Dynamics of Autonomous Hamiltonian Systems

Lectures given by

Alberto Abbondandolo

Gabriel P. Paternain

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Periodic Hamiltonian orbits by variational methods

Alberto Abbondandolo [*]

Notes taken by Jungsoo Kang[†]

Introduction

The aim of these notes is to discuss the question of the existence of periodic orbits of prescribed energy for classical Hamiltonian systems on compact configuration spaces. More precisely, we consider a closed manifold M and a smooth Lagrangian L on the tangent bundle TM of M, which is assumed to be fiberwise strictly convex and superlinear. Such a Lagrangian induces a flow on TM which preserves the energy function $E : TM \to \mathbb{R}$. Given a number $\kappa \in [\min E, +\infty)$, the problem is to prove the existence of a periodic orbit on $E^{-1}(\kappa)$.

Such periodic orbits can be found as critical points of the free period action functional

$$\mathbb{A}_\kappa(\gamma) = \int_0^T \Big(L\big(\gamma(t), \gamma'(t)\big) + \kappa \Big)\, dt,$$

on a suitable space of closed curves γ in M of arbitrary period T. The geometric and the compactness properties of this functional depend on the value of the energy κ and change drastically when κ crosses some special values, which are known as the Mañé critical values of L. Our knowledge about the existence of periodic orbits on $E^{-1}(\kappa)$ varies accordingly.

Most of the results of these notes are due to Contreras and are contained in the long paper [Con06], along with many other results. These notes are meant to be a gentle introduction to the part of [Con06] which concerns periodic orbits.

1 The minimax principle

The mountain pass theorem Let H be a real Hilbert space and let f be a continuously differentiable real function on H. We assume that a certain sublevel $\{f < a\}$ is not connected, say $\{f < a\} = A \cup B$, with A and B disjoint non-empty open sets. We may think of A and B as two valleys, and consider the set of paths going from one valley to the other one, that is

$$\Gamma := \{\text{curves in } H \text{ with one end in } A \text{ and the other in } B\}.$$

We can define the minimax value of f in Γ as

$$c := \inf_{\gamma \in \Gamma} \max_{x \in \gamma} f(x),$$

[*]Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I-56127 Pisa, Italy. E-mail: abbondandolo@dm.unipi.it

[†]Department of Mathematical Sciences, Seoul National University, Seoul, Republic of Korea. E-mail: hoho159@snu.ac.kr

and we notice that $a \leq c < +\infty$, because Γ is non empty and each of its elements intersects the set $H \setminus (A \cup B) = \{f \geq a\}$. One would expect this mountain pass level c to be a critical value of f. The next simple example shows that this is not always the case.

Example 1.1. Consider the smooth function f on $\mathbb{R}^2$ defined by

$$f(x, y) = e^x - y^2.$$

Then $\{f < 0\}$ has two connected components, $c = 0$, but f has no critical points. The problem here is that the critical point is pushed to infinity: indeed, $f(-n, 0) = e^{-n}$ converges to the mountain pass level $c = 0$ and $df(-n, 0) = e^{-n}dx$ tends to zero.

This example suggests the following definition.

Definition 1.2. A sequence $(x_n)_{n\in\mathbb{N}} \subset H$ is called a *Palais-Smale sequence* at level c ($(\mathrm{PS})_c$ for short) if

$$\lim_{n\to\infty} f(x_n) = c \quad \text{and} \quad \lim_{n\to\infty} df(x_n) = 0.$$

The function f is said to satisfy $(\mathrm{PS})_c$ if all $(\mathrm{PS})_c$ sequences are compact. It is said to satisfy (PS) if it satisfies $(\mathrm{PS})_c$ for every $c \in \mathbb{R}$.

Notice that limiting points of $(\mathrm{PS})_c$ sequences are critical points at level c. We can now state the celebrated mountain pass theorem of Ambrosetti and Rabinowitz.

Theorem 1.3 (Mountain Pass Theorem)**.** *Let $f \in C^{1,1}(H)$ be such that $\{f < a\}$ is not connected and let c be defined as above. Then f admits a $(\mathrm{PS})_c$ sequence. In particular, if f satisfies $(\mathrm{PS})_c$, then c is a critical value.*

Here $C^{1,1}$ denotes the set of functions whose differential is locally Lipschitz.

Proof. By contradiction, suppose that there exists $\epsilon > 0$ such that $||df|| \geq \epsilon$ on the set $\{|f - c| \leq \epsilon\}$. We denote by ∇f the gradient of f and we assume for sake of simplicity that the locally Lipschitz vector field $-\nabla f$ is positively complete, meaning that its flow ϕ, that is the solution of

$$\begin{cases} \dfrac{\partial}{\partial t}\phi_t(u) = -\nabla f\big(\phi_t(u)\big), \\ \phi_0(u) = u, \end{cases}$$

is defined for every $t \geq 0$ and every $u \in H$. This holds, for instance, if ∇f is globally Lipschitz (in this case the flow of $-\nabla f$ is defined on the whole $\mathbb{R} \times H$). See Remark 1.4 below for a hint on how to remove this extra assumption. Notice that

$$\frac{d}{dt} f\big(\phi_t(u)\big) = df\big(\phi_t(u)\big)\big[-\nabla f\big(\phi_t(u)\big)\big] = -\big\|df\big(\phi_t(u)\big)\big\|^2, \tag{1.1}$$

so the function $t \mapsto f(\phi_t(u))$ is decreasing. If $|f(\phi_t(u)) - c| \leq \epsilon$ for all $t \in [0, T]$, we have

$$2\epsilon \geq f(u) - f(\phi_T(u)) = -\int_0^T \frac{d}{dt} f(\phi_t(u))dt = \int_0^T \big\|df\big(\phi_t(u)\big)\big\|^2 dt \geq \epsilon^2 T,$$

from which we conclude that $T \leq 2/\epsilon$. Choose $\gamma \in \Gamma$ such that $\max_\gamma f \leq c + \epsilon$ and set

$$\tilde{\gamma} = \phi_T(\gamma), \qquad \text{for some } T > \frac{2}{\epsilon}.$$

The fact that f decreases along the orbits of ϕ implies that $\tilde{\gamma}$ belongs to Γ. Since we have chosen $f \leq c+\epsilon$ on γ, any $x \in \gamma$ satisfies either (i) $|f(x)-c| \leq \epsilon$ or (ii) $f(x) < c-\epsilon$. Let $x \in \gamma$. If (i) holds, then $f(\phi_T(x)) < c-\epsilon$ because $T > 2/\epsilon$. If (ii) holds, then $f(\phi_T(x)) < c-\epsilon$ because f decreases along the orbits of ϕ. Therefore we conclude that $\tilde{\gamma} \subset \{f < c-\epsilon\}$, which contradicts the definition of c. □

Remark 1.4. If the vector field $-\nabla f$ is not positively complete, we can replace it by the complete one $-\nabla f/\sqrt{||\nabla f||^2+1}$. The above proof goes through with minor adjustments.

Remark 1.5. The mountain pass theorem holds also for $f \in C^{1,1}(\mathcal{M})$ where $(\mathcal{M}, g)$ is a Hilbert manifold equipped with a complete Riemannian metric g. In this case, $(x_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ is a $(\mathrm{PS})_c$ sequence if $\lim_{n\to\infty} f(x_n) = c$ and $\lim_{n\to\infty} ||df(x_n)|| = 0$, where $\|\cdot\|$ denotes the dual norm induced by g. Notice that the (PS) condition and the completeness of g are somehow antagonist requirements: One may always achieve the completeness of an arbitrary Riemannian metric g by multiplying it by a positive function which diverges at infinity (such an operation reduces the set of the Cauchy sequences), while the (PS) condition could be achieved by multiplying g by a positive function which is infinitesimal at infinity (since the dual norm is multiplied by the inverse of this function, this operation reduces the set of the (PS) sequences).

Remark 1.6. The mountain pass theorem holds also if f is just continuously differentiable. In this case, its negative gradient vector field is just continuous and may not induce a continuous flow. In order to prove the above theorem, one needs to construct a locally Lipschitz pseudo-gradient vector field for f, see for instance [Str00, Lemma 3.2]. The same construction allows to prove the mountain pass theorem for continuously differentiable functions on Banach manifolds.

Remark 1.7. When dealing with functions on manifolds, it is sometimes useful to have a formulation of the mountain pass theorem which does not involve the choice of a metric. Here is such a formulation. Assume that f is a continuously differentiable function on a Hilbert manifold $\mathcal{M}$ and that V is a positively complete locally Lipschitz vector field such that $df[V] < 0$ on $\mathcal{M} \setminus \mathrm{Crit} f$. Then the mountain pass theorem holds, provided that we define $(x_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ to be a $(\mathrm{PS})_c$ sequence if $f(x_n)$ tends to c and $df(x_n)[V(x_n)]$ is infinitesimal. Now the antagonism is between this form of the (PS) condition and the positive completeness of V.

The general minimax principle In the proof of Theorem 1.3 we have not used the fact that Γ is a set of curves, but rather that Γ is positively invariant with respect to the negative gradient flow ϕ of f, meaning that $\phi_t(\gamma) \in \Gamma$ for all $\gamma \in \Gamma$ and $t \geq 0$. Here ϕ is either the flow of $-\nabla f$, when this vector field is positively complete, or the flow of some conformally equivalent positively complete vector field, such as $-\nabla f/\sqrt{\|\nabla f\|^2+1}$, in the general case. This simple observation leads to the following powerful generalization of the mountain pass theorem.

Theorem 1.8 (**General Minimax Principle**)**.** *Let f be a $C^{1,1}$ function on the complete Riemannian Hilbert manifold $(\mathcal{M}, g)$ and let Γ be a set of subsets of $\mathcal{M}$ which is positively invariant with respect to the negative gradient flow of f. If the number*

$$c = \inf_{\gamma\in\Gamma} \sup_{\gamma} f$$

is finite, then f admits a $(\mathrm{PS})_c$ *sequence. In particular, if f satisfies* $(\mathrm{PS})_c$*, then c is a critical value.*

The proof is a straightforward modification of the proof of Theorem 1.3.

Exercise 1.9. Let $f \in C^{1,1}(H)$, where H is a Hilbert space. If $\pi_k(\{f < a\}) \neq 0$ for some $k \geq 0$ and f satisfies (PS), then f has a critical point.

Remark 1.10. If Γ is the class of all one-point sets in $\mathcal{M}$, then c is the infimum of f. Therefore, the general minimax principle has as a particular case the following existing result for minimizers. Assume that $f \in C^{1,1}(\mathcal{M})$ is bounded from below, has complete sublevels and satisfies $(\mathrm{PS})_c$ at the level $c = \inf f$. Then f admits minimizers.

Remark 1.11. It is sometimes useful to replace the negative gradient flow by a flow which fixes a certain sublevel of f. Let $\rho : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a smooth bounded function such that $\rho = 0$ on $(-\infty, b]$ and $\rho > 0$ on $(b, +\infty)$. Then we consider the vector field $V = -\rho(f) \cdot \nabla f$ (or $V = -\rho(f)\nabla f / \sqrt{\|\nabla f\|^2 + 1}$ in the non-positively complete case) and denote its flow by ϕ. It is a negative gradient flow truncated below level b: The function $t \mapsto f(\phi_t(u))$ is constant if $u \in \mathrm{Crit} f \cup \{f \leq b\}$ and it is strictly decreasing otherwise. If Γ is positively invariant with respect to this negative gradient flow truncated below level b and the minimax value c is strictly larger than b, then f has a $(\mathrm{PS})_c$ sequence.

2 A Hilbert manifold of loops

Let (M, g) be a closed Riemannian manifold of dimension n and consider the Sobolev space of loops

$$W^{1,2}(\mathbb{T}, M) := \left\{ x : \mathbb{T} \longrightarrow M \;\middle|\; x \text{ is absolutely continuous and } \int_{\mathbb{T}} |x'(s)|^2_{x(s)} ds < \infty \right\},$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $|\cdot|_\cdot$ denotes the norm induced by g. This set of loops is clearly independent from the choice of the Riemannian metric g.

The smooth structure of $\mathbf{W^{1,2}(T, M)}$ Let us recall the construction of the smooth Hilbert manifold structure on $W^{1,2}(\mathbb{T}, M)$. Fix $x_0 \in C^\infty(\mathbb{T}, M)$. Assume for simplicity that x_0 preserves the orientation, so that $x_0^*(TM)$ has a trivialization

$$\Phi : \mathbb{T} \times \mathbb{R}^n \longrightarrow x_0^*(TM).$$

Let B_r be the open ball of radius r about 0 in $\mathbb{R}^n$. Consider a smooth map

$$\varphi : \mathbb{T} \times B_r \longrightarrow M,$$

such that $\varphi(t, 0) = x_0(t)$ and $\varphi(t, \cdot)$ is a diffeomorphism onto an open subset in M, for every $t \in \mathbb{T}$. For instance, the map

$$\varphi(t, \xi) = \exp_{x_0(t)} \big(\Phi(t, \xi)\big),$$

satisfies the above requirements if r is small enough.

The map φ induces the following parameterization:

$$\varphi_* : W^{1,2}(\mathbb{T}, B_r) \longrightarrow W^{1,2}(\mathbb{T}, M), \quad \zeta \mapsto \varphi\big(\cdot, \zeta(\cdot)\big), \tag{2.1}$$

where $W^{1,2}(\mathbb{T}, B_r)$ denotes the open subset of the Hilbert space $W^{1,2}(\mathbb{T}, \mathbb{R}^n)$ which consists of loops taking values into B_r. The collection of all these parameterizations, for every $x_0 \in C^\infty(\mathbb{T}, M)$ and every φ as above, defines a smooth atlas for $W^{1,2}(\mathbb{T}, M)$, which is then a smooth manifold modeled on the Hilbert space $W^{1,2}(\mathbb{T}, \mathbb{R}^n)$. Indeed, the smoothness of the transition maps is an immediate consequence of the chain rule. It is worth noticing that the image of the parameterization φ_* is C^0-open.

Remark 2.1. If x_0 is not orientation preserving, the natural model for the connected component of $W^{1,2}(\mathbb{T}, M)$ which contains x_0 is the space of $W^{1,2}$ sections of the vector bundle $x_0^*(TM)$.

The tangent space of $W^{1,2}(\mathbb{T}, M)$ at x is naturally identified with the space of $W^{1,2}$ sections of $x^*(TM)$. Therefore, we can define a Riemannian metric on $W^{1,2}(\mathbb{T}, M)$ by setting

$$\langle \xi, \eta \rangle_x := \int_{\mathbb{T}} \Big(g(\xi, \eta) + g(\nabla_t \xi, \nabla_t \eta) \Big)\, dt, \quad \forall \xi, \eta \in T_x W^{1,2}(\mathbb{T}, M), \tag{2.2}$$

where ∇_t denotes the Levi-Civita covariant derivative along x. The distance induced by this Riemannian metric is compatible with the topology of $W^{1,2}(\mathbb{T}, M)$.

The fact that M is compact implies that this metric on $W^{1,2}(\mathbb{T}, M)$ is complete (more generally, this metric is complete whenever g is complete).

Remark 2.2. If φ is the restriction of a smooth map $B_{r'} \times \mathbb{T} \to M$ with the same properties, for some $r' > r$, then the parameterization φ_* is bi-Lipschitz.

The homotopy type of $\mathbf{W^{1,2}(T, M)}$ The inclusions

$$C^\infty(\mathbb{T}, M) \hookrightarrow W^{1,2}(\mathbb{T}, M) \hookrightarrow C(\mathbb{T}, M)$$

are dense homotopy equivalences. These facts can be proved by embedding M into a Euclidean space $\mathbb{R}^N$, by regularizing the loops $x : \mathbb{T} \to M \subset \mathbb{R}^N$ by convolution, and by projecting the regularized loop back to M using the tubular neighborhood theorem. In particular, the connected components of $W^{1,2}(\mathbb{T}, M)$ are in one-to-one correspondence with the conjugacy classes of $\pi_1(M)$.

3 The free period action functional

The setting Let M be a closed manifold. A function $L \in C^\infty(TM)$ is called a *Tonelli Lagrangian* if:

(i) L is fiberwise C^2-strictly convex, i.e. $d_{vv}L(x, v) > 0$ for every $(x, v) \in TM$, where $d_{vv}L$ denotes the fiberwise second differential of L;

(ii) L has superlinear growth on each fiber, i.e.

$$\lim_{|v| \to +\infty} \frac{L(x, v)}{|v|_x} = +\infty.$$

The main example of Tonelli Lagrangians are the *electromagnetic Lagrangians*, that is functions of the form

$$L(x,v)=\frac{1}{2}|v|_x^2+\theta(x)[v]-V(x), \tag{3.1}$$

where $|\cdot|_x$ denotes the norm associated to a Riemannian metric (the kinetic energy) on M, θ is a smooth one-form (the magnetic potential) and V is a smooth function (the scalar potential) on M. We shall omit the subscript x in $|\cdot|_x$ when the point x is clear from the context. The Tonelli assumptions imply that the Euler-Lagrange equation, which in local coordinates can be written as

$$\frac{d}{dt}\big(\partial_v L(\gamma(t),\gamma'(t))\big)=\partial_x L(\gamma(t),\gamma'(t)), \tag{3.2}$$

is well-posed and defines a smooth flow on TM. This flow preserves the energy

$$E:TM\to\mathbb{R},\quad E(x,v)=d_vL(x,v)[v]-L(x,v),$$

where d_v denotes the fiberwise differential. When L has the form (3.1), then

$$E(x,v)=\frac{1}{2}|v|_x^2+V(x). \tag{3.3}$$

Exercise 3.1. More generally, the energy function of a Tonelli Lagrangian satisfies the following properties:

(i) E is fiberwise C^2-strictly convex and superlinear.

(ii) For any $x\in M$, the restriction of E to T_xM achieves its minimum at $v=0$.

(iii) The point $(\bar{x},0)$ is singular for the Euler-Lagrange flow if and only if $(\bar{x},0)$ is a critical point of E.

We are interested in proving the existence of periodic orbits on a given energy level $E^{-1}(\kappa)$. Since such an energy level is compact, up to the modification of L outside it, we may assume that the Tonelli Lagrangian $L(x,v)$ is electromagnetic for $|v|$ large enough. In particular, we have the inequalities

$$L(x,v)\geq L_0|v|^2-L_1,\qquad \forall(x,v)\in TM, \tag{3.4}$$

$$d^2_{vv}L(x,v)[u,u]\geq 2L_0|u|^2,\qquad \forall(x,v)\in TM,\ u\in T_xM, \tag{3.5}$$

for some numbers $L_0>0$ and $L_1\in\mathbb{R}$. Moreover, E has the form (3.3) for $|v|$ large.

The free period action functional Let $\gamma:\mathbb{R}/T\mathbb{Z}\longrightarrow M$ be an absolutely continuous T-periodic curve and define $x:\mathbb{T}\to M$ as $x(s)=\gamma(sT)$. Then the action of γ on the time interval $[0,T]$ is the number

$$\int_0^T L\big(\gamma(t),\gamma'(t)\big)\,dt=T\int_0^1 L\big(x(s),x'(s)/T\big)\,ds.$$

Fix a real number κ, the value of the energy for which we would like to find periodic solutions. Consider the *free period action functional* corresponding to the energy κ

$$\mathbb{A}_\kappa(\gamma)=\mathbb{A}_\kappa(x,T):=T\int_0^1\Big(L\big(x(s),x'(s)/T\big)+\kappa\Big)\,ds=\int_0^T\Big(L\big(\gamma(t),\gamma'(t)\big)+\kappa\Big)\,dt.$$

The fact that L is electromagnetic outside a compact subset of TM implies that $\mathbb{A}_\kappa(x,T)$ is well-defined when $x \in W^{1,2}(\mathbb{T},M)$. Hence, we get a functional

$$\mathbb{A}_\kappa : W^{1,2}(\mathbb{T},M) \times (0,+\infty) \to \mathbb{R}.$$

The Hilbert manifold $W^{1,2}(\mathbb{T},M) \times (0,+\infty)$ is denoted by $\mathcal{M}$.

