi^2}{256\pi^2},\ \mathrm{Tr}\,\rho^6 = \frac{36-2\pi-3\pi^2}{1536\pi^2},\ \ldots$$

$$Z(1) = \frac{1}{4},\ Z(2) = \frac{1}{16\pi},\ Z(3) = \frac{-3+\pi}{64\pi},\ Z(4) = \frac{-\pi^2+10}{1024\pi^2},$$
$$Z(5) = \frac{26+20\pi-9\pi^2}{4096\pi^2}, Z(6) = \frac{78-121\pi^2+36\pi^3}{147456\pi^3},\ \ldots \text{(up to } Z(19)\text{)}$$

M2-brane instantons

One can use these exact values of $Z(N)$ and to extract (numerically) 1-instanton correction:

$$\frac{Z^{(\text{1-inst})}}{Z^{(\text{pert})}} = \left(2N + c_1\sqrt{N} + c_2 + \dots\right) e^{-2\pi\sqrt{2N}}$$

't Hooft limit, $k \to \infty$, $\lambda = N/k$	$k = 1$
IIA string theory ($g_s \sim 1/k$) on $\mathrm{AdS}_4 \times \mathbb{CP}^3$	M-theory on $\mathrm{AdS}_4 \times S^7$

<table>
<tr><td>worldsheet instantons wrapping $\mathbb{CP}^1 \subset \mathbb{CP}^3$,
$\sim e^{-2\pi\sqrt{2\lambda}}$</td><td rowspan="2">M2-brane instantons wrapping $S^3 \subset S^7$
$\sim e^{-2\pi\sqrt{2N}}$</td></tr>
<tr><td>D2-brane instantons wrapping $\mathbb{RP}^3 \subset \mathbb{CP}^3$,
$\sim e^{-S_{\text{D2}}/g_s} = e^{-k\pi\sqrt{2\lambda}}$</td></tr>
</table>

Summary

- Fermi approach provides simple physical derivation of $N^{3/2}$ scaling.
- For $\mathcal{N} \geq 3$ Fermi gas is free, system is described in terms of single-particle Hamiltonian.
- Provides a method to resum all perturbative $1/N$ corrections.
- TBA method for ABJM allows easy exact calculation of $Z(N)$ for different N.

Some open problems:

- Relate $1/N$ corrections to higher curvature corrections of M-theory effective action.
- Compute contributions from membrane instantons directly in M/string theory.