Exercise 3.2. (Regularity properties of $\mathbb{A}_\kappa$, see e.g. [AS09])

(i) $\mathbb{A}_\kappa \in C^{1,1}(\mathcal{M})$ and it has second Gateaux differential at every point.

(ii) $\mathbb{A}_\kappa$ is twice Fréchét differentiable at every point if and only if L is electromagnetic on the whole TM. In this case, $\mathbb{A}_\kappa$ is actually smooth on $\mathcal{M}$.

If d_x denotes the horizontal differential with respect to the some horizontal-vertical splitting of TTM, the differential of $\mathbb{A}_\kappa$ with respect to the first variable at some $(x,T) \in \mathcal{M}$ has the form

$$\begin{aligned} d\mathbb{A}_\kappa(x,T)\big[(\xi,0)\big] &= T\int_0^1 \Big(d_x L\big(x,x'/T\big)[\xi] + d_v L\big(x,x'/T\big)\big[\xi'/T\big]\Big)\,ds \\ &= \int_0^T \Big(d_x L\big(\gamma,\gamma'\big)[\zeta] + d_v L\big(\gamma,\gamma'\big)[\zeta']\Big)\,dt, \end{aligned} \tag{3.6}$$

where $\xi \in T_x W^{1,2}(\mathbb{T},M)$, $\gamma(t) = x(t/T)$ and $\zeta(t) := \xi(t/T)$. Let (x,T) be a critical point of $\mathbb{A}_\kappa$. The above formula and an integration by parts imply that γ is a T-periodic solution of (3.2). Moreover

$$\begin{aligned} \frac{\partial \mathbb{A}_\kappa}{\partial T}(x,T) &= \int_0^1 \Big(L\big(x(s),x'(s)/T\big) + \kappa + T\,d_v L\big(x(s),x'(s)/T\big)\big[-x'(s)/T^2\big]\Big)\,ds \\ &= \int_0^1 \Big(\kappa - E\big(x(s),x'(s)/T\big)\Big)\,ds = \frac{1}{T}\int_0^T \Big(\kappa - E\big(\gamma(t),\gamma'(t)\big)\Big)\,dt. \end{aligned} \tag{3.7}$$

Together with the fact that E is constant along the orbits of the Euler-Lagrange flow, the above identity shows that the T-periodic orbit γ belongs to the energy levek $E^{-1}(\kappa)$. We conclude that (x,T) is a critical point of $\mathbb{A}_\kappa$ on $\mathcal{M}$ if and only if $\gamma(t) := x(t/T)$ is a T-periodic orbit of energy κ (T is not necessarily the minimal period).

The gradient vector field It is useful to reduce the set of the (PS) sequences (x_h,T_h) for which $T_h \to 0$, so that only the ones with a well understood limiting behavior remain (see Lemma 4.1 below). For this reason, we choose a smooth function $\rho : (0,+\infty) \longrightarrow \mathbb{R}$ such that

$$\rho(T) = T^2 \quad \forall T \le 1/2, \qquad \rho(T) = 1 \quad \forall T \ge 1,$$

and we consider the following metric on $\mathcal{M} = W^{1,2}(\mathbb{T},M) \times (0,+\infty)$:

$$\big\langle(\xi_1,\tau_1),(\xi_2,\tau_2)\big\rangle_{(x,T)} := \tau_1\tau_2 + \rho(T)\langle\xi_1,\xi_2\rangle_x, \qquad \forall\, \xi_1,\ \xi_2 \in T_x W^{1,2}(\mathbb{T},M),\ \tau_1,\ \tau_2 \in \mathbb{R}, \tag{3.8}$$

where $\langle\cdot,\cdot\rangle_x$ is the $W^{1,2}$ Hilbert product defined in (2.2). Since $\rho(T)$ is infinitesimal for $T \to 0$, this metric has more non-converging Cauchy sequences than the product one and is a fortiori not complete.

The gradient vector field of $\mathbb{A}_\kappa$ with respect to the above metric has the form

$$\nabla\mathbb{A}_\kappa(x,T) = \frac{\partial \mathbb{A}_\kappa}{\partial T}(x,T)\frac{\partial}{\partial T} + \frac{1}{\rho(T)}\nabla_x \mathbb{A}_k(x,T),$$

where ∇_x denotes the gradient with respect to the $W^{1,2}$ metric (2.2).

Behavior of $\mathbb{A}_\kappa$ for $\mathbf{T} \to \mathbf{0}$ We decompose $\mathcal{M} = W^{1,2}(\mathbb{T}, M) \times \mathbb{R}^+$ into the contractible component $\mathcal{M}^{\mathrm{contr}}$ and the non-contractible component $\mathcal{M}^{\mathrm{noncontr}}$, the latter being empty if and only if $\mathcal{M}$ is simply connected.

Lemma 3.3. (i) On $\mathcal{M}^{\mathrm{noncontr}}$ the sublevels $\{\mathbb{A}_\kappa \leq c\}$ are complete.

(ii) If $(x_h, T_h) \in \mathcal{M}^{\mathrm{contr}}$ and $T_h \to 0$, then $\liminf_h \mathbb{A}_\kappa(x_h, T_h) \geq 0$.

Proof. By (3.4), we have the chain of inequalities

$$\begin{aligned} \mathbb{A}_\kappa(x, T) &= T \int_0^1 \Big(L\big(x, x'/T\big) + \kappa \Big)\, ds \geq T \int_0^1 \Big(L_0 \frac{|x'|^2}{T^2} - L_1 + \kappa \Big)\, ds \\ &= \frac{L_0}{T} \int_0^1 |x'|^2\, ds - (L_1 - \kappa) T \geq \frac{L_0}{T} \ell(x)^2 - (L_1 - \kappa) T, \end{aligned} \tag{3.9}$$

where $\ell(x)$ denotes the length of the loop x. The length of the non-contractible loops in M is bounded away from zero. Therefore, the estimate (3.9) implies that for every $c \in \mathbb{R}$ the number T is bounded away from zero on

$$\big\{ (x, T) \in \mathcal{M}^{\mathrm{noncontr}} \mid \mathbb{A}_k(x, T) \leq c \big\},$$

proving (i). Statement (ii) is also an immediate consequence of (3.9). □

Lemma 3.4. Let $(x, T) : [0, \sigma^*) \longrightarrow \mathcal{M}^{\mathrm{contr}}$, $0 < \sigma^* < \infty$, be a flow line of $-\nabla \mathbb{A}_\kappa$ such that

$$\liminf_{\sigma \to \sigma^*} T(\sigma) = 0.$$

Then

$$\lim_{\sigma \to \sigma^*} \mathbb{A}_\kappa\big(x(\sigma), T(\sigma)\big) = 0.$$

Proof. Since both E and L are quadratic in v for $|v|$ large, we have the estimate

$$E(x, v) \geq L_2\, L(x, v) - L_3,$$

for some $L_2 > 0$ and $L_3 \in \mathbb{R}$. From (3.7) we obtain the inequality

$$\begin{aligned} \frac{\partial \mathbb{A}_\kappa}{\partial T}(x, T) &= \frac{1}{T} \int_0^T \big(\kappa - E(\gamma, \gamma') \big)\, dt \leq \frac{1}{T} \int_0^T \big(\kappa - L_2\, L(\gamma, \gamma') + L_3 \big)\, dt \\ &= \kappa + L_3 - \frac{L_2}{T} \int_0^T \big(L(\gamma, \gamma') + \kappa \big)\, dt + L_2 \kappa = (L_2 + 1)\kappa + L_3 - \frac{L_2}{T} \mathbb{A}_\kappa(x, T), \end{aligned}$$

which can be rewritten as

$$\mathbb{A}_\kappa(x, T) \leq \frac{T}{L_2} \Big(C - \frac{\partial \mathbb{A}_\kappa}{\partial T}(x, T) \Big), \tag{3.10}$$

for a suitable constant C. By the assumption, there exists an increasing sequence (σ_h) which converges to σ^* and satisfies $T'(\sigma_h) \leq 0$ and $T(\sigma_h) \to 0$. Since $\sigma \mapsto (x(\sigma), T(\sigma))$ is a flow line of $-\nabla \mathbb{A}_\kappa$,

$$0 \geq T'(\sigma_h) = -\frac{\partial \mathbb{A}_\kappa}{\partial T}\big(x(\sigma_h), T(\sigma_h) \big),$$

and by (3.10) we have

$$\mathbb{A}_\kappa\big(x(\sigma_h),T(\sigma_h)\big) \le \frac{T(\sigma_h)}{L_2}\Big(C - \frac{\partial \mathbb{A}_\kappa}{\partial T}\big(x(\sigma_h),T(\sigma_h)\big)\Big) \le \frac{C}{L_2}T(\sigma_h).$$

Since $T(\sigma_h)$ is infinitesimal, we obtain

$$\limsup_{h\to\infty} \mathbb{A}_k\big(x(\sigma_h),T(\sigma_h)\big) \le 0.$$

Together with statement (ii) of Lemma 3.3 and the monotonicity of the function $\sigma \longmapsto \mathbb{A}_\kappa(x(\sigma),T(\sigma))$, this concludes the proof. □

4 Palais-Smale sequences

(PS) sequences with $(\mathbf{T_h})$ infinitesimal In the following lemmas, (PS) sequences are meant with respect to the metric on $\mathcal{M}$ defined in (3.8). The following lemma justifies the choice of this metric.

Lemma 4.1. Let (x_h,T_h) be a $(\mathrm{PS})_c$ sequence for $\mathbb{A}_\kappa$ with $T_h \to 0$. Then a subsequence of (x_h) converges in $W^{1,2}(\mathbb{T},M)$ to a constant loop $\bar{x}$, where $(\bar{x},0) \in TM$ is a singular point of the Euler-Lagrange flow with energy $E(\bar{x},0)=\kappa$.

Proof. We may assume that $T_h \le 1/2$ and $\mathbb{A}_\kappa(x_h,T_h) \le c+1$. Since M is compact, up to a subsequence we may assume that $x_h(0) \to \bar{x}$ for some $\bar{x} \in M$. Due to (3.9), we have

$$\int_0^1 \big|x_h'(s)\big|^2\,ds \le \frac{T_h}{L_0}\big(\mathbb{A}_\kappa(x_h,T_h)+(L_1-\kappa)T_h\big) \le \frac{T_h}{L_0}\big(c+1+(L_1-\kappa)T_h\big),$$

so the L^2-norm of (x_h') is infinitesimal. Therefore, (x_h) converges in $W^{1,2}(\mathbb{T},M)$ to the constant loop $\bar{x}$. It remains to show that $(\bar{x},0) \in TM$ is a singular point of the Euler-Lagrange flow with energy $E(\bar{x},0)=\kappa$.

If we set, as usual, $\gamma_h(t) := x_h(t/T_h)$, the above inequality implies that

$$\int_0^{T_h} \big|\gamma_h'(t)\big|^2\,dt = \frac{1}{T_h}\int_0^1 \big|x_h'(s)\big|^2\,ds = O(1) \quad \text{for } h \to \infty. \tag{4.1}$$

Actually, more is true:

$$\int_0^{T_h} \big|\gamma_h'(t)\big|^2\,dt = O\big(T_h^2\big) \quad \text{for } h\to\infty. \tag{4.2}$$

Let us prove this fact. Since (x_h,T_h) is a (PS) sequence for $\mathbb{A}_\kappa$, we have

$$\epsilon_h := \big\|d\mathbb{A}_\kappa(x_h,T_h)\big\|^*_{(x_h,T_h)} = o(1) \quad \text{for } h\to\infty, \tag{4.3}$$

where $\|\cdot\|^*_{(x,T)}$ is dual to the norm $\|\cdot\|_{(x,T)}$ defined in (3.8). Since (x_h) converges uniformly to the constant loop $\bar{x}$, we may assume that all the curves $\gamma_h(\mathbb{T})$ lie in a ball B_r of $\mathbb{R}^n$. Since $L(x,v)$ is electromagnetic for $|v|$ large, we have the following bounds:

$$\begin{aligned} \big|d_xL(x,v)\big| \le C_0(1+|v|^2), \quad \big|d_{xx}L(x,v)\big| \le C_1(1+|v|^2),\\ \big|d_{xv}L(x,v)\big| \le C_2(1+|v|), \quad \big|d_{vv}L(x,v)\big| \le C_3, \end{aligned} \tag{4.4}$$

for every $(x,v) \in B_r \times \mathbb{R}^n$. We define

$$\xi_h(s) := x_h(s) - x_h(0), \quad \zeta_h(t) := \gamma_h(t) - \gamma_h(0) = \xi_h\big(t/T_h\big),$$

so that

$$\xi_h(0) = \xi_h(1) = 0 = \zeta_h(0) = \zeta_h(T_h), \quad \zeta_h'(t) = \gamma_h'(t).$$

By (4.3) and by the definition (3.8) of the norm $\|\cdot\|_{(x,t)}$, we have

$$\begin{aligned} \big|d\mathbb{A}_\kappa(x_h,T_h)[\xi_h,0]\big| \leq \epsilon_h \|(\xi_h,0)\|_{(x_h,T_h)} &= \epsilon_h \left(T_h^2 \int_{\mathbb{T}} |\xi_h'(s)|^2\, ds\right)^{1/2} \\ &= \epsilon_h T_h^{3/2} \left(\int_0^{T_h} \big|\gamma_h'(t)\big|^2\, dt\right)^{1/2}. \end{aligned} \tag{4.5}$$

For every curve $\gamma : [0,T] \to B_r$, the Taylor formula with Lagrange remainder produces the estimate

$$\begin{aligned} d_vL\big(\gamma(t),\gamma'(t)\big)\big[\gamma'(t)\big] - d_vL\big(\gamma(0),0\big)\big[\gamma'(t)\big] \\ = \; & d_{vv}L\big(\gamma(0)+\lambda(\gamma(t)-\gamma(0)),\lambda\gamma'(t)\big)\big[\gamma'(t),\gamma'(t)\big] \\ & + d_{xv}L\big(\gamma(0)+\lambda(\gamma(t)-\gamma(0)),\lambda\gamma'(t)\big)\big[\gamma(t)-\gamma(0),\gamma'(t)\big] \\ \geq \; & 2L_0|\gamma'(t)|^2 - C_2\big(1+|\gamma'(t)|\big)\big|\gamma(t)-\gamma(0)\big|\,\big|\gamma'(t)\big|, \end{aligned}$$

where we have used (3.5) and the third bound in (4.4). By this inequality and by the first bound in (4.4), formula (3.6) yields

$$\begin{aligned} d\mathbb{A}_\kappa(x_h,T_h)\big[(\xi_h,0)\big] &= \int_0^{T_h} \Big(d_vL(\gamma_h,\gamma_h')[\gamma_h'] + d_xL(\gamma_h,\gamma_h')[\zeta_h]\Big)\, dt \\ \geq \; & \int_0^{T_h} d_vL\big(\gamma_h(0),0\big)\big[\gamma_h'(t)\big]\, dt + 2L_0 \int_0^{T_h} \big|\gamma_h'(t)\big|^2\, dt \\ & - C_2 \int_0^{T_h} \big(1+|\gamma_h'(t)|\big)\big|\gamma_h(t)-\gamma_h(0)\big|\,\big|\gamma_h'(t)\big|\, dt - C_0 \int_0^{T_h} \big(1+|\gamma_h'|^2\big)|\gamma_h(t)-\gamma_h(0)|\, dt. \end{aligned}$$

Since γ_h is a closed curve, the first integral in the last expression vanishes. By combining the above estimate with the elementary inequalities

$$\begin{aligned} \int_0^T \big|\gamma(t)-\gamma(0)\big|\, dt &\leq T\ell(\gamma), \\ \int_0^T \big|\gamma'(t)\big|\,\big|\gamma(t)-\gamma(0)\big|\, dt &\leq \ell(\gamma)^2, \\ \int_0^T \big|\gamma'(t)\big|^2\big|\gamma(t)-\gamma(0)\big|\, dt &\leq \ell(\gamma) \int_0^T \big|\gamma'(t)\big|^2\, dt, \end{aligned}$$

where $\ell(\gamma)$ denotes the length of the curve γ, we obtain

$$\begin{aligned} d\mathbb{A}_\kappa(x_h,T_h)\big[(\xi_h,0)\big] \geq \; & 2L_0 \int_0^{T_h} \big|\gamma_h'(t)\big|^2\, dt - C_2\ell\big(\gamma_h\big)^2 - (C_0+C_2)\ell\big(\gamma_h\big) \int_0^{T_h} \big|\gamma_h'(t)\big|^2\, dt \\ & - C_0T_h\ell\big(\gamma_h\big). \end{aligned}$$

Together with (4.5), this yields

$$\Big(2L_0-(C_0+C_2)\ell(\gamma_h)\Big)\int_0^{T_h}\big|\gamma_h'(t)\big|^2\,dt\le C_2\ell(\gamma_h)^2+C_0T_h\ell(\gamma_h)+\epsilon_hT_h^{3/2}\left(\int_0^{T_h}\big|\gamma_h'(t)\big|^2\,dt\right)^{1/2}.$$

By (4.1), we have

$$\ell(\gamma_h)\le T_h^{1/2}\left(\int_0^{T_h}\big|\gamma_h'(t)\big|^2\,dt\right)^{1/2}=O\big(T_h^{1/2}\big)\quad\text{for } h\to\infty,$$

hence the previous estimate implies

$$\begin{aligned}\big(2L_0+o(1)\big)\int_0^{T_h}\big|\gamma_h'(t)\big|^2\,dt&\le\Big(C_2\ell(\gamma_h)T_h^{1/2}+C_0T_h^{3/2}+\epsilon_hT_h^{3/2}\Big)\left(\int_0^{T_h}\big|\gamma_h'(t)\big|^2\,dt\right)^{1/2}\\&=O(T_h)\left(\int_0^{T_h}\big|\gamma_h'(t)\big|^2\,dt\right)^{1/2},\end{aligned}$$

and dividing by the L^2 norm of γ_h' we obtain (4.2).

Let us show that $(\bar{x},0)$ is a singular point for the Euler-Lagrange flow, that is that the vector $\eta:=\partial_xL(\bar{x},0)$ is zero. By Taylors formula and by (4.4), for every curve $\gamma:[0,T]\to B_r$ we have

$$\begin{aligned}&\big|d_xL\big(\gamma(t),\gamma'(t)\big)[\eta]-d_xL(\bar{x},0)[\eta]\big|\\&=\big|d_{xx}L\big(\bar{x}+\lambda(\gamma(t)-\bar{x}),\lambda\gamma'(t)\big)\big[\gamma(t)-\bar{x},\eta\big]+d_{vx}L\big(\bar{x}+\lambda(\gamma(t)-\bar{x}),\lambda\gamma'(t)\big)\big[\gamma'(t),\eta\big]\big|\\&\le C_1\big(1+|\gamma'(t)|^2\big)|\gamma(t)-\bar{x}|\,|\eta|+C_2\big(1+|\gamma'(t)|\big)|\gamma'(t)|\,|\eta|.\end{aligned}$$

If we apply the above inequality to γ_h and we integrate it over $[0,T_h]$, by (4.2) we obtain

$$\begin{aligned}&\int_0^{T_h}\big|d_xL\big(\gamma_h(t),\gamma_h'(t)\big)[\eta]-d_xL(\bar{x},0)[\eta]\big|\,dt\\&\qquad\le\int_0^{T_h}\Big(C_1\big(1+|\gamma_h'(t)|^2\big)|\gamma_h(t)-\bar{x}|\,|\eta|+C_2\big(1+|\gamma_h'(t)|\big)|\gamma_h'(t)|\,|\eta|\Big)\,dt\\&\qquad\le C_1\big(T_h+O(T_h^2)\big)\|\gamma_h-\bar{x}\|_\infty|\eta|+C_2\ell(\gamma_h)|\eta|+O(T_h^2)|\eta|.\end{aligned}$$

Since $\ell(\gamma_h)=O(T_h^{3/2})$, by (4.2), the above estimate implies that

$$\int_0^{T_h}\big|d_xL\big(\gamma_h(t),\gamma_h'(t)\big)[\eta]-d_xL(\bar{x},0)[\eta]\big|\,dt=|\eta|o(T_h).\tag{4.6}$$

By (3.6) and (4.3),

$$\begin{aligned}\left|\int_0^{T_h}d_xL\big(\gamma_h(t),\gamma_h'(t)\big)[\eta]\,dt\right|&=\Big|d\mathbb{A}_\kappa(x_h,T_h)\big[(\eta,0)\big]\Big|\le\epsilon_h\|\eta\|_{(x_h,T_h)}\\&=\epsilon_h\left(T_h^2\int_0^1|\eta|^2\,ds\right)^{1/2}=\epsilon_hT_h|\eta|.\end{aligned}\tag{4.7}$$

By (4.6) and (4.7),

$$T_h|\eta|^2 = \int_0^{T_h} d_x L(\bar{x},0)[\eta]\,dt \le \int_0^{T_h} d_x L\big(\gamma_h(t),\gamma_h'(t)\big)[\eta]\,dt + |\eta| o(T_h)$$
$$\le \epsilon_h T_h|\eta| + |\eta| o(T_h),$$

and dividing by T_h we deduce that η is the zero vector, as claimed.

It remains to show that $E(\bar{x},0)=\kappa$. Since for every $x\in B_r$ the smooth function $v\mapsto E(x,v)$ achieves its minimum at $v=0$, the last bound in (4.4) and Taylor's formula imply

$$0\le E(x,v)-E(x,0)\le \frac{1}{2}C_3|v|^2, \quad \forall (x,v)\in B_r\times\mathbb{R}^n.$$

Therefore,

$$\big|E(\gamma_h(t),\gamma_h'(t))-E(\bar{x},0)\big| \le \big|E(\gamma_h(t),\gamma_h'(t))-E(\gamma_h(t),0)\big| + \big|E(\gamma_h(t),0)-E(\bar{x},0)\big|$$
$$\le \frac{1}{2}C_3|\gamma_h'(t)|^2 + C\|\gamma_h-\bar{x}\|_\infty,$$

where C is a Lipschitz constant for the restriction of E to $B_r\times\{0\}$. By averaging this inequality over the interval $[0,T_h]$ and by using (4.2), we obtain

$$\frac{1}{T_h}\int_0^{T_h}\big|E\big(\gamma_h(t),\gamma_h'(t)\big)-E(\bar{x},0)\big|\,dt = O(T_h)+C\|\gamma_h-\bar{x}\|_\infty = o(1). \tag{4.8}$$

On the other hand, by (3.7), by the form (3.8) of the Riemannian metric on $\mathcal{M}$ and by (4.3),

$$\left|\frac{1}{T_h}\int_0^{T_h}\Big(E\big(\gamma_h(t),\gamma_h'(t)\big)-\kappa\Big)\,dt\right| = \left|\frac{\partial\mathbb{A}_\kappa}{\partial T}(x_h,T_h)\right| = \left|d\mathbb{A}_\kappa(x_h,T_h)\Big[\frac{\partial}{\partial T}\Big]\right| \le \epsilon_h. \tag{4.9}$$

The inequalities (4.8) and (4.9) imply that $E(\bar{x},0)=\kappa$, as claimed. □

(PS) sequences with $(\mathbf{T_h})$ bounded and bounded away from zero Together with the previous one, the following lemma shows that the only (PS) sequences which may cause difficulties are those for which the sequence (T_h) of periods is not bounded from above.

Lemma 4.2. Let (x_h,T_h) be a $(\mathrm{PS})_c$ sequence for $\mathbb{A}_\kappa$ with $0<T_*\le T_h\le T^*<\infty$. Then (x_h,T_h) is compact in $\mathcal{M}$.

PROOF. Up to a subsequence, we may assume that (T_h) converges to some $T\in[T_*,T^*]$. By (3.4) we have

$$\begin{aligned} c+o(1) \ge \mathbb{A}_\kappa(x_h,T_h) &\ge T_h\int_0^1\Big(L\big(x_h,x_h'/T_h\big)+\kappa\Big)\,ds \\ \ge T_h\int_0^1\Big(L_0\frac{|x_h'|^2}{T_h^2}-(L_1-\kappa)\Big)\,ds &\ge \frac{L_0}{T^*}\|x_h'\|_2^2 - T^*|L_1-\kappa|, \end{aligned} \tag{4.10}$$

where $\|\cdot\|_2$ denotes the L^2 norm with respect to the fixed Riemannian metric on M. Therefore, $\|x_h'\|_2$ is uniformly bounded and (x_h) is 1/2-equi-Hölder-continuous:

$$\mathrm{dist}\big(x_h(s'),x_h(s)\big) \le \int_s^{s'}|x_h'(r)|\,dr \le |s'-s|^{1/2}\|x_h'\|_2.$$

By the Ascoli-Arzelà theorem, up to a subsequence (x_h) converges uniformly to some $x \in C(\mathbb{T}, M)$. In particular, (x_h) eventually belongs to the image of the parameterization φ_* induced by a smooth map

$$\varphi : \mathbb{T} \times B_r \to M.$$

See (2.1) and recall that the image of this parameterization is C^0-open. Then $x_h = \varphi_*(\xi_h)$, where ξ_h belongs to $W^{1,2}(\mathbb{T}, B_r)$ and is a (PS) sequence for the functional

$$\widetilde{\mathbb{A}}(\xi, T) = T \int_0^1 \widetilde{L}\big(s, \xi, \xi'/T\big)\, ds,$$

with respect to the standard Hilbert product on $W^{1,2}(\mathbb{T}, \mathbb{R}^n)$, where the Lagrangian $\widetilde{L} \in C^\infty(\mathbb{T} \times B_r \times \mathbb{R}^n)$ is obtained by pulling back $L + \kappa$ by φ. Moreover, (ξ_h) converges uniformly and, since $\|\xi_h'\|_2$ is bounded, weakly in $W^{1,2}$ to some ξ in $W^{1,2}(\mathbb{T}, B_r)$. We must prove that this convergence is actually strong in $W^{1,2}$.

Since $\widetilde{L}(s, x, v)$ is electromagnetic for $|v|$ large, we have the bounds

$$\big|d_x \widetilde{L}(s, x, v)\big| \leq C(1 + |v|^2), \quad \big|d_v \widetilde{L}(s, x, v)\big| \leq C(1 + |v|), \tag{4.11}$$

for a suitable constant C. Since (ξ_h, T_h) is a (PS) sequence with (T_h) bounded away from zero and (ξ_h) is bounded in $W^{1,2}$, we have by (3.6)

$$\begin{aligned} o(1) &= d\widetilde{\mathbb{A}}(\xi_h, T_h)[(\xi_h - \xi, 0)] \\ &= T_h \int_0^1 d_x \widetilde{L}\big(s, \xi_h, \xi_h'/T_h\big)[\xi_h - \xi]\, ds + T_h \int_0^1 d_v \widetilde{L}\big(s, \xi_h, \xi_h'/T_h\big)\big[(\xi_h' - \xi')/T_h\big]\, ds. \end{aligned}$$

By the first bound in (4.11) and the uniform convergence $\xi_h \to \xi$, the first integral is infinitesimal. Therefore

$$\int_0^1 d_v \widetilde{L}\big(s, \xi_h, \xi_h'/T_h\big)\big[(\xi_h' - \xi')/T_h\big]\, ds = o(1). \tag{4.12}$$

From the fiberwise C^2 strict convexity of $\widetilde{L}$, we have the bound

$$d_{vv} \widetilde{L}(s, x, v)[u, u] \geq \delta |u|^2, \quad \forall (s, x, v) \in \mathbb{T} \times B_r \times \mathbb{R}^n,\ u \in \mathbb{R}^n,$$

for a suitable positive number δ. It follows that

$$\begin{aligned} d_v \widetilde{L}\left(s, \xi_h, \frac{\xi_h'}{T_h}\right)\left[\frac{\xi_h' - \xi'}{T_h}\right] - d_v \widetilde{L}\left(s, \xi_h, \frac{\xi'}{T_h}\right)\left[\frac{\xi_h' - \xi'}{T_h}\right] \\ = \int_0^1 d_{vv} \widetilde{L}\left(s, \xi_h, \frac{\xi'}{T_h} + \sigma \frac{\xi_h' - \xi'}{T_h}\right)\left[\frac{\xi_h' - \xi'}{T_h}, \frac{\xi_h' - \xi'}{T_h}\right] d\sigma \geq \frac{\delta}{T_h^2} |\xi_h' - \xi'|^2. \end{aligned}$$

By integrating this inequality over $s \in [0, 1]$ and by (4.12), we obtain

$$o(1) - \int_0^1 d_v \widetilde{L}\big(s, \xi_h, \xi'/T_h\big)\big[(\xi_h' - \xi')/T_h\big]\, ds \geq \frac{\delta}{T_h^2} \|\xi_h' - \xi'\|_2^2.$$

By the second bound in (4.11), the sequence

$$d_v \widetilde{L}\big(s, \xi_h, \xi'/T_h\big)$$

converges strongly in L^2. By the weak L^2 convergence to 0 of $(\xi_h' - \xi)$, we deduce that the integral on the left-hand side of the above inequality is infinitesimal. We conclude that (ξ_h) converges to ξ strongly in $W^{1,2}$. □

5 Periodic orbits with high energy

The Mañé critical values The following numbers should be interpreted as energy levels and mark important dynamical and geometric changes for the Euler-Lagrange flow induced by the Tonelli Lagrangian L:

$$\begin{aligned} c_0(L) &:= \inf\{\kappa \in \mathbb{R} \,|\, \mathbb{A}_\kappa(x,T) \geq 0 \; \forall (x,T) \in \mathcal{M} \text{ with } x \text{ homologous to zero}\} \\ &= -\inf\Big\{ \frac{1}{T}\int_0^T L(\gamma(t),\gamma'(t))\,dt \,\Big|\, \gamma \in C^\infty(\mathbb{R}/T\mathbb{Z}, M) \text{ homologous to zero, } T>0 \Big\}, \\ c_u(L) &:= \inf\{\kappa \in \mathbb{R} \,|\, \mathbb{A}_\kappa(x,T) \geq 0 \; \forall (x,T) \in \mathcal{M} \text{ with } x \text{ contractible}\} \\ &= -\inf\Big\{ \frac{1}{T}\int_0^T L(\gamma(t),\gamma'(t))\,dt \,\Big|\, \gamma \in C^\infty(\mathbb{R}/T\mathbb{Z}, M) \text{ contractible, } T>0 \Big\}, \\ e_0(L) &:= \max_{x\in M} E(x,0). \end{aligned}$$

The number $c_0(L)$ is known as the *strict Mañé critical value*, while $c_u(L)$ is the *Mañé critical value associated to the universal covering of* M (see [Mañ97]). When the fundamental group of M is rich, there are other Mañé critical values, which are associated to the different coverings to M, but the above ones are those which are more relevant for the question of existence of periodic orbits. It is easy to see that

$$\min E \leq e_0(L) \leq c_u(L) \leq c_0(L).$$

When L has the form (3.1), $\min E$ is the minimum of the scalar potential V, while $e_0(L)$ is its maximum. When the magnetic potential θ vanishes, the identities $e_0(L) = c_u(L) = c_0(L)$ hold, but in general $e_0(L)$ is strictly lower than the other two numbers. The values $c_u(L)$ and $c_0(L)$ coincide when the fundamental group of M is Abelian and, more generally, when it is ameanable (see [FM07]).

Lemma 5.1. If $\kappa \geq c_u(L)$, then $\mathbb{A}_\kappa$ is bounded from below on every connected component of $\mathcal{M}$.

Proof. Choose $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ in some connected component of the free loop space and let $\tilde\gamma : [0,T] \longrightarrow \widetilde{M}$ be the its lift to the universal covering $\pi : \widetilde{M} \to M$. We lift the metric of M to $\widetilde{M}$ and notice that the fact of having fixed the connected component of the free loop space implies that $\mathrm{dist}\big(\tilde\gamma(T), \tilde\gamma(0)\big)$ is uniformly bounded. Therefore, there exists a path $\tilde\alpha : [0,1] \to \widetilde{M}$ which joins $\tilde\gamma(T)$ to $\tilde\gamma(0)$ and has uniformly bounded action

$$\widetilde{\mathbb{A}}_\kappa(\tilde\alpha) = \int_0^1 \Big(\widetilde{L}\big(\tilde\alpha(t), \tilde\alpha'(t)\big) + \kappa \Big)\, dt \leq C,$$

where $\widetilde{L}$ denotes the Lagrangian on $\widetilde{M}$ which is obtained by lifting L to $\widetilde{M}$. If $\alpha := \pi \circ \tilde\alpha$, the juxtaposition $\gamma \# \alpha$ is a contractible loop in M. Since $\kappa \geq c_u(L)$, we have

$$0 \leq \mathbb{A}_\kappa(\gamma\#\alpha) = \mathbb{A}_\kappa(\gamma) + \mathbb{A}_\kappa(\alpha) = \mathbb{A}_\kappa(\gamma) + \widetilde{\mathbb{A}}_\kappa(\tilde\alpha) \leq \mathbb{A}_\kappa(\gamma) + C,$$

from which $\mathbb{A}_\kappa(\gamma) \geq -C$. □

Lemma 5.2. If $\kappa > c_u(L)$, then any (PS) sequence (x_h, T_h) in a given connected component of $\mathcal{M}$ with $T_h \geq T_* > 0$ is compact.

PROOF. By Lemma 4.2, it is enough to show that (T_h) is bounded from above. Since

$$\mathbb{A}_\kappa(x,T) = \mathbb{A}_{c_u(L)}(x,T) + \big(\kappa - c_u(L)\big)T,$$

the period

$$T_h = \frac{1}{\kappa - c_u(L)}\big(\mathbb{A}_\kappa(x_h,T_h) - \mathbb{A}_{c_u(L)}(x_h,T_h)\big)$$

is bounded from above, because $\mathbb{A}_\kappa$ is bounded on the (PS) sequence (x_h,T_h) and $\mathbb{A}_{c_u(L)}(x_h,T_h)$ is bounded from below by Lemma 5.1. □

Existence of periodic orbits for energies above $\mathbf{c_u(L)}$ The following result shows that the energy levels above $c_u(L)$ have always periodic orbits.

Theorem 5.3. *If $\kappa > c_u(L)$, then:*

(i) if M is not simply connected, then the energy level $E^{-1}(\kappa)$ has a periodic orbit in each non-trivial homotopy class of free loops, which minimizes the action $\mathbb{A}_\kappa$ among the free loops in that class;

(ii) if M is simply connected, then the energy level $E^{-1}(\kappa)$ has a periodic orbit with positive action $\mathbb{A}_\kappa$.

PROOF. (i) Assume that M is not simply connected. Let $\alpha \in [\mathbb{T}, M]$ be a non-trivial homotopy class and let $\mathcal{M}_\alpha$ be the connected component of $\mathcal{M}^{\mathrm{noncontr}}$ corresponding to α. By Lemma 5.1, the functional $\mathbb{A}_\kappa$ is bounded from below on $\mathcal{M}_\alpha$. By Lemma 3.3 (i), the sublevels

$$\{(x,T) \in \mathcal{M}_\alpha \,|\, \mathbb{A}_\kappa(x,T) \leq c\}$$

are complete. Let $(x_h,T_h) \subset \mathcal{M}_\alpha$ be a (PS) sequence for $\mathbb{A}_\kappa$. By Lemma 4.1, (T_h) is bounded away from zero, so Lemma 5.2 implies that $\mathbb{A}_\kappa$ satisfies the (PS) condition on $\mathcal{M}_\alpha$. By Remark 1.10, we conclude that $\mathbb{A}_\kappa$ has a minimizer on $\mathcal{M}_\alpha$, as we wished to prove.

(ii) Assume that M is simply connected, so that $\mathcal{M} = \mathcal{M}^{\mathrm{contr}}$. In this case, $\mathbb{A}_\kappa$ is strictly positive everywhere, because $\kappa > c_u(L)$, but the infimum of $\mathbb{A}_\kappa$ is zero, as one readily checks by looking at sequences of the form (x_0,T_h), with x_0 a constant loop and $T_h \to 0$. So the infimum is not achieved. We will find the periodic orbit by considering the same minimax class which Lusternik and Fet considered in their proof of the existence of a closed geodesic on a simply connected compact manifold.

Since the closed manifold M is simply connected, there exists $l \geq 2$ such that $\pi_l(M) \neq 0$ (a manifold all of whose homotopy groups vanish is contractible, but closed manifolds are never contractible, for instance because their n-dimensional homology group with $\mathbb{Z}_2$ coefficients does not vanish). We fix a non-zero homotopy class $\mathcal{G} \in \pi_l(M)$. Thanks to the isomorphism $\pi_{l-1}(C^0(\mathbb{T},M)) \cong \pi_l(M)$, we have an induced non-zero homotopy class

$$\mathcal{H} \in [S^{l-1}, C^0(\mathbb{T},M)] \cong [S^{l-1}, \mathcal{M}],$$

and we consider the minimax value

$$c = \inf_{\substack{h:S^{l-1}\to\mathcal{M} \\ h\in\mathcal{H}}} \max_{\xi\in S^{l-1}} \mathbb{A}_\kappa(h(\xi)).$$

Let us show that $c > 0$. Since $\mathcal{H}$ is non-trivial, there exists a positive number a such that for every map $h = (x, T) : S^{l-1} \to \mathcal{M}$ belonging to the class $\mathcal{H}$ there holds

$$\max_{\xi \in S^{l-1}} \ell(x(\xi)) \geq a,$$

where $\ell(x(\xi))$ denotes the length of the loop $x(\xi)$ (see [Kli78, Theorem 2.1.8]). If (x, T) is an element of $\mathcal{M}$ with $\ell(x) \geq a$, then (3.4) implies

$$\begin{aligned}\mathbb{A}_\kappa(x, T) = T \int_0^1 \Big(L\big(x, x'/T\big) + \kappa \Big)\, ds &\geq T \int_0^1 \Big(L_0 \frac{|x'|^2}{T^2} - L_1 + \kappa \Big)\, ds \\ &\geq \frac{L_0}{T} \ell(x)^2 - T(L_1 - \kappa) \geq \frac{L_0}{T} a^2 - T(L_1 - \kappa).\end{aligned}$$

Since $a > 0$, the above chain of inequalities implies that there exists $T_0 > 0$ such that for every $(x, T) \in \mathcal{M}$ with $\ell(x) \geq a$ and $\mathbb{A}_\kappa(x, T) \leq c + 1$, the period T is at least T_0. Now let $h \in \mathcal{H}$ be such that the maximum of $\mathbb{A}_\kappa$ on $h(S^{l-1})$ is less than $c + 1$. By the above considerations, there exists (x, T) in $h(S^{l-1})$ with $T \geq T_0$, whence

$$\mathbb{A}_\kappa(x, T) = \mathbb{A}_{c_u(L)}(x, T) + \big(\kappa - c_u(L)\big) T \geq \big(\kappa - c_u(L)\big) T_0 > 0.$$

This shows that the minimax value c is strictly positive.

Theorem 1.8, together with Remark 1.11 and Lemma 3.4, implies the existence of a $(\mathrm{PS})_c$ sequence (x_h, T_h). Lemma 4.1 guarantees that (T_h) is bounded away from zero, so by Lemma 5.2 the sequence (x_h, T_h) has a limiting point in $\mathcal{M}$, which gives us the required periodic orbit. □

Remark 5.4. If M is not simply connected and $\kappa > c_u(L)$, the energy level $E^{-1}(\kappa)$ might have no contractible periodic orbits. Consider for instance the Lagrangian $L(x, v) = |v|^2/2$ on the torus $\mathbb{T}^n$, equipped with the flat metric. The corresponding Euler-Lagrange flow is the geodesic flow on $T\mathbb{T}^n$, $c_u(L) = 0$, and all the non-constant closed geodesics on the flat torus are non-contractible.

Remark 5.5. If $\kappa > c_0(L)$, the existence of a periodic orbit on $E^{-1}(\kappa)$ also follows from the fact that every Finsler metric on a closed manifold has a closed geodesic. In fact, the strict Mañé critical value $c_0(L)$ can be characterized as

$$c_0(L) = \inf \left\{ \max_{x \in M} H(x, \alpha(x)) \,\middle|\, \alpha \text{ smooth closed one-form on } M \right\},$$

where $H : T^*M \to \mathbb{R}$ is the Hamiltonian associated to the Lagrangian L via Legendre duality (see [CIPP98]). So, if $\kappa > c_0(L)$, there is a smooth closed one-form α whose image is contained in the sublevel $\{H < \kappa\}$. Since α is closed, the diffeomorphism of T^*M defined by $(x, p) \mapsto (x, p + \alpha(x))$ is symplectic and conjugates the Hamiltonian flow of H to that of $K(x, p) := H(x, p + \alpha(x))$. The energy level $K^{-1}(\kappa)$ is now the boundary of a fiberwise strictly convex bounded open set which contains the zero section of T^*M. Therefore, there exists a fiberwise convex and 2-homogeneous function $F : T^*M \to [0, +\infty)$ such that $F^{-1}(1) = K^{-1}(\kappa)$. Thus, the Hamiltonian flow of F on $F^{-1}(1) = K^{-1}(\kappa)$ is related to that of K - hence to that of H on $H^{-1}(\kappa)$ - by a time reparameterization. But the Legendre dual of the fiberwise convex and 2-homogeneous Hamiltonian F is a Finsler structure on M. In particular, the closed orbits of the Hamiltonian vector field of F on $F^{-1}(1)$ are precisely the closed Finsler geodesics. We conclude that the periodic orbits of the Euler-Lagrange flow of L of energy κ are reparametrized closed Finsler geodesics.

Remark 5.6. Most of the multiplicity results for closed Finsler geodesics hold also Hamiltonian orbits at energy levels $\kappa > c_u(L)$. In fact, as the proof of Theorem 5.3 suggests, the (PS) condition and the topology of the sublevels of the functional $\mathbb{A}_\kappa$ are analogous to the corresponding properties of the geodesic energy functional (with the notable exception of the zero level). By the above remark, when $\kappa > c_0(L)$, such results follow even more directly from the Finsler case.

6 Topology of the free period action functional for low energies

When $\kappa < c_u(L)$, the functional $\mathbb{A}_\kappa$ is unbounded from below on each connected component of $\mathcal{M}$. In fact, if α is a contractible closed curve with $\mathbb{A}_\kappa(\alpha) < 0$, we can modify any closed curve γ within its free homotopy class and make it have arbitrarily low action $\mathbb{A}_\kappa$: Join $\gamma(0)$ to $\alpha(0)$ by some path, wind around α several times, come back to $\gamma(0)$ by the inverse path, and finally go once around γ. The aim of this section is to show that, nevertheless, the sublevels of $\mathbb{A}_\kappa$ have a sufficiently rich topology.

We start by proving a simple lemma about the integral of a one form. The integral of a given one-form on a curve x is clearly bounded by a constant times the length of x. When the support of the curve is contained in a ball of M, one may also take the square of the length in this bound, which is a better estimate for short curves. More precisely, we have the following:

Lemma 6.1. *Let θ be a smooth one-form on M and let $U \subset M$ be an open set whose closure is diffeomorphic to a closed ball in $\mathbb{R}^n$. Then there exists a number $\Theta > 0$ such that*

$$\left| \int_{\mathbb{T}} x^*(\theta) \right| \leq \Theta \cdot \ell(x)^2,$$

for every closed curve $x : \mathbb{T} \to U$.

Proof. Up to the change of the constant Θ, we may assume that $U = B_r$ is the ball of radius r around the origin in $\mathbb{R}^n$, equipped with the Euclidean metric. Given the closed curve $x : \mathbb{T} \to B_r$, we consider the map

$$X : [0,1] \times \mathbb{T} \to B_r, \quad X(s,t) = x(0) + s\big(x(t) - x(0)\big).$$

Then $X(1,t) = x(t)$ and $X(0,t) = x(0)$, hence by Stokes theorem

$$\begin{aligned}
\left| \int_{\mathbb{T}} x^*(\theta) \right| &= \left| \int_{[0,1]\times\mathbb{T}} X^*(d\theta) \right| = \left| \int_0^1 ds \int_{\mathbb{T}} d\theta\big(X(s,t)\big)\Big[\frac{\partial X}{\partial s}, \frac{\partial X}{\partial t}\Big]\, dt \right| \\
&\leq \|d\theta\|_\infty \int_0^1 ds \int_{\mathbb{T}} \left|\frac{\partial X}{\partial s}\right| \left|\frac{\partial X}{\partial t}\right| dt = \|d\theta\|_\infty \int_0^1 ds \int_{\mathbb{T}} \big|x(t) - x(0)\big| s \big|x'(t)\big|\, dt \\
&\leq \frac{1}{2}\|d\theta\|_\infty\, \ell(x) \int_0^1 ds \int_{\mathbb{T}} s\big|x'(t)\big|\, dt = \frac{1}{4}\|d\theta\|_\infty\, \ell(x)^2,
\end{aligned}$$

as claimed. □

The energy range $(\mathbf{e_0(L)}, \mathbf{c_u(L)})$ If $\kappa < c_u(L)$, there are contractible closed curves with negative action $\mathbb{A}_\kappa$. Since the space of contractible loops is connected, we can consider the following class of continuous paths in $\mathcal{M}$:

$$\mathcal{Z}_0 := \big\{(x,T) : [0,1] \to \mathcal{M} \,\big|\, x(0) \text{ is a constant loop and } \mathbb{A}_\kappa(x(1), T(1)) < 0\big\}. \tag{6.1}$$

Notice that if x_0 is a constant loop and $T > 0$, then

$$\mathbb{A}_\kappa(x_0, T) = T\big(L(x_0, 0) + \kappa\big) = T\big(\kappa - E(x_0, 0)\big). \tag{6.2}$$

When $\kappa > e_0(L) = \max_{x\in M} E(x,0)$, the above quantity is strictly positive (and tends to zero for $T \to 0$). The next lemma shows that when $e_0(L) < \kappa < c_u(L)$, $\mathbb{A}_\kappa$ has a sort of mountain pass geometry:

Lemma 6.2. Assume that $e_0(L) < \kappa < c_u(L)$. Then there exists $a > 0$ such that for every $z \in \mathcal{Z}_0$ there holds

$$\max_{\sigma\in[0,1]} \mathbb{A}_\kappa\big(z(\sigma)\big) \geq a.$$

Proof. Consider the smooth one-form on M,

$$\theta(x)[v] := d_v L(x,0)[v].$$

By taking a Taylor expansion and by using the bound (3.5), we get the estimate

$$L(x,v) = L(x,0) + d_v L(x,0)[v] + \frac{1}{2} d_{vv} L(x, sv)[v,v] \geq -E(x,0) + \theta(x)[v] + L_0|v|^2, \tag{6.3}$$

where $s \in [0,1]$. Let $\{U_1, \dots, U_N\}$ be a finite covering of M consisting of open sets whose closures are diffeomorphic to closed Euclidean balls, and let $\Theta > 0$ be such that the conclusion of Lemma 6.1 holds for the one-form θ, for each of the open sets U_j's. Let r_0 be a Lebesgue number for this covering, meaning that every ball of radius r_0 is contained in one of the U_j's.

We claim that if $\mathbb{A}_\kappa(x,T) < 0$ then

$$\ell(x) > \min\left\{r_0, \frac{\sqrt{L_0(\kappa - e_0(L))}}{\Theta}\right\} =: r_1. \tag{6.4}$$

In fact, assuming that $\ell(x) \leq r_0$, we have that $x(\mathbb{T})$ is contained in some U_j, for $1 \leq j \leq N$. Set as usual $\gamma(t) = x(t/T)$. By Lemma 6.1 and by (6.3), we obtain the chain of inequalities

$$\begin{aligned} 0 > \mathbb{A}_\kappa(x,T) = A_\kappa(\gamma) &= \int_0^T \big(L(\gamma,\gamma') + \kappa\big)\,dt \\ &\geq \int_0^T \big(-E(\gamma,0) + \theta(\gamma)[\gamma'] + L_0|\gamma'|^2 + \kappa\big)\,dt \\ &= \int_0^T \big(\kappa - E(\gamma,0)\big)\,dt + \int_{\mathbb{R}/T\mathbb{Z}} \gamma^*(\theta) + L_0 \int_0^T |\gamma'|^2\,dt \\ &\geq \big(\kappa - e_0(L)\big)T - \Theta\cdot\ell(\gamma)^2 + \frac{L_0}{T}\ell(\gamma)^2. \end{aligned} \tag{6.5}$$

Since we are assuming $\kappa > e_0(L)$, the above estimate implies that $T > L_0/\Theta$ and that

$$\ell(\gamma)^2 > \frac{\big(\kappa - e_0(L)\big)T}{\Theta - L_0/T} > \frac{\big(\kappa - e_0(L)\big)T}{\Theta} > \frac{\big(\kappa - e_0(L)\big)L_0}{\Theta^2},$$

which proves (6.4).

Fix some number r in the open interval $(0, r_1)$. Since $z = (x, T) \in \mathcal{Z}_0$, $\mathbb{A}_\kappa(x(1), T(1))$ is negative, so by (6.4) the length of $x(1)$ is larger than r_1. By continuity, using the fact that $x(0)$ is a constant loop, we get the existence of $\sigma \in (0,1)$ for which $\ell(x(\sigma)) = r$. Then (6.5) implies

$$\mathbb{A}_\kappa(x(\sigma), T(\sigma)) \geq \big(\kappa - e_0(L)\big)T + \Big(\frac{L_0}{T} - \Theta\Big) r^2.$$

Minimization in T yields

$$\mathbb{A}_\kappa(x(\sigma), T(\sigma)) \geq r\big(\sqrt{L_0(\kappa - e_0(L))} - \Theta r\big) =: a.$$

The number a is positive because $r < r_1$. This concludes the proof. $\square$

The energy range $(\min \mathbf{E}, \mathbf{e_0(L)})$ When $\kappa < e_0(L)$, the identity (6.2) shows that $\mathbb{A}_\kappa$ takes negative values on some constant loops, and the conclusion of Lemma 6.2 cannot hold. Instead than considering the class of paths which go from some constant loop to a loop of negative action, one has to consider the class of deformations of the space of constant loops - which is diffeomorphic to M - into the space of loops with negative action. More precisely, we consider the set of continuous maps

$$\mathcal{Z}_M = \big\{(x, T) : [0,1] \times M \to \mathcal{M} \,\big|\, x(0, x_0) = x_0 \text{ and } \mathbb{A}_\kappa\big(x(1, x_0), T(1, x_0)\big) < 0, \ \forall x_0 \in M\big\}.$$

Lemma 6.3. If $\kappa < c_u(L)$, then the set $\mathcal{Z}_M$ is not empty.

We just sketch the proof, referring to [Tai83] for more details (see also [Tai10]). The argument follows closely Bangert's technique of "pulling one loop at a time" (see [Ban80] and [BK83]).

Let $M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a CW-complex decomposition of M. Since $\kappa < c_u(L)$ and since the 0-skeleton M_0 is a finite set, it is easy to construct a continuous map

$$z_0 : [0,1] \times M \to \mathcal{M}, \quad z_0(\sigma, x_0) = \big(y_0(\sigma, x_0), T_0(\sigma, x_0)\big),$$

such that

(i) $y_0(0, x_0) = x_0$ for every $x_0 \in M$;

(ii) $\mathbb{A}_\kappa \circ z_0(1, x_0) < 0$ for every $x_0 \in M_0$.

Given a positive integer h, we may iterate each loop h times and obtain the map

$$z_0^h : [0,1] \times M \to \mathcal{M}, \quad z_0^h(\sigma, x_0) = \big(y_0^h(\sigma, x_0), hT_0(\sigma, x_0)\big),$$

where

$$y_0^h(\sigma, x_0)(s) := y_0(\sigma, x_0)(hs), \quad \forall (\sigma, x_0) \in [0,1] \times M, \ \forall s \in \mathbb{T}.$$

Consider an edge S in M_1 with end-points $x_0, x_1 \in M_0$. The map $z_0^h(1, \cdot)$ maps the the end-points of S into the h-th iterates α^h and β^h of two loops α and β with negative action $\mathbb{A}_\kappa$. By pulling one of the h loops at a time from α^h to β^h, one can construct a new map from S into $\mathcal{M}$ with end-points α^h and β^h and such that $\mathbb{A}_\kappa$ is negative on its image, provided that h is large enough. By relying on the map z_0^h, this construction can be done globally, and one ends up with a continuous map

$$z_1 : [0,1] \times M \to \mathcal{M}, \quad z_1(\sigma, x_0) = \big(y_1(\sigma, x_0), T_1(\sigma, x_0)\big),$$

such that

(i') $y_1(0, x_0) = x_0$ for every $x_0 \in M$;

(ii') $\mathbb{A}_\kappa \circ z_1(1, x_0) < 0$ for every $x_0 \in M_1$.

Iterating this process, one can construct continuous maps

$$z_k : [0,1] \times M \to \mathcal{M}, \quad z_k(\sigma, x_0) = \big(y_k(\sigma, x_0), T_k(\sigma, x_0)\big),$$

such that

(i") $y_k(0, x_0) = x_0$ for every $x_0 \in M$;

(ii") $\mathbb{A}_\kappa \circ z_k(1, x_0) < 0$ for every $x_0 \in M_k$.

The map z_n is an element of $\mathcal{Z}_M$. This concludes our sketch of the proof of Lemma 6.3. The proof of the following result is analogous to the proof of Lemma 6.2.

Lemma 6.4. Assume that $\min E < \kappa < c_u(L)$. Then there exists $a > 0$ such that for every $z \in \mathcal{Z}_M$ there holds

$$\max_{(\sigma, x_0) \in [0,1] \times M} \mathbb{A}_\kappa\big(z(\sigma, x_0)\big) \geq a.$$

Remark 6.5. The minimax class $\mathcal{Z}_M$ will be used in the next section to prove the existence of periodic orbits for almost energy level in the interval $(\min E, e_0(L))$. This result can be proved also by an argument from symplectic topology. In fact, let $H : T^*M \to \mathbb{R}$ be the Hamiltonian which is Legendre dual to L. The fact that $\kappa < e_0(L)$ implies that the restriction of the projection $T^*M \to M$ to $H^{-1}(\kappa)$ is not surjective. Therefore, one can build a Hamiltonian diffeomorphism of T^*M which displaces $H^{-1}(\kappa)$ from itself (see [Con06, Proposition 8.2]). Sets which are displaceable by a Hamiltonian diffeomorphism have finite π_1-sensitive Hofer-Zehnder capacity (see [Sch06] and [FS07]) and this fact implies the almost everywhere existence result for periodic orbits (see [HZ94]). See [Con06, Corollary 8.3] for more details.

7 Periodic orbits with low energy

The Struwe monotonicity argument When $\kappa \leq c_u(L)$, the periods in a (PS) sequence need not be bounded anymore. Because of this fact, the question of the existence of periodic orbits for every energy κ in the interval $[\min E, c_u(L)]$ is open, although no counterexamples are known. The horocycle flow on a closed surface M with constant negative curvature has no periodic orbits, but it can be seen as an Euler-Lagrange flow associated to some Lagrangian L only when lifted to the (non-compact) universal covering of M. In this case, the relevant value of the energy would be exactly $c_u(L)$.

The following argument is a version of an argument of Struwe, which says that when dealing with a minimax value associated to a family of functionals depending on a real parameter in a suitable monotone way, there exist compact (PS) sequences for almost every value of the parameter. This argument has applications both to Hamiltonian periodic orbits and to semilinear elliptic equations (see [Str90], [Str00, section II.9] and references therein).

Let us assume that $\min E < c_u(L)$, otherwise the interval of low energies collapses to a single level and there is nothing to prove. Given $\kappa \in (\min E, c_u(L))$, let Γ be the set of the images of the maps either in $\mathcal{Z}_0$ or in $\mathcal{Z}_M$, which were introduced in the previous section: If

$e_0(L) < \kappa < c_u(L)$ we may take $\mathcal{Z}_0$, while in general we should take $\mathcal{Z}_M$. Let I be either the interval $(e_0(L), c_u(L))$ - if we are dealing with $\mathcal{Z}_0$ - or the interval $(\min E, c_u(L))$ - if we are dealing with $\mathcal{Z}_M$. For every $\kappa \in I$, consider the minimax value

$$c(\kappa) := \inf_{K \in \Gamma} \max_{(x,T) \in K} \mathbb{A}_\kappa(x,T). \tag{7.1}$$

By Lemmas 6.2, 6.3, and 6.4, $c(\kappa)$ is finite and positive, and since $\mathbb{A}_\kappa$ depends monotonically on κ, the function

$$c : I \to (0, +\infty)$$

is weakly increasing. By Lebesgue Theorem, the set of points of I at which c has a linear modulus of continuity, that is

$$J := \big\{ \bar{\kappa} \in I \,\big|\, \exists \delta > 0,\ \exists M > 0 \text{ s.t. } |c(\kappa) - c(\bar{\kappa})| \leq M|\kappa - \bar{\kappa}| \text{ for every } \kappa \in I \text{ with } |\kappa - \bar{\kappa}| < \delta \big\},$$

has full Lebesgue measure in I.

Lemma 7.1. If $\bar{\kappa} \in J$, then $\mathbb{A}_\kappa$ admits a bounded (PS) sequence at level $c(\bar{\kappa})$, which consists of contractible loops.

Proof. First recall that Γ is a class of subsets of $\mathcal{M}^{\mathrm{contr}}$. Let $(\kappa_h) \subset I$ be a strictly decreasing sequence which converges to $\bar{\kappa}$, and set $\epsilon_h := \kappa_h - \bar{\kappa} \downarrow 0$. We pick $K_h \in \Gamma$ such that

$$\max_{K_h} \mathbb{A}_{\kappa_h} \leq c(\kappa_h) + \epsilon_h.$$

Let $z = (x,T) \in K_h$ be such that $\mathbb{A}_{\bar{\kappa}}(z) > c(\bar{\kappa}) - \epsilon_h$. Since $\bar{\kappa}$ belongs to J, we have

$$T = \frac{\mathbb{A}_{\kappa_h}(z) - \mathbb{A}_{\bar{\kappa}}(z)}{\kappa_h - \bar{\kappa}} \leq \frac{c(\kappa_h) + \epsilon_h - c(\bar{\kappa}) + \epsilon_h}{\epsilon_h} \leq M + 2.$$

Moreover,

$$\mathbb{A}_{\bar{\kappa}}(z) \leq \mathbb{A}_{\kappa_h}(z) \leq c(\kappa_h) + \epsilon_h \leq c(\bar{\kappa}) + (M+1)\epsilon_h.$$

By the above considerations,

$$K_h \subset A_h \cup \big\{ \mathbb{A}_{\bar{\kappa}} \leq c(\bar{\kappa}) - \epsilon_h \big\},$$

where

$$A_h := \big\{ (x,T) \,\big|\, T \leq M+2 \text{ and } \mathbb{A}_{\bar{\kappa}}(x,T) \leq c(\bar{\kappa}) + (M+1)\epsilon_h \big\}.$$

If (x,T) belongs to A_h, we have the estimate

$$\mathbb{A}_{\bar{\kappa}}(x,T) \geq \frac{L_0}{M+2} \|x'\|_2^2 - (M+2)(L_1 - \bar{\kappa}),$$

(see (4.10)), which shows that A_h is bounded in $\mathcal{M}$, uniformly in h. Let ϕ be the flow of the vector field obtained by multiplying $-\nabla \mathbb{A}_{\bar{\kappa}}$ by a suitable non-negative function, whose role is to make the vector field bounded on $\mathcal{M}$ and vanishing on the sublevel $\{\mathbb{A}_{\bar{\kappa}} \leq c(\bar{\kappa})/4\}$, while keeping the uniform decrease condition

$$\frac{d}{d\sigma} \mathbb{A}_{\bar{\kappa}}\big(\phi_\sigma(z)\big) \leq -\frac{1}{2} \min\big\{ \|d\mathbb{A}_{\bar{\kappa}}(\phi_\sigma(z))\|^2, 1 \big\}, \quad \text{if } \mathbb{A}_{\bar{\kappa}}(\phi_\sigma(z)) \geq c(\bar{\kappa})/2. \tag{7.2}$$

See (1.1) and Remarks 1.4, 1.11. Then Lemma 3.4 implies that ϕ is well-defined on $[0,+\infty[\times\mathcal{M}$, and the class of sets Γ is positively invariant with respect to ϕ. Since ϕ maps bounded sets into bounded sets, we have

$$\phi([0,1]\times K_h)\subset B_h\cup\big\{\mathbb{A}_{\bar\kappa}\le c(\bar\kappa)-\epsilon_h\big\}, \tag{7.3}$$

for some uniformly bounded set

$$B_h\subset\big\{\mathbb{A}_{\bar\kappa}\le c(\bar\kappa)+(M+1)\epsilon_h\big\}. \tag{7.4}$$

We claim that there exists a sequence $(z_h)\subset\mathcal{M}^{\mathrm{contr}}$ with

$$z_h\in B_h\cap\big\{\mathbb{A}_{\bar\kappa}\ge c(\bar\kappa)-\epsilon_h\big\},$$

and $\|d\mathbb{A}_{\bar\kappa}(z_h)\|$ infinitesimal. Such a sequence is clearly a bounded (PS) sequence at level $c(\bar\kappa)$. Assume, by contradiction, the above claim to be false. Then there exists $0<\delta<1$ which satisfies

$$\|d\mathbb{A}_{\bar\kappa}\|\ge\delta\quad\text{on } B_h\cap\big\{\mathbb{A}_{\bar\kappa}\ge c(\bar\kappa)-\epsilon_h\big\},$$

for every h large enough. Together with (7.2), (7.3) and (7.4), this implies that, for h large enough, for any $z\in K_h$ such that

$$\phi([0,1]\times\{z\})\subset\big\{\mathbb{A}_\kappa\ge c(\bar\kappa)-\epsilon_h\big\},$$

there holds

$$\mathbb{A}_{\bar\kappa}\big(\phi_1(z)\big)\le\mathbb{A}_{\bar\kappa}(z)-\frac{1}{2}\delta^2\le c(\bar\kappa)+(M+1)\epsilon_h-\frac{1}{2}\delta^2.$$

It follows that

$$\max_{\phi_1(K_h)}\mathbb{A}_{\bar\kappa}\le c(\bar\kappa)-\epsilon_h,$$

for h large enough. Since $\phi_1(K_h)$ belongs to Γ, this contradicts the definition of $c(\bar\kappa)$ and concludes the proof. $\square$

Existence of periodic orbits of low energy We are finally ready to prove the following:

Theorem 7.2. *For almost every $\kappa\in(\min E, c_u(L))$, there is a contractible periodic orbit γ of energy $E(\gamma,\gamma')=\kappa$ and positive action $\mathbb{A}_\kappa(\gamma)=c(\kappa)$.*

Proof. Let κ be an element of the full measure set $J\subset I$. We may also assume that κ does not belong to the set of critical values of the smooth function $x_0\mapsto E(x_0,0)$, which by Sard theorem has zero measure. By Lemma 7.1, $\mathbb{A}_\kappa$ admits a (PS) sequence $(x_h,T_h)\subset\mathcal{M}^{\mathrm{contr}}$ with (T_h) bounded. By Lemma 4.1, (T_h) is bounded away from zero, because otherwise κ would be a critical value of the function $x_0\mapsto E(x_0,0)$. By Lemma 4.2, the sequence (x_h,T_h) has a limiting point in $\mathcal{M}^{\mathrm{contr}}$, which gives us the required contractible periodic orbit. $\square$

Contact type and stable energy hypersurfaces Let $H \in C^\infty(T^*M)$ be the Hamiltonian which is Legendre dual to the Lagrangian L:

$$H(x,p) := \max_{v \in T_xM} \big(p[v] - L(x,v) \big).$$

Then H is a Tonelli Hamiltonian, meaning that it is fiberwise superlinear and C^2-strictly convex (see the beginning of Section 3). Let X_H be the induced Hamiltonian vector field on T^*M, which is defined by the identity

$$\omega(X_H(z), \zeta) = -dH(z)[\zeta], \quad \forall z \in T^*M, \ \zeta \in T_zT^*M,$$

where $\omega = dp \wedge dx$ is the standard symplectic form on T^*M. The flow of X_H preserves each level $H^{-1}(\kappa)$, where it is conjugated to the Euler-Lagrange flow of L on $E^{-1}(\kappa)$ by the Legendre transform

$$TM \to T^*M, \quad (x,v) \mapsto \big(x, d_vL(x,v)\big).$$

Assume that κ is a regular value of H, so that $\Sigma := H^{-1}(\kappa)$ is a hypersurface. Up to a time reparametrization, the dynamics on Σ is determined only by the geometry of Σ and not by the Hamiltonian of which Σ is an energy level: in fact the nowhere vanishing vector field $X_H|_\Sigma$ belongs to the one-dimensional distribution

$$\mathcal{L}_\Sigma := \ker \omega|_\Sigma,$$

whose integral curves are hence the orbits of $X_H|_\Sigma$.

The energy level Σ is said to be of *contact type* if there is a one-form η on Σ which is a primitive of $\omega|_\Sigma$ and is such that η does not vanish on $\mathcal{L}_\Sigma$. Equivalently, there is a smooth vector field Y in a neighborhood of Σ which is transverse to Σ and such that $L_Y\omega = \omega$ (the vector field Y and the one-form η are related by the identity $\imath_Y\omega|_\Sigma = \eta$).

Remark 7.3. If $\kappa > c_0(L)$, then $H^{-1}(\kappa)$ is of contact type (actually, it is of restricted contact type, meaning that the one-form η extends to a primitive of ω on the whole T^*M, as one can deduce from the considerations of Remark 5.5). If $c_u(L) \leq \kappa \leq c_0(L)$ and M is not the 2-torus, $H^{-1}(\kappa)$ is not of contact type (see [Con06, Proposition B.1]), and it is conjectured that the same is true for $e_0(L) < \kappa < c_u(L)$. If $\min E < \kappa < e_0(L)$, $H^{-1}(\kappa)$ might or might not be of contact type: For instance, if the one-form $\theta(x)[v] := d_vL(x,0)[v]$ is closed, then every regular energy level is of contact type (see [Con06, Proposition C.2], in this case $e_0(L) = c_u(L) = c_0(L)$).

The contact condition has the following important consequence: if Σ is a contact type compact hypersurface in a symplectic manifold (W,ω) (in our case, $W = T^*M$), then there is a diffeomorphism

$$(-\epsilon, \epsilon) \times \Sigma \to W, \qquad (r,x) \mapsto \psi_r(x),$$

onto an open neighborhood of Σ such that ψ_0 is the identity on Σ and

$$\psi_r : \Sigma \to \Sigma_r := \psi_r(\Sigma)$$

induces an isomorphism between the line bundles $\mathcal{L}_\Sigma$ and $\mathcal{L}_{\Sigma_r}$ (the hypersurface Σ_r is the image of Σ by the flow at time r of the vector field Y given by the contact condition, see

[HZ94, page 122]). Therefore, if the hypersurfaces Σ_r are level sets of a Hamiltonian K, the dynamics of X_K on Σ_r is conjugate to the one on $\Sigma_0 = \Sigma$ up to a time reparametrization.

Hypersurfaces with the above propery are called *stable* (see [HZ94, page 122]). The stability condition is weaker than the contact condition, as the following characterization, which is due to K. Cieliebak and K. Mohnke [CM05, Lemma 2.3], shows:

Proposition 7.4. *Let Σ be a compact hypersurface in the symplectic manifold (W, ω). Then the following facts are equivalent:*

(i) Σ is stable;

(ii) there is a vector field Y on a neighborhood of Σ which is transverse to Σ and satisfies $\mathcal{L}_\Sigma \subset \ker(L_Y \omega|_\Sigma)$;

(iii) there is a one-form η on Σ such that $\mathcal{L}_\Sigma \subset \ker d\eta$ and η does not vanish on $\mathcal{L}_\Sigma$.

Proof. (i) $\Rightarrow$ (ii). By stability, a neighborhood of Σ can be identified with $(-\epsilon, \epsilon) \times \Sigma$ in such a way that $\mathcal{L}_{\{r\}\times\Sigma}$ does not depend on r. Set $Y := \partial/\partial r$ and denote by $\phi_t(r, x) = (r + t, x)$ its flow. Then $\ker(\phi_t^* \omega|_{\{0\}\times\Sigma})$ does not depend on t and differentiating in t at $t = 0$ we get

$$\mathcal{L}_\Sigma = \ker \omega|_\Sigma \subset \ker(L_Y \omega|_\Sigma).$$

(ii) $\Rightarrow$ (iii). If we set $\eta := \imath_Y \omega|_\Sigma$, by Cartan's identity we have

$$d\eta = d\imath_Y \omega|_\Sigma = (L_Y \omega - \imath_Y d\omega)|_\Sigma = L_Y \omega|_\Sigma,$$

so $\mathcal{L}_\Sigma \subset \ker(L_Y \omega|_\Sigma) = \ker d\eta$. If $\xi \neq 0$ is a vector in $\mathcal{L}_\Sigma$, then

$$\eta(\xi) = \omega(Y, \xi) \neq 0,$$

because $Y \notin T\Sigma$.
(iii) $\Rightarrow$ (i). Consider the closed two-form on $(-\epsilon, \epsilon) \times \Sigma$

$$\tilde{\omega} = \omega|_\Sigma + d(r\eta) = \omega|_\Sigma + r d\eta + dr \wedge \eta.$$

If ϵ is small enough, the form $\omega|_\Sigma + r d\eta$ is non-degenerate on $\ker \eta$ for every $r \in (-\epsilon, \epsilon)$, from which we deduce that $\tilde{\omega}$ is a symplectic form. Since $\tilde{\omega}|_{\{0\}\times\Sigma}$ coincides with $\omega|_\Sigma$, by the coisotropic neighborhood theorem (see [Got82], or [MS98, Exercise 3.36] for the particular case of a hypersurface), a neighborhood of Σ in W is symplectomorphic to $((-\epsilon, \epsilon) \times \Sigma, \tilde{\omega})$, up to the choice of a smaller ϵ. Since for $\xi \in \mathcal{L}_\Sigma(x)$ and $\zeta \in T_{(r,x)}(\{r\} \times \Sigma) = (0) \times T_x\Sigma$ there holds

$$\tilde{\omega}(\xi, \zeta) = \omega(\xi, \zeta) + r d\eta(\xi, \zeta) = 0,$$

we deduce that $\ker(\tilde{\omega}|_{\{r\}\times\Sigma}) = \mathcal{L}_\Sigma$ does not depend on r. Therefore, $\{0\} \times \Sigma$ is stable in $((-\epsilon, \epsilon) \times \Sigma, \tilde{\omega})$ and hence Σ is stable in (W, ω). □

Remark 7.5. L. Macarini and G. Paternain have constructed examples of Tonelli Lagrangians on the tangent bundle of $\mathbb{T}^n$ such that $H^{-1}(\kappa)$ is stable for $\kappa = c_u(L) = c_0(L)$, see [MP10].

After these preliminaries, we can prove that stable energy levels of Tonelli Hamiltonians posses periodic orbits. In particular, the same is true for contact type energy levels.

Corollary 7.6. *Assume that κ is a regular value of the Tonelli Hamiltonian $H \in C^\infty(T^*M)$ and that the hypersurface $\Sigma = H^{-1}(\kappa)$ is stable. Then Σ carries a periodic orbit.*

PROOF. By stability, we can find a diffeomorphism

$$(-\epsilon, \epsilon) \times \Sigma \to T^*M, \qquad (r, x) \mapsto \psi_r(x),$$

onto an open neighborhood of Σ such that ψ_0 is the identity on Σ and

$$\psi_r : \Sigma \to \Sigma_r := \psi_r(\Sigma)$$

induces an isomorphism between the line bundles $\mathcal{L}_\Sigma$ and $\mathcal{L}_{\Sigma_r}$. Up to the choice of a smaller ϵ, we may assume that all the hypersurfaces Σ_r are levels of a C^2-strictly convex function. Therefore, they are the level sets of a Tonelli Hamiltonian $K \in C^\infty(T^*M)$. Since the Legendre transform of K is a Tonelli Lagrangian, Theorems 5.3 and 7.2 imply that $K^{-1}(\kappa)$ has periodic orbits for almost every κ. In particular, Σ_r has periodic orbits for almost every r, but since the dynamics on Σ_r and on Σ are conjugated, the same is true for Σ. □

Remark 7.7. The above proof shows the usefulness of having a theory which works with Tonelli Lagrangians, rather than just electromagnetic ones. In fact even if the stable hypersurface Σ is the level set of an electromagnetic Hamiltonian (that is, it is fiberwise an ellipsoid), the hypersurfaces Σ_r given by the stability assumption may be more general fiberwise C^2-strictly convex hypersurfaces.

Remark 7.8. It can be proved that when M is a closed surface, there are periodic orbits on *every* energy level (see [Tai92a], [Tai92b], [Tai92c] and [CMP03]). In fact, the advantage of dealing with a surface is that when $\kappa < c_u(L)$ one can minimize $\mathbb{A}_\kappa$ on a suitable space of *embedded* closed curves.

The two Lyapunov functions argument We conclude these notes by discussing an alternative argument to deal with the lack of (PS) which is exhibited by $\mathbb{A}_\kappa$ when $\kappa < c_u(L)$. It allows to prove that the set of energy levels κ such that the Euler-Lagrange flow has a periodic orbit of energy κ is *dense* in $(\min E, c_u(L))$, a weaker statement than Theorem 7.2. However, it has some advantages, which are discussed in Remark 7.12 below. This argument is used, in a different context, in [AM08]. Here we shall use it in order to prove the following weaker version of Theorem 7.2:

Theorem 7.9. *Let $\min E < \bar{\kappa} < c_u(L)$ and assume that there are no contractible periodic orbits of energy $\bar{\kappa}$ and non-negative action $\mathbb{A}_{\bar{\kappa}}$. Then there exists a strictly decreasing sequence (κ_h) which converges to $\bar{\kappa}$ and is such that the Euler-Lagrange flow has has a contractible periodic orbit γ_h with energy κ_h and period T_h, which satisfies $\mathbb{A}_{\kappa_h}(\gamma_h)/T_h \downarrow 0$.*

PROOF. We argue by contradiction and assume that there exists $\tilde{\kappa} > \bar{\kappa}$ and $\delta > 0$ such that for any $\kappa \in [\bar{\kappa}, \tilde{\kappa}]$ all the periodic orbits γ of energy κ and period T satisfy either $\mathbb{A}_\kappa(\gamma)/T \geq \delta$ or $\mathbb{A}_\kappa(\gamma) \leq 0$. Fix real numbers $a > c(\bar{\kappa})$ and $\kappa^* \in (\bar{\kappa}, \tilde{\kappa}]$. Assume that we can find $\lambda \in [0,1]$ and $(x, T) \in \mathcal{M}$ such that

$$\lambda \, d\mathbb{A}_{\bar{\kappa}}(x, T) + (1 - \lambda) \, d\mathbb{A}_{\kappa^*}(x, T) = 0, \quad 0 < \mathbb{A}_{\bar{\kappa}}(x, T) \leq a.$$

Then (x, T) is a critical point of $\mathbb{A}_{\lambda\bar{\kappa}+(1-\lambda)\kappa^*}$, hence it is a T-periodic orbit with energy $\lambda\bar{\kappa} + (1-\lambda)\kappa^*$. By what we have assumed at the beginning, we have

$$\delta \leq \frac{1}{T}\mathbb{A}_{\lambda\bar{\kappa}+(1-\lambda)\kappa^*}(x,T) = \frac{1}{T}\mathbb{A}_{\bar{\kappa}}(x,T) + (1-\lambda)(\kappa^* - \bar{\kappa}) \leq \frac{a}{T} + \kappa^* - \bar{\kappa}.$$

Up to the choice of a smaller $\kappa^* > \bar{\kappa}$, we may assume that $\kappa^* - \bar{\kappa} \leq \delta/2$. Then the above estimate implies that

$$T \leq \frac{2a}{\delta} =: T^*.$$

With these choices of κ^* and T^*, we can restate what we have proved so far as:

Lemma 7.10. If $T > T^*$ and $0 < \mathbb{A}_{\bar{\kappa}}(x,T) \leq a$, then the segment

$$\operatorname{conv}\big\{d\mathbb{A}_{\bar{\kappa}}(x,T), d\mathbb{A}_{\kappa^*}(x,T)\big\} \subset T^*_{(x,T)}\mathcal{M}$$

does not contain 0.

The above lemma allows us to construct a negative pseudo-gradient vector field for $\mathbb{A}_{\bar{\kappa}}$ which has all the good properties of $-\nabla\mathbb{A}_{\bar{\kappa}}$ and moreover has $\mathbb{A}_{\kappa^*}$ as a Lyapunov function on the open set

$$A := \big\{T > T^*\big\} \cap \big\{0 < \mathbb{A}_{\bar{\kappa}} < a\big\}.$$

In fact, the only obstruction to finding a vector field W whose flow make both $\mathbb{A}_{\bar{\kappa}}$ and $\mathbb{A}_{\kappa^*}$ decrease in A, is that the differentials of $\mathbb{A}_{\bar{\kappa}}$ and $\mathbb{A}_{\kappa^*}$ point in opposite directions in some point of A, and this is precisely what is excluded by Lemma 7.10. More precisely, one can prove the following:

Lemma 7.11. There exists a locally Lipschitz vector field W on $\mathcal{M}$ such that:

(i) $d\mathbb{A}_{\bar{\kappa}}[W] < 0$ on $\{\mathbb{A}_{\bar{\kappa}} > 0\}$;

(ii) W is forward complete and vanishes on $\{\mathbb{A}_{\bar{\kappa}} \leq 0\}$;

(iii) let $z_h = (x_h, T_h)$ be a sequence in $\mathcal{M}^{\mathrm{contr}}$ such that

$$0 < \inf \mathbb{A}_{\bar{\kappa}}(z_h) \leq \sup \mathbb{A}_{\bar{\kappa}}(z_h) < +\infty, \quad \lim_{h\to\infty} d\mathbb{A}_{\bar{\kappa}}(z_h)[W(z_h)] = 0,$$

and (T_h) is bounded from above; then (z_h) has a subsequence which converges in $\mathcal{M}^{\mathrm{contr}}$;

(iv) $d\mathbb{A}_{\kappa^*}[W] < 0$ on A.

In fact, one can choose W to be given by the vector field

$$\nabla\mathbb{A}_{\bar{\kappa}} + \chi\frac{\|\nabla\mathbb{A}_{\bar{\kappa}}\|}{\|\nabla\mathbb{A}_{\kappa^*}\|}\nabla\mathbb{A}_{\kappa^*}$$

multiplied by a suitable non-positive function. Here χ is a suitable cut-off function. See [AM08, Lemmas 5.1 and 5.4] for a similar construction.

We can now prove Theorem 7.9. By the definition of $c(\bar{\kappa})$, there is a set K in Γ such that

$$\max_K \mathbb{A}_{\bar{\kappa}} < a.$$

By Lemma 7.11 (i) and (ii), for every $\sigma_0 > 0$ we have

$$\inf_{\sigma\in\sigma_0}\Big|d\mathbb{A}_{\bar\kappa}\big(\phi_\sigma(z)\big)\big[W(\phi_\sigma(z))\big]\Big| \leq \frac{1}{\sigma_0}\int_0^{\sigma_0}\Big|d\mathbb{A}_{\bar\kappa}\big(\phi_\sigma(z)\big)\big[W(\phi_\sigma(z))\big]\Big|\,d\sigma = \frac{\mathbb{A}_{\bar\kappa}(z)-\mathbb{A}_{\bar\kappa}(\phi_{\sigma_0}(z))}{\sigma_0},$$

and, by the definition of $c(\bar\kappa)$,

$$\max_{z\in K}\mathbb{A}_{\bar\kappa}\big(\phi_{\sigma_0}(z)\big) \geq c(\bar\kappa).$$

By taking a limit for $\sigma_0 \to +\infty$, thanks to Lemma 7.11 (ii), the above facts imply that $\phi_{\mathbb{R}^+}(K)\cap\{\mathbb{A}_{\bar\kappa}>0\}$ contains a sequence $z_h = (x_h, T_h)$ such that

$$0 < c(\bar\kappa) \leq \inf\mathbb{A}_{\bar\kappa}(z_h) \leq \sup\mathbb{A}_{\bar\kappa}(z_h) < a \quad \text{and} \quad \lim_{h\to\infty} d\mathbb{A}_{\bar\kappa}(z_h)[W(z_h)] = 0.$$

It is enough to show that (T_h) is bounded from above: Indeed, in this case Lemma 7.11 (iii) implies that (z_h) has a limiting point, which is a critical point of $\mathbb{A}_{\bar\kappa}$ with positive action, contradicting the hypothesis of Theorem 7.9.

The upper bound on (T_h) is a consequence of the following claim: the period T is bounded on the set $\phi_{\mathbb{R}^+}(K)\cap\{\mathbb{A}_{\bar\kappa}>0\}$. In order to prove this claim, we first notice that

$$\mathbb{A}_{\bar\kappa}(x,T)\leq a,\;\; T\leq T^* \quad\Rightarrow\quad \mathbb{A}_{\kappa^*}(x,T)\leq a+(\kappa^*-\bar\kappa)T^* =: b. \tag{7.5}$$

Since K is compact, we can find $d > b$ such that $K \subset \{\mathbb{A}_{\kappa^*} < d\}$. Let ϕ be the flow of the vector field W. We claim that

$$\phi_{\mathbb{R}^+}(K)\cap\big\{\mathbb{A}_{\bar\kappa}>0\big\}\subset\big\{\mathbb{A}_{\kappa^*}<d\big\}. \tag{7.6}$$

In fact, let $z \in K$ and let $\sigma_0 > 0$ be the first instant such that $\mathbb{A}_{\kappa^*}(\phi_{\sigma_0}(z)) = d$, while $\mathbb{A}_{\bar\kappa}(\phi_{\sigma_0}(z)) > 0$. By Lemma 7.11 (i), $\mathbb{A}_{\bar\kappa}(\phi_{\sigma_0}(z)) \leq \mathbb{A}_{\bar\kappa}(z) < a$. By Lemma 7.11 (iv), the point $\phi_{\sigma_0}(z)$ cannot belong to A, so $\phi_{\sigma_0}(z) = (x,T)$ with $T \leq T^*$ and (7.5) implies that $\mathbb{A}_{\kappa^*}(\phi_{\sigma_0}(z)) \leq b < d$. This contradiction proves (7.6).

If $\mathbb{A}_{\bar\kappa}(x,T) > 0$ and $\mathbb{A}_{\kappa^*}(x,T) < d$, then

$$d > \mathbb{A}_{\kappa^*}(x,T) = \mathbb{A}_{\bar\kappa}(x,T) + (\kappa^*-\bar\kappa)T > (\kappa^*-\bar\kappa)T.$$

This shows that the period T is bounded on the set

$$\big\{\mathbb{A}_{\bar\kappa}>0\big\}\cap\big\{\mathbb{A}_{\kappa^*}<d\big\},$$

and by (7.6) it is bounded also on

$$\phi_{\mathbb{R}^+}(K)\cap\big\{\mathbb{A}_{\bar\kappa}>0\big\},$$

as claimed. □

Remark 7.12. In the Struwe monotonicity argument, one gets the existence of bounded (PS) sequences at level $c(\bar\kappa)$, but has no control on the (PS) sequences at other levels. Therefore, it is not clear whether the space of negative gradient flow lines for $\mathbb{A}_{\bar\kappa}$ which connect two given critical points - say with positive action - is bounded. An advantage of the two Lyapunov functions argument, is that the latter fact is true for the flow lines of the vector field W constructed in Lemma 7.11: the second Lyapunov function $\mathbb{A}_{\kappa^*}$ allows to exclude the existence of flow lines which go arbitrarily far and come back. This fact would allow to develop some global critical point theory for $\mathbb{A}_{\bar{\kappa}}$, such as Morse theory or Lusternik-Schnirelmann theory. This is not useful here, because the a priori estimates which lead to the existence of the pseudo-gradient vector field W come from a contradiction argument. However, it might be useful in situations where these a priori bounds have a different origin, such as for example in the case of tame energy levels (see [CFP10] for the definition of tameness and for motivating examples).

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Magnetic flows and Mañé's critical value

Gabriel P. Paternain*

Notes taken by Youngjin Bae†

Introduction

Assume (M, g) is a closed Riemannian manifold with the cotangent bundle $\tau : T^*M \to M$. Suppose ω_0 denotes the canonical symplectic form $dp \wedge dq$ on T^*M. Let $\sigma \in \Omega^2(M)$ be a closed 2-form on M which represents a magnetic field. We consider an autonomous Hamiltonian system defined by a Hamiltonian

$$H : T^*M \to \mathbb{R}$$
$$H(q,p) = \frac{1}{2}|p|^2 + U(q)$$

and a twisted symplectic form

$$\omega_\sigma = \omega_0 + \tau^*\sigma.$$

Note that ω_σ is really a symplectic form on T^*M. Obviously it is closed and one can check that it is nondegenerate. Now consider the Hamiltonian flow of H with respect to ω_σ, the motion of a particle on M moving under the conservative force $-\nabla U$ and the magnetic field σ. Let X_H be its Hamiltonian vector field and $\phi_t : T^*M \to T^*M$ be its flow. Since the Hamiltonian flow preserves the energy level, the restriction

$$\phi|_{\Sigma_k} : \Sigma_k \to \Sigma_k$$

is well-defined, where $\Sigma_k = H^{-1}(k)$. Our goal is to understand the dynamics of $\phi|_{\Sigma_k}$.

In local coordinates $q_1, \ldots, q_n$ on M and dual coordinates $p_1, \ldots, p_n$ the Hamiltonian system is given by

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{n} \sigma_{ij}(q)\frac{\partial H}{\partial p_j}, \end{aligned} \tag{0.1}$$

where

$$\sigma = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij}(q) dq_i \wedge dq_j, \qquad \sigma_{ij} = -\sigma_{ji}.$$

*Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge CB3 0WB, UK.
E-mail:g.p.paternain@dpmms.cam.ac.uk

†Department of Mathematical Sciences, Seoul National University, Seoul, Republic of Korea.
E-mail:jini0919@snu.ac.kr

In particular, the q-components of the Hamiltonian vector field X_H are independent of σ,

$$X_H = \left(\frac{\partial H}{\partial p}, *\right).$$

1 Critical values

Assume that there is a covering $\pi : \widehat{M} \to M$ of (M, σ) such that $\pi^*\sigma = d\theta$ for some $\theta \in \Omega^1(\widehat{M})$. The coverings we are interested in are the universal cover and the *abelian cover* which will be defined later.

Example 1.1. Consider $(\mathbb{T}^2, \sigma = dq_1 \wedge dq_2)$ and its universal cover $\pi : \mathbb{R}^2 \to \mathbb{T}^2$. Then σ is not exact but $\pi^*\sigma$ is exact.

Definition 1.2. The Mañé critical value of the cover is defined by

$$c(\widehat{H}) = \inf_{\substack{\theta \\ d\theta = \pi^*\sigma}} \sup_{q \in \widehat{M}} \widehat{H}(q, \theta_q),$$

where $\widehat{H}$ is the lift of H to the cover $\widehat{M}$.

Remark 1.3.

1. If $\overline{M}$ covers $\widehat{M}$ then $c(\overline{H}) \leq c(\widehat{H})$.
2. In the above case, equality holds if the deck transformation group is *amenable*, see Definition 2.5.

Now consider a covering $\pi : (\widehat{M}, \pi^*\sigma) \to (M, \sigma)$ and a primitive 1-form $\theta \in \Omega^1(\widehat{M})$ of $\pi^*\sigma$. Consider the Lagrangian $\widehat{L} : T\widehat{M} \to \mathbb{R}$ which is defined by

$$\widehat{L}(q, v) = \frac{1}{2}|v|^2 - U(q) + \theta_q(v).$$

The action of the Lagrangian $\widehat{L}$ on an absolutely continuous curve $\gamma : [0, T] \to \widehat{M}$ is given by

$$A_{\widehat{L}}(\gamma) = \int_0^T \widehat{L}(\gamma(t), \dot{\gamma}(t))dt.$$

We define the critical value of the Lagrangian $\widehat{L}$ as

$$c(\widehat{L}) = \inf\{k \in \mathbb{R} : A_{\widehat{L}+k}(\gamma) \geq 0 \text{ for any a.c.}^{1} \text{ closed curve } \gamma : [0, T] \to \widehat{M},\ \forall T > 0\}.$$

Theorem 1.4. *If $\widehat{M}$ is any covering of the closed manifold M, then*

$$c(\widehat{H}) = \inf_{[\varpi] \in \mathrm{H}^1(\widehat{M}, \mathbb{R})} c(\widehat{L} - \varpi).$$

Here a closed 1-form ϖ can be considered as a function $\varpi : T\widehat{M} \to \mathbb{R}$ defined by $\varpi(q, v) = \varpi_q(v)$.

[1] a.c. means absolutely continuous

Sketch of proof. For simplicity, let us denote by

$$\widehat{L}_0(q,v) = \frac{1}{2}|v|^2 - U(q).$$

First we claim that

$$c(\widehat{L}_0) = \inf_{u\in C^\infty(\widehat{M},\mathbb{R})} \sup_{x\in\widehat{M}} \widehat{H}(x, d_x u). \tag{1.1}$$

We begin by showing that

$$c(\widehat{L}_0) \le \inf_{u\in C^\infty(\widehat{M},\mathbb{R})} \sup_{x\in\widehat{M}} \widehat{H}(x, d_x u).$$

If there exists a smooth function $u : \widehat{M} \to \mathbb{R}$ such that $\widehat{H}(x, d_x u) \le k < \infty$ for all $x \in \widehat{M}$, we need to show that $c(\widehat{L}_0) \le k$. Observe that

$$\widehat{H}(x,p) = \max_{v\in T_x\widehat{M}} \{p(v) - \widehat{L}_0(x,v)\}.$$

Since $\widehat{H}(x, d_x u) \le k$ for all $x \in \widehat{M}$, it follows that for all $(x,v) \in T\widehat{M}$

$$d_x u(v) - \widehat{L}_0(x,v) \le k.$$

Therefore, along any absolutely continuous closed curve $\gamma : [0,T] \to \widehat{M}$, we have

$$A_{\widehat{L}_0+k}(\gamma) = \int_0^T (\widehat{L}_0(\gamma,\dot\gamma) + k)dt = \int_0^T (\widehat{L}_0(\gamma,\dot\gamma) + k - d_\gamma u(\dot\gamma))dt \ge 0,$$

and thus $c(\widehat{L}_0) \le k$.

We now prove the reverse inequality. For each $k \in \mathbb{R}$, we define the *action potential* $\Phi_k : \widehat{M} \times \widehat{M} \to \mathbb{R}$ by

$$\Phi_k(x,y) = \inf\{A_{\widehat{L}_0+k}(\gamma) : \gamma \text{ is an absolutely continuous curve from } x \text{ to } y\}.$$

Triangle inequality holds straight from the definition

$$\Phi_k(x_1,x_3) \le \Phi_k(x_1,x_2) + \Phi_k(x_2,x_3) \qquad \forall x_i \in \widehat{M}.$$

When $c(\widehat{L}_0) \le k$, the function Φ_k is locally Lipschitz. If we define a function $u : \widehat{M} \to \mathbb{R}$ by

$$u(x) = \Phi_k(q,x),$$

for a fixed point $q \in \widehat{M}$, then u is locally Lipschitz and u is differentiable almost everywhere by Rademacher's theorem. For a differentiable point $x \in \widehat{M}$, take a differentiable curve $\gamma(t)$ on $\widehat{M}$ with $(\gamma(0), \dot\gamma(0)) = (x,v)$. Then

$$\limsup_{t\to 0^+} \frac{u(\gamma(t)) - u(x)}{t} \le \limsup_{t\to 0^+} \frac{\Phi_k(\gamma(0),\gamma(t))}{t} \le \limsup_{t\to 0^+} \frac{1}{t}\int_0^t [\widehat{L}_0(\gamma,\dot\gamma) + k]ds.$$

Hence we get

$$d_x u(v) \le \widehat{L}_0(x,v) + k.$$

Since $v \in T_x\widehat{M}$ is arbitrary, it follows that

$$\widehat{H}(x, d_x u) = \max_{v \in T_x\widehat{M}} \{d_x u(v) - \widehat{L}_0(x, v)\} \leq k.$$

By a regularization process, there is a smooth function $\widehat{u} : \widehat{M} \to \mathbb{R}$ that approximates u well enough so that $H(x, d_x\widehat{u}) \leq k$ for all $x \in \widehat{M}$. This proves the claim.

If we consider $\theta + du$ instead of du, then we obtain

$$c(\widehat{L}) = \inf_{u \in C^\infty(\widehat{M},\mathbb{R})} \sup_{x \in \widehat{M}} \widehat{H}(x, \theta_x + d_x u).$$

If we choose another primitive θ' such that $d\theta' = \pi^*\sigma$. Then $\varpi = \theta - \theta'$ determines a class $\mathrm{H}^1(\widehat{M}, \mathbb{R})$. Now we consider the all non-zero classes in $\mathrm{H}^1(\widehat{M}, \mathbb{R})$. This conclude that

$$c(\widehat{H}) = \inf_{[\varpi] \in \mathrm{H}^1(\widehat{M},\mathbb{R})} c(\widehat{L} - \varpi).$$

□

The followings are examples of Mañé's critical value for coverings of manifolds. All examples will have $U \equiv 0$.

Example 1.5. Let $(\mathbb{T}^2, g)$ be a 2-torus with flat metric and $\sigma = dq_1 \wedge dq_2$ be its standard area form. Consider the universal cover $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ with Euclidean metric. Then $\pi^*\sigma = d(q_1 \wedge dq_2)$ is exact. In this case $c = \infty$. Indeed, suppose there is $\theta \in \Omega^1(\mathbb{R}^2)$ such that $d\theta = dq_1 \wedge dq_2$ and $\|\theta\|_\infty \leq C < \infty$. Then

$$\pi r^2 = \int_{\mathbb{D}_r} d\theta = \int_{C_r} \theta \leq C \cdot 2\pi r,$$

which cannot happen as $r \to \infty$.

Example 1.6. Let $\left(\Sigma_{g\geq 2}, ds^2 = \frac{1}{y^2}(dx^2 + dy^2)\right)$ be a genus $g \geq 2$ surface with hyperbolic metric and $\sigma = \frac{1}{y^2} dx \wedge dy$ be its standard area form. Now consider the universal cover $\pi : \mathbb{H}^2 \to \Gamma \setminus \mathbb{H}^2 = \Sigma_{g\geq 2}$, then

$$\frac{1}{2}|\theta|^2 = \frac{1}{2}y^2\left(\frac{1}{y^2} + 0\right) = \frac{1}{2}.$$

Hence we get

$$c = \inf_\theta \sup_{q \in \mathbb{H}^2} \frac{1}{2}|\theta|^2 \leq \frac{1}{2}.$$

In order to show that $c = \frac{1}{2}$, we use the Lagrangian definition

$$c = \inf\{k : A_{L+k}(\gamma) \geq 0 \text{ for every absolutely continuous closed curve } \gamma\},$$

where $L = \frac{1}{2}|v|^2 + \theta$. Note that the length of any radius r-circle $l(C_r)$ is $2\pi \sinh r$ and the area of any radius r-disk $area(\mathbb{D}_r)$ is $2\pi(\cosh r - 1)$. Let us parametrize C_r clockwise with

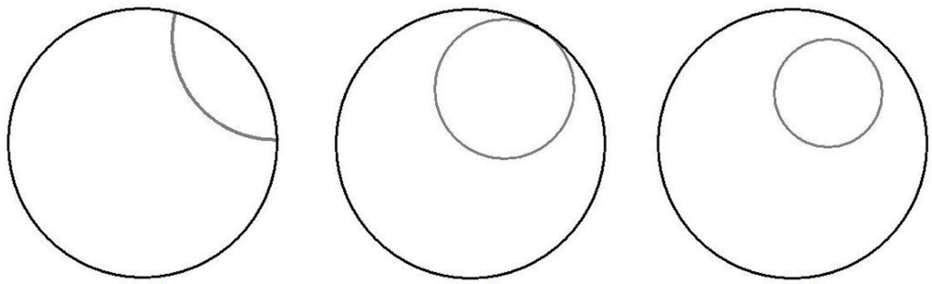

Figure 1: Geodesic flows

speed one, then

$$\begin{aligned} A_{L+k}(C_r) &= \left(k + \frac{1}{2}\right) 2\pi \sinh r + \int_{C_r} \theta \\ &= \left(k + \frac{1}{2}\right) 2\pi \sinh r - area(\mathbb{D}_r) \\ &= 2\pi \left[\left(k + \frac{1}{2}\right) \sinh r - \cosh r + 1\right]. \end{aligned} \tag{1.2}$$

If $k < \frac{1}{2}$, $A_{L+k}(C_r) \to -\infty$ as $r \to \infty$. Therefore we conclude that $c(L) \geq \frac{1}{2}$. Finally let us remark that if $k = \frac{1}{2}$, then

$$A_{L+\frac{1}{2}}(C_r) = 2\pi(1 - e^{-r}) \to 2\pi \qquad \text{as } r \to \infty.$$

2 Contact and virtually contact hypersurfaces

Definition 2.1. Let Σ be a closed odd-dimensional manifold. A *Hamiltonian structure on* Σ is a closed 2-form ω such that $\ker \omega$ is 1-dimensional everywhere.

Example 2.2. Let (T^*M, ω_σ) be a twisted cotangent bundle with an autonomous Hamiltonian $H(q,p) = \frac{1}{2}|p|^2 + U(q)$. Then $\left(\Sigma_k := H^{-1}(k), \omega_\sigma|_{\Sigma_k}\right)$ is a Hamiltonian structure.

Definition 2.3. The Hamiltonian structure is of *contact type* if there exists a 1-form λ such that $\omega = d\lambda$ and $\lambda(v) \neq 0$ for all $0 \neq v \in \ker \omega$.

A Hamiltonian structure is *virtually contact type* if on the universal covering $\pi : \widetilde{\Sigma} \to \Sigma$ there exists a primitive λ of $\pi^*\omega$ such that

1. $|\lambda|_{C^0}$ is bounded with respect to the lift of one and hence every Riemannian metric on the compact manifold Σ;

2. $|\lambda(v)| \geq C|v|$ for all $v \in \ker \pi^*\omega$, where C is positive constant and the norms again are taken with respect to the lift of a Riemannian metric on Σ.

Question 2.4. Does the Weinstein conjecture hold for (3-dimensional) virtually contact Hamiltonian structure?

Definition 2.5. A group Γ is said to be *amenable* if it has a right-invariant mean in $l^\infty(\Gamma)$ i.e. a bounded linear functional

$$m : l^\infty(\Gamma) \to \mathbb{R}$$

such that

1. $m(a) = a$ for constant functions,
2. $m(a_1) \geq m(a_2)$ if $a_1 \geq a_2$,
3. $m(\varphi_* a) = m(a)$, where $(\varphi_* a)(\psi) := a(\psi \circ \varphi)$.

Example 2.6. All abelian, nilpotent, and solvable groups are amenable. A group containing a free subgroup of at least 2 generators and the fundamental group of a negatively curved manifold are non-amenable.

Proposition 2.7. *If (Σ, ω) is virtually contact and $\pi_1(\Sigma)$ is amenable, then (Σ, ω) is of contact type.*

Proof. Let $\pi : \widetilde{\Sigma} \to \Sigma$ be the universal cover and $\lambda \in \Omega^1(\widetilde{\Sigma})$ be a primitive of $\pi^*\omega$ satisfying the virtually contact assumption. Consider a function $a : \Gamma = \pi_1(\Sigma) \to \mathbb{R}$ defined by

$$a_{(x,v)}(\varphi) = \lambda_{\varphi(x)}(d\varphi(v)),$$

for $x \in \widetilde{\Sigma}$ and $v \in T_x\widetilde{\Sigma}$. Since φ acts by isometries and $|\lambda|_{C^0}$ is bounded,

$$|a_{(x,v)}(\varphi)| = |\lambda_{\varphi(x)}(d\varphi(v))| \leq |\lambda|_{C^0}|d\varphi(v)| = |\lambda|_{C^0}|v|.$$

This implies that $a_{(x,v)} \in l^\infty(\Gamma)$. Hence we can set

$$\lambda'_x(v) := m(a_{(x,v)}).$$

For $\psi \in \Gamma$,

$$\psi^*\lambda'_x(v) = \lambda'_{\psi(x)}(d\psi(v)) = m(b_{(x,v)}),$$

where

$$b_{(x,v)}(\varphi) := \lambda_{\varphi(\psi(x))}(d\varphi d\psi(v)) = a_{(x,v)}(\varphi\psi) = (\psi_* a_{(x,v)})(\varphi).$$

Then we get

$$m(b_{(x,v)}) = m(\psi_* a_{(x,v)}) = m(a_{(x,v)}) = \lambda'_x(v)$$

which implies

$$\psi^*\lambda'_x(v) = \lambda'_x(v).$$

This means that λ' is Γ-invariant so it descends to Σ. In order to verify the contact property of λ', choose $X \in \ker\omega$ with norm 1. Let $\widetilde{X}$ denote a lift of X such that $\lambda(\widetilde{X}) \geq C > 0$. Then

$$\begin{aligned} \lambda'_x\big(\widetilde{X}(x)\big) &= m\big(\varphi \mapsto \lambda_{\varphi(x)}[d_x\varphi(\widetilde{X}(x))]\big) \\ &= m\big(\varphi \mapsto \lambda_{\varphi(x)}[\widetilde{X}(\varphi(x))]\big) \\ &\geq m(\varphi \mapsto C) \\ &= C. \end{aligned} \tag{2.1}$$

The linearity and continuity of m implies that λ' is a smooth 1-form and

$$\int_\gamma \lambda' = m\left(\varphi \mapsto \int_\gamma \varphi^*\lambda\right),$$

for any closed curve γ. Now we show that $\int_\gamma \varphi^*\lambda$ is independent of φ. Since $\widetilde{M}$ is simply connected, there exists a smooth map $F : \mathbb{D} \to \widetilde{M}$, where $\mathbb{D}$ is a 2-disk and the restriction of F to $\partial\mathbb{D}$ is γ. Hence

$$\int_\gamma \varphi^*\lambda = \int_{\partial\mathbb{D}} F^*\varphi^*\lambda = \int_{\mathbb{D}} F^*\varphi^* d\lambda = \int_{\mathbb{D}} F^*\varphi^*\pi^*\omega.$$

Since $\pi^*\omega$ is Γ-invariant, the last integral is independent of φ. Thus

$$\int_\gamma \lambda' = m\left(\varphi \mapsto \int_\gamma \varphi^*\lambda\right) = m\left(\varphi \mapsto \int_\gamma \lambda\right) = \int_\gamma \lambda.$$

This implies that $\lambda' - \lambda$ is an exact form. Finally we conclude that $d\lambda' = \pi^*\omega$. $\square$

Remark 2.8. If $\overline{M}$ is a covering of $\widehat{M}$, then $c(\overline{H}) \le c(\widehat{H})$. A very similar argument shows that if the deck transformation group of this covering is amenable, then $c(\overline{H}) = c(\widehat{H})$.

Remark 2.9. By using a similar argument we conclude the following. If $0 \neq [\sigma] \in \mathrm{H}^2(M, \mathbb{R})$, $\sigma|_{\pi_2(M)} = 0$, and $\pi_1(M)$ is amenable, then

$$c = \inf_{\substack{\theta \\ d\theta = \widetilde{\sigma}}} \sup_{q \in \widetilde{M}} \frac{1}{2}|\theta_q|^2 = \infty.$$

Lemma 2.10. If $k > c$, then Σ_k is virtually contact.

Proof. For simplicity, let us assume that $U \equiv 0$. If $k > c$, we may choose $\epsilon > 0$ and a primitive θ of $\pi^*\sigma$ such that

$$\epsilon + |\theta_q| \le \sqrt{2k}$$

for all $q \in \widetilde{M}$. Let $\lambda = pdq$ be the Liouville form on $\widetilde{M}$. Then we may write $\widetilde{\omega} = d(\lambda + \widetilde{\tau}^*\theta)$, where $\widetilde{\tau} : T^*\widetilde{M} \to \widetilde{M}$. Since $X_{\widetilde{H}} = \left(\frac{\partial \widetilde{H}}{\partial p}, *\right)$, on $\widetilde{\Sigma}_k$ we have

$$\begin{aligned}
(\lambda + \widetilde{\tau}^*\theta)(X_{\widetilde{H}}) &= |p|^2 + \theta_q(\widetilde{H}_p) \\
&= 2k + \theta_q(\widetilde{H}_p) \\
&\ge 2k - |\theta_q|\sqrt{2k} \\
&\ge \sqrt{2k}(\sqrt{2k} - |\theta_q|) \\
&\ge \epsilon^2.
\end{aligned} \tag{2.2}$$

Note also that $|\lambda + \widetilde{\tau}^*\theta|_{C^0} < \infty$ on Σ_k. $\square$

3 The exact case

Now suppose that σ is exact on M. Let us define the *strict critical value*

$$c_0 = \inf_{\substack{\theta \\ d\theta=\sigma}} \max_{q\in M} \widehat{H}(q,\theta_q).$$

We note that $c_0 = c(\widehat{L}) = c(\widehat{H})$, where $\widehat{L}, \widehat{H}$ are the lifts of L, H to the abelian cover $\widehat{M}$. A covering $\widehat{M}$ is called an abelian cover if $\pi_1(\widehat{M})$ is the kernel of the Hurewicz homomorphism $\pi_1(M) \to \mathrm{H}_1(M,\mathbb{Z})$.

Lemma 3.1. If $k > c_0$, then Σ_k is of (restricted) contact type.

PROOF. Same as the proof of Lemma 2.10. □

Theorem 3.2. *The hypersurface Σ_{c_0} is not of contact type if $M \neq \mathbb{T}^2$. For $M = \mathbb{T}^2$, Σ_{c_0} is not of restricted contact type, but there are example on $\mathbb{T}^2$ for which Σ_{c_0} is of contact type.*

Theorem 3.3. *If $M \neq \mathbb{T}^2$ and $c < k \leq c_0$, then Σ_k is never of contact type. So $(c, c_0]$ is virtually contact but not contact.*

Question 3.4. For $e_0 < k \leq c_0$, is it true that Σ_k is never contact, where $e_0 = \inf\{k : \tau(\Sigma_k) = M\}$? This is open even if we assume that Σ_k is displaceable.

Definition 3.5. *Mather's α-function* is the convex superlinear function $\alpha : \mathrm{H}^1(M,\mathbb{R}) \to \mathbb{R}$ defined by

$$\alpha([\varpi]) = c(L - \varpi),$$

where $L(x,v) = \frac{1}{2}|v|^2 + \theta_x(v)$.

Clearly $c_0 = \inf_{[\varpi]\in \mathrm{H}^1(M,\mathbb{R})} \alpha([\varpi])$.

Theorem 3.6 (Mather, Mañé)**.** *Mather's α-function satisfies*

$$\alpha([\varpi]) = -\min_{\mu\in\mathcal{M}(L)} \int_{TM} (L-\varpi)d\mu,$$

where $\mathcal{M}(L)$ is the set of Borel probability measures on TM invariant under the Euler-Lagrange flow of L.

PROOF. It suffices to check that

$$c(L) = -\min_{\mu\in\mathcal{M}(L)} \int_{TM} L\ d\mu.$$

Let $\mu \in \mathcal{M}(L)$ be ergodic. Let $(q,v) \in TM$ be such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T L(f_t(q,v))dt = \int_{TM} L\ d\mu,$$

where $f_t(\gamma(t),\dot\gamma(t))$ is the *Lagrangian flow* on TM which is the solution of the Euler-Lagrange equation with $\gamma(0) = q$ and $\dot\gamma(0) = v$. Let $B > 0$ be such that

$$|L(q,v)| < B \qquad \text{if } |v| \leq 2.$$

For $N > 0$ let $q_N := \pi(f_N(q, v))$ and let $\gamma_N : [0, d(q, q_N)] \to M$ be a geodesic joining q_N to q. Let $x_N : [0, N] \to M$ be defined by $x_N(t) = \pi(f_t(q, v))$. Then $f_t(q, v) = (x_N(t), \dot{x}_N(t))$. For $k \in \mathbb{R}$, we have that

$$A_{L+k}(\gamma_N) = \int_0^{d(q,q_N)} [L(\gamma_N(t), \dot{\gamma}_N(t)) + k]dt \leq (B + k)\mathrm{diam}(M),$$

$$\lim_{N\to\infty} \frac{1}{N} A_{L+k}(x_N * \gamma_N) = \lim_{N\to\infty} \frac{1}{N} A_{L+k}(x_N) + 0 = A_{L+k}(\mu) = A_L(\mu) + k.$$

If $k < -A_L(\mu)$, then

$$\Phi_k(q, q) \leq \lim_N A_{L+k}(x_N * \gamma_N) = -\infty.$$

Hence $k \leq c(L)$. Therefore

$$\begin{aligned} c(L) &\geq \sup\{-A_L(\mu) \,|\, \mu \in \mathcal{M}_{\mathrm{erg}}(L)\} \\ &\geq -\min\{A_L(\mu) \,|\, \mu \in \mathcal{M}(L)\}. \end{aligned} \tag{3.1}$$

Now let $k < c(L)$ and $q, q' \in M$. Then $\Phi_k(q, q') = -\infty$ and there exists a sequence of absolutely continuous curves $x_n : [0, T_n] \to M$ such that

$$\lim_{n\to\infty} A_{L+k}(x_n) = -\infty.$$

Since L is bounded from below, we have that

$$\lim_{n\to\infty} T_n = +\infty.$$

Let $y_n : [0, T_n] \to M$ be a minimizer of the action among the absolutely continuous curves q to q' with time T_n. Then $(y_n(t), \dot{y}_n(t))$ is a segment of f_t ; note that $|\dot{y}|$ is uniformly bounded in n. Let ν_n be the probability measure defined by

$$\begin{aligned} \int_{TM} h \, d\nu_n &= \frac{1}{T_n} \int_0^{T_n} h(y_n(t), \dot{y}_n(t))dt \\ &= \frac{1}{T_n} \int_0^{T_n} h(f_t(y_n(0), \dot{y}_n(0)))dt, \end{aligned} \tag{3.2}$$

for all $h : TM \to \mathbb{R}$ continuous. There exists a subsequence ν_{n_i} which converges weakly to a probability measure μ. Since $\lim_{n\to\infty} T_n = +\infty$, μ is invariant under the flow f_t. Since $|\dot{y}|$ is bounded we have that

$$\lim_{n_i} \frac{1}{T_{n_i}} A_{L+k}(y_{n_i}) = A_{L+k}(\mu) = A_L(\mu) + k.$$

Since $\lim_n A_{L+k}(y_n) = \Phi_k(q, q') = -\infty$ and $T_n > 0$ for all n, it follows that $A_L(\mu) + k \leq 0$. Thus for any $k < c(L)$ we have found an invariant measure μ such that $k \leq -A_L(\mu)$. Therefore

$$\begin{aligned} c(L) &\leq \sup\{-A_L(\mu) \,|\, \mu \in \mathcal{M}(L)\} \\ &\leq -\min\{A_L(\mu) \,|\, \mu \in \mathcal{M}(L)\}. \end{aligned} \tag{3.3}$$

□

Definition 3.7. *Mather's β-function* $\beta : \mathrm{H}_1(M, R) \to \mathbb{R}$ is defined by

$$\begin{aligned}\beta(\gamma) &= \max_{[\varpi]\in \mathrm{H}^1(M,\mathbb{R})} \{\langle [\varpi], \gamma\rangle - \alpha([\varpi])\} \\ &= \inf_{\rho(\mu)=\gamma} \int_{TM} L \, d\mu,\end{aligned} \tag{3.4}$$

where $\rho(\mu) \in \mathrm{H}_1(M, \mathbb{R})$ is uniquely determined by the condition

$$\langle [\varpi], \rho(\mu)\rangle = \int_{TM} \varpi \, d\mu,$$

for all closed 1-forms on M. In the integral on the right-hand side ϖ is considered as a function $\varpi : TM \to \mathbb{R}$. A *minimizing measure* is a measure achieving the infimum of Mather's β-function, i.e.

$$\beta(\rho(\mu)) = \int_{TM} L \, d\mu.$$

Note that

$$\begin{aligned}\beta(0) &= \max_{[\varpi]\in \mathrm{H}^1(M,\mathbb{R})} \{0 - \alpha([\varpi])\} \\ &= -\min_{[\varpi]\in \mathrm{H}^1(M,\mathbb{R})} \alpha([\varpi]) \\ &= -c_0.\end{aligned} \tag{3.5}$$

Theorem 3.8 (Dias Carneiro [5])**.** *If μ is a minimizing measure, then*

$$\mathrm{supp}(\mu) \subset E^{-1}(c_0).$$

Lemma 3.9. Let $\Theta = \mathscr{L}^*(\Theta_{\mathrm{can}})$, where $\mathscr{L} : TM \to T^*M$ is the *Legendre transform* and X_E is the Euler-Lagrange vector field, then

$$(L + k)|_{E^{-1}(k)} = \Theta(X_E)|_{E^{-1}(k)}.$$

Proof. Since the projection of $X_E(x, v)$ to M is v, we get

$$\begin{aligned}\Theta_{\mathrm{can}}(\mathscr{L}_*(X_E(x, v))) &= \mathscr{L}(x, v)(d\pi_{T^*M}(\mathscr{L}_*(X_E(x, v)))) \\ &= \frac{\partial L}{\partial v}(x, v)v \\ &= L(x, v) + E(x, v) \\ &= L(x, v) + k,\end{aligned} \tag{3.6}$$

on $E^{-1}(k)$. □

Proof of Theorem 3.2. Suppose that Σ_{c_0} is contact i.e. there exists a 1-form α on $E^{-1}(c_0)$ such that $d\Theta = d\alpha$ and $\alpha(X_E) > 0$. Then $\Theta = \alpha + \varphi$, where φ is a closed 1-form on $E^{-1}(c_0)$. Note that each fiber of $\tau : E^{-1}(c_0) \to M^n$ is S^{n-1}. If $M \neq \mathbb{T}^2$, then the Euler class of the sphere bundle $\tau : E^{-1}(c_0) \to M$ is non-zero, and thus looking at the Gysin sequence of the τ one sees that the map

$$\tau^* : \mathrm{H}^1(M, \mathbb{R}) \to \mathrm{H}^1(E^{-1}(c_0), \mathbb{R})$$

is an isomorphism. Now we set

$$\varphi = \tau^*\varpi + dF,$$

where $\varpi \in \Omega^1(M)$, $F : E^{-1}(c_0) \to \mathbb{R}$. By applying X_E to both sides we get

$$\Theta(X_E) = \alpha(X_E) + \tau^*\varpi(X_E) + dF(X_E).$$

Let μ be a minimizing measure, then

$$\begin{aligned}\int_{TM} \Theta(X_E)d\mu &= \int_{TM} \alpha(X_E)d\mu + \int_{TM} \tau^*\varpi(X_E)d\mu + \int_{TM} dF(X_E)d\mu \\ &> 0 + \int_{TM} \varpi d\mu + \int_{TM} dF(X_E)d\mu \\ &= \underbrace{\rho(\mu)([\varpi])}_{=0} + \underbrace{\int_{TM} dF(X_E)d\mu}_{=0}.\end{aligned} \tag{3.7}$$

This is a contradiction, since by Lemma 3.9 one has

$$\int_{TM} \Theta(X_E)d\mu = \int_{TM} (L + c_0)d\mu = 0,$$

The latter equality holding as μ is assumed to be minimizing. □

Theorem 3.10 (McDuff-Sullivan criterion for contact type)**.** *Let $(\Sigma, \omega = d\alpha)$ be an exact Hamiltonian structure. Then Σ is contact if and only if $\int_{TM} \alpha(X)d\mu \neq 0$ for every invariant measure μ with zero homology and for every non-zero vector field $X \in \ker \omega$.*

Example 3.11. Consider $\mathbb{T}^2$ with its standard flat metric. Fix a vector field Z on $\mathbb{T}^2$ with a closed contractible orbit γ such that $|\dot{\gamma}| = 1$. Let $\psi : \mathbb{T}^2 \to \mathbb{R}$ be a C^∞-function such that $\psi \geq 0$ and $\psi(x) = 0$ if and only if $x \in \gamma$. Let us define

$$\theta_x(v) := \langle v, Z(x)\rangle, \qquad \varphi(x) := |Z(x)|^2 + 2\psi(x) > 0,$$

and

$$L(x, v) = \frac{1}{2}\varphi(x)|v|^2 - \theta_x(v).$$

Then we get

$$L + \frac{1}{2} = \frac{1}{2}\varphi(x)\left|v - \frac{Z(x)}{\varphi(x)}\right|^2 + \frac{\psi(x)}{\varphi(x)} \geq 0$$

and $L + \frac{1}{2} = 0$ if and only if $x \in \gamma$ and $v \in Z(x)$. Hence we conclude that γ is an orbit of the Euler-Lagrange flow and $c(L) = c_0(L) = \frac{1}{2}$.

We claim that the $E^{-1}(\frac{1}{2})$ is of contact type. By virtue of Theorem 3.10 and Lemma 3.9, it suffices to show that for any probability measure μ supported in $E^{-1}(\frac{1}{2})$ and with homology $S(\mu) = 0$ in $E^{-1}(\frac{1}{2})$, we have

$$\int_{TM} (L + \frac{1}{2})d\mu > 0.$$

Here $S(\mu) \in \mathrm{H}_1(E^{-1}(\frac{1}{2}), \mathbb{R})$ is given by

$$\langle[\varphi], S(\mu)\rangle = \int_{TM} \varphi(X)d\mu$$

for any $[\varphi] \in \mathrm{H}^1(E^{-1}(\frac{1}{2}), \mathbb{R})$, where X is the Euler-Lagrange field restricted to $E^{-1}(\frac{1}{2})$. Suppose there exists such a μ for which

$$\int_{TM} (L + \frac{1}{2}) d\mu = 0.$$

Then μ has to be supported on $t \mapsto (\gamma(t), \dot{\gamma}(t))$. But the curve $t \mapsto (\gamma(t), \dot{\gamma}(t))$ is not homologous to zero in $E^{-1}(\frac{1}{2})$. Therefore there is no μ with $S(\mu) = 0$ for which $\int_{TM}(L + \frac{1}{2})d\mu = 0$, so the energy level $E^{-1}(\frac{1}{2})$ is of contact type.

Question 3.12. What is the Rabinowitz Floer homology RFH $\left(E^{-1}(\frac{1}{2}), T\mathbb{T}^2\right)$ and the symplectic homology SH $\left(E^{-1}(\frac{1}{2}), T\mathbb{T}^2\right)$? Are they nonzero?

Example 3.13. Let (M, g) be a Riemannian manifold and fix a vector field X on M. The associated *Mañé Lagrangian* $L : TM \to \mathbb{R}$ is defined by

$$L(x, v) = \frac{1}{2}|v - X(x)|_x^2 \geq 0.$$

Note that $L(x, v) = 0$ if and only if $v = X(x)$.
Thus $c(L) = 0$ and $\int_{TM} L \; d\mu = 0$ for minimizing measure μ. This implies that $\tau_* \mu$ is X-invariant measure. Moreover, if X has invariant measure with zero homology, then $c(L) = c_0(L) = 0$.

Definition 3.14. A *cross section* for a vector field X is a closed codimension 1 submanifold which at every point is transversal to X and cuts every orbit of X.

Theorem 3.15 (Schwartzman [10])**.** *A vector field X has a measure with zero homology if and only if X admits no* cross section*.*

Definition 3.16. A Hamiltonian structure (Σ, ω) is said to be *stable* if there exists a 1-form λ on Σ such that

1. $\lambda(v) \neq 0$, for all $0 \neq v \in \ker \omega$;
2. $\ker \omega \subset \ker(d\lambda)$.

Note that the contact property implies the virtually contact and the stable property.

Theorem 3.17 (Wadsley [12])**.** *A Hamiltonian structure (Σ, ω) is stable if and only if its characteristic foliation is* geodesible*, i.e. there exists a Riemannian metric such that all leaves are geodesics.*

Theorem 3.18. *Assume X does not vanish anywhere and that X admits no cross section. Let L denote the associated Mañé Lagrangian. Then if X is not geodesible then the energy level Σ_{c_0} is not stable.*

Proof. Since X is nowhere vanishing and admits no cross section, $c_0(L) = 0$ and Σ_{c_0} is a regular hypersurface. Note that the dynamics of X sit inside the dynamics of the hypersurface. If Σ_{c_0} is stable then by Theorem 3.17 Σ_{c_0} is geodesible, and hence so is X. This is a contradiction.

□

Example 3.19. (Gap $[c, c_0]$) Let M be a closed manifold and Ω be a non-zero closed 2-form with bounded primitive in the universal covering. Consider a S^1-fiber bundle $p : P \to M$ and a 1-form ψ such that $p^*\Omega = -d\psi$. Choose a metric h on M and define a metric g_ϵ on P

$$g_\epsilon(u, v) = \frac{1}{\epsilon} h(dp(u), dp(v)) + \psi(u)\psi(v),$$

and a Lagrangian $L : TP \to \mathbb{R}$

$$L(x, v) = \frac{1}{2}|v|_\epsilon^2 - \psi_x(v).$$

Claim : *$c_0 = \frac{1}{2}$ and $c \to 0$ as $\epsilon \to 0$.*
By the Gysin sequence, we get an isomorphism

$$p^* : \mathrm{H}^1(M, \mathbb{R}) \to \mathrm{H}^1(P, \mathbb{R}).$$

Let V be the vector field dual to ψ using g_ϵ. The orbits of V are circles of length 2π homologous to zero. Let $\gamma : [0, 2\pi] \to P$ be an integral curve of V. Then

$$\begin{aligned} A_{L+k}(\gamma) &= \frac{1}{2}\int_0^{2\pi} 1 - \int_\gamma \psi + 2\pi k \\ &= 2\pi k + \pi - 2\pi \\ &= 2\pi\left(k - \frac{1}{2}\right). \end{aligned} \tag{3.8}$$

Since $\frac{1}{2}|\psi|_\epsilon^2 = \frac{1}{2}$, the strict Mañé critical value c_0 equals to $\frac{1}{2}$. Note that γ is homologous to zero, but not homotopic to zero. Now consider the universal covering $\pi : \widetilde{M} \to M$, and the following commutative diagram

$$\begin{tikzcd} \pi^* P \arrow[r, "\widehat{\pi}"] \arrow[d, "\widehat{p}"'] & P \arrow[d, "p"] \\ \widetilde{M} \arrow[r, "\pi"] & M \end{tikzcd}$$

Let θ denote a primitive of $\pi^*\Omega$ with $\|\theta\|_{C^0}$. Then

$$d(\widehat{p}^*\theta) = \widehat{p}^* d\theta = \widehat{p}^*\pi^*\Omega = \widehat{\pi}^* p^*\Omega = -\widehat{\pi}^* d\psi.$$

By definition of c,

$$c \leq \sup_{x \in \pi^* P} \frac{1}{2}|(\widehat{p}^*\theta)_x|_\epsilon^2 = \sup_{x \in \widetilde{M}} \frac{\epsilon}{2}|\theta_x|^2 = \frac{\epsilon}{2}|\theta|_{C^0}^2 \to 0$$

as $\epsilon \to 0$.

Example 3.20. Let us consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ endowed with the round metric g. We use spherical coordinates (r, ϕ) where r is the distance from the North pole and ϕ measures the rotation degree with respect to the z-axis. Let θ be a 1-form dual to the Killing vector field. Now consider the following Lagrangian with real parameter λ

$$L_\lambda = \frac{1}{2}(\dot{r}^2 + \sin^2 r \dot{\phi}^2) - \lambda \sin^2 r \ \dot{\phi}.$$

Since we work with speed 1, i.e. energy $\frac{1}{2}$, we conclude

$$\frac{1}{2}|\theta|^2 = \frac{\sin^2 r}{2} \leq \frac{1}{2}$$

which implies that $c = c_0 \leq \frac{1}{2}$. Now recall the Euler-Lagrange equations for L: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$. In our case we have

$$\begin{cases} \partial_{\dot{r}} L_\lambda = \dot{r} \\ \partial_{\dot{\phi}} L_\lambda = \sin^2 r \dot{\phi} - \lambda \sin^2 r \\ \partial_r L_\lambda = \sin r \cos r \dot{\phi}^2 - 2\lambda \sin r \cos r \dot{\phi} \\ \partial_\phi L_\lambda = 0. \end{cases} \tag{3.9}$$

Hence we obtain for a constant C

$$\begin{cases} \sin^2 r(\dot{\phi} - \lambda) = C \\ \ddot{r} = \sin r \cos r \dot{\phi}(\dot{\phi} - 2\lambda). \end{cases} \tag{3.10}$$

For simplicity we look at parallels only, i.e. curves γ of the form $\gamma(t) = (r_0, \dot{\phi}_0 t)$ for some $r_0, \dot{\phi}_0$. A parallel is an orbit when

$$r_0 = \frac{\pi}{2}, \qquad \dot{\phi}_0 = \pm 1.$$

This gives the two orbits $\gamma^\pm$.

We compute

$$A_{L_\lambda + \frac{1}{2}}(\gamma^+) = 2\pi(1-\lambda), \quad A_{L_\lambda + \frac{1}{2}}(\gamma^-) = 2\pi(1+\lambda),$$

and the sign changes at $\lambda = 1$. This implies that $c(L_1) = c_0(L_1) = \frac{1}{2}$. The only Mather minimizing measure is carried by γ^+. For L_1, Σ_k is contact for $k > \frac{1}{2}$ and not contact for $k \leq \frac{1}{2}$.

Question 3.21. For $\lambda = 1$, is $\Sigma_{1/2}$ stable?

Question 3.22. Can Σ_c be virtually contact for a surface of higher genus?

4 Holonomic measures

Let C_l^0 be the set of continuous functions which have at most linear growth, i.e.,

$$C_l^0 = \left\{ f : TM \to \mathbb{R} \;\middle|\; \sup_{(x,v)\in TM} \frac{f(x,v)}{1+|v|} < +\infty \right\},$$

and let $\mathcal{M}_l$ denote the set of probability measures on the Borel σ-algebra of TM such that $\int_{TM} |v| d\mu < \infty$, equipped with the topology given by:

$$\lim_n \mu_n = \mu \iff \int_{TM} f d\mu_n = \int_{TM} f d\mu \quad \forall f \in C_l^0.$$

Note that the above topology on $\mathcal{M}_l$ is metrizable and is called *the weak-$\star$ topology*. We will consider a certain subset $\overline{\mathcal{C}} \subseteq \mathcal{M}_l$ of probability measures, defined as follows. If $\gamma : [0,T] \to M$ is a closed absolutely continuous curve, let μ_γ be such that

$$\int_{TM} f(x,v) d\mu_\gamma = \frac{1}{T}\int_0^T f(\gamma(t), \dot{\gamma}(t)) dt \quad \forall f \in C_l^0.$$

Since γ is absolutely continuous, $\int_0^T |\dot{\gamma}(t)| dt < +\infty$. This implies that $\mu_\gamma \in \mathcal{M}_l$.

Definition 4.1. We set

$$\mathcal{C} = \{\mu_\gamma \in \mathcal{M}_l \mid \gamma \text{ is an absolutely continuous curve}\}.$$

We call the closure $\overline{\mathcal{C}} \subseteq \mathcal{M}_l$ of $\mathcal{C}$, the set of *holonomic measures* on M.

Remark 4.2.

1. $\overline{\mathcal{C}}$ is convex.
2. Let $\mathcal{M}(L)$ be the set of all probability measures which are invariant with respect to the Euler-Lagrange flow of L and have compact support. Then $\mathcal{M}(L) \subset \overline{\mathcal{C}}$.

Let $\mathcal{C}_0 := \{\mu_\gamma : \gamma \text{ is contractible}\}$. Note that $\mathcal{C}_0 \subset \mathcal{C} \subset \mathcal{M}_l$. We call the closure $\mathcal{H} := \overline{\mathcal{C}_0}$ of $\mathcal{C}_0$, the set of *holonomic measures with zero homology.* Note that

$$c = -\inf_{\mu_\gamma \in \mathcal{C}_0} A_L(\mu_\gamma).$$

It is known that for a given $\mu \in \mathcal{H}$ there exists $\mu_{\gamma_n} \in \mathcal{C}_0$ such that $\mu_{\gamma_n} \to \mu$ and such that $\lim \int_{TM} L d\mu_{\gamma_n} = \int_{TM} L d\mu$. This is not obvious since L is *not* of linear growth. Thus we get

$$c = -\inf_{\mu_\gamma \in \mathcal{H}} A_L(\mu_\gamma).$$

Proposition 4.3. *Suppose $\mu \in \mathcal{H}$ is such that $c = -A_L(\mu)$. Then*

$$\int_{TM} E d\mu = c,$$

where $E(x,v) = L_v(x,v) \cdot v - L(x,v)$.

Proof. Consider the family $\{\mu_\lambda\} \subseteq \mathcal{H}$ of measures defined by

$$\int_{TM} f(x,v) d\mu_\lambda = \int_{TM} f(x, \lambda v) d\mu \qquad \forall f \in C_l^0.$$

We further define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(\lambda) = \int_{TM} L(x, \lambda_v) d\mu.$$

Since $\mu \in \mathcal{H}$ is a minimizing measure, we have $\phi'(1) = 0$. By definition of ϕ,

$$\phi'(1) = \int_{TM} \frac{\partial L}{\partial v}(x,v)[v] d\mu = \int_{TM} (E + L) d\mu = \int_{TM} E d\mu - c$$

which implies that $\int_{TM} E d\mu = c$. $\square$

Theorem 4.4. *Let M be a closed manifold with* $\dim M \geq 3$ *and $L : TM \to \mathbb{R}$ be a* Tonelli Lagrangian. *Then $E^{-1}(c)$ is not virtually contact.*

Proof. Assume that $E^{-1}(c)$ is virtually contact and c is a regular value of E. Recall that

$$(L + k)|_{E^{-1}(k)} = \Theta(X_E)|_{E^{-1}(k)},$$

where $\Theta = \mathscr{L}^* \lambda$, $\mathscr{L} : TM \to T^*M$ is the Legendre transform and $\lambda = pdq$. Let $\pi : \widetilde{M} \to M$ be the universal covering. Since $\dim M \geq 3$, $\widetilde{E}^{-1}(c) \to \widetilde{M}$ is a sphere fibration over a simply connected manifold with simply connected fibers. Thus $\widetilde{E}^{-1}(c)$ is simply connected and hence $\widetilde{E^{-1}(c)} = \widetilde{E}^{-1}(c)$. Then we get

$$\begin{tikzcd} \widetilde{E}^{-1}(c) \arrow[r, hook] \arrow[d] & T\widetilde{M} \arrow[d] \\ E^{-1}(c) \arrow[r, hook] & TM, \end{tikzcd} \qquad \begin{tikzcd} S^{n-1} \arrow[r] & E^{-1}(c) \arrow[d] \\ & M. \end{tikzcd}$$

By definition of virtually contact, there exists a smooth 1-form α on $\widetilde{E}^{-1}(c)$ such that $d\alpha = d\Theta$ and such that α satisfies

1. $|\alpha|_{C^0} < \infty$;
2. $\alpha(X_{\widetilde{E}}) \geq \epsilon$ for some $\epsilon > 0$.

Since $\pi_1(\widetilde{E}^{-1}(0)) = 0$, there exists a smooth function $f : \widetilde{E}^{-1}(0) \to \mathbb{R}$ such that $\alpha = \Theta + df$ on $\widetilde{E}^{-1}(0)$. Consider a tubular neighborhood $\rho : E^{-1}(c-\delta, c+\delta) \to E^{-1}(c)$ where $\delta > 0$ is chosen sufficiently small so that, if $\widetilde{\rho} : \widetilde{E}^{-1}(c-\delta, c+\delta) \to \widetilde{E}^{-1}(c)$ denotes the lift of ρ to $\widetilde{E}^{-1}(c)$ then :

1. $|\Theta(X_{\widetilde{E}})(x,v) - \Theta(X_{\widetilde{E}})(\widetilde{\rho}(x,v))| < \frac{\epsilon}{4} \qquad \forall\ (x,v) \in \widetilde{E}^{-1}(c-\delta, c+\delta)$;
2. $|df_{\widetilde{\rho}(x,v)}[d\widetilde{\rho}(X_{\widetilde{E}}(x,v))] - df_{\widetilde{\rho}(x,v)}[X_{\widetilde{E}}(\rho(x,v))]| < \frac{\epsilon}{4} \qquad \forall\ (x,v) \in \widetilde{E}^{-1}(c-\delta, c+\delta)$.

By using this condition, we get the following estimate

$$\begin{aligned} (\Theta + df \circ d\widetilde{\rho})(X_{\widetilde{E}})(x,v) \geq & \Theta(X_{\widetilde{E}})(\widetilde{\rho}(x,v)) - |\Theta(X_{\widetilde{E}})(x,v) - \Theta(X_{\widetilde{E}})(\widetilde{\rho}(x,v))| \\ & + df \circ d\widetilde{\rho}(X_{\widetilde{E}})(x,v) \\ = & (\alpha - df)(X_{\widetilde{E}})(\widetilde{\rho}(x,v)) - |\Theta(X_{\widetilde{E}})(x,v) - \Theta(X_{\widetilde{E}})(\widetilde{\rho}(x,v))| \\ & + df \circ d\widetilde{\rho}(X_{\widetilde{E}})(x,v) \\ \geq & \alpha(X_{\widetilde{E}})(\widetilde{\rho}(x,v)) - |\Theta(X_{\widetilde{E}})(x,v) - \Theta(X_{\widetilde{E}})(\widetilde{\rho}(x,v))| \\ & - |df \circ d\widetilde{\rho}(X_{\widetilde{E}})(x,v) - df(X_{\widetilde{E}})(\widetilde{\rho}(x,v))| \\ \geq & \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{4} = \frac{\epsilon}{2}, \end{aligned} \tag{4.1}$$

for all $(x,v) \in \widetilde{E}^{-1}(c-\delta, c+\delta)$.

Since $c = -\inf_{\gamma \in \mathcal{C}_0} A_L(\mu_\gamma)$, we can choose a sequence (γ_n, T_n) such that

$$A_L(\mu_{\gamma_n}) \to -c, \qquad 0 \leq \frac{1}{T_n} A_{L+c}(\gamma_n) \to 0 \qquad \text{as } n \to \infty,$$

where γ_n are closed contractible orbits with energy k_n.

Claim : $k_n \to c$.

First we show that the k_n are uniformly bounded. Since L is super-linear, there exists $D \in \mathbb{R}$ such that

$$L(x,v) \geq |v| + D, \qquad \forall\ (x,v) \in TM,$$

which implies

$$\frac{1}{T_n} A_{L+c}(\gamma_n) \geq \frac{1}{T_n} \int_0^{T_n} |\dot{\gamma}_n| dt + D + c.$$

The mean-value theorem in $[0, T_n]$ tells us that there exist $t_0^n \in [0, T_n]$ such that

$$|\dot{\gamma}_n(t_0^n)| + D + c \leq \frac{1}{T_n} A_{L+c}(\gamma_n) \to 0.$$

Thus we conclude that $k_n = E(\gamma_n(t_0^n), \dot{\gamma}_n(t_0^n))$ is uniformly bounded, and they converge $k_n \to k_\infty$. Let μ denote a weak limit of the μ_{γ_n}. Then the following holds:

1. $\mu \in \mathcal{H}$, where $\mathcal{H}$ is the set of holonomic measures with zero homology;
2. $\operatorname{supp}(\mu) \subset E^{-1}(k_\infty)$;
3. $A_L(\mu) = -c$.

By Proposition 4.3, we conclude that $c = k_\infty$. For n large enough, the closed orbits $\widetilde{\Gamma}_n = (\widetilde{\gamma}_n(t), \dot{\widetilde{\gamma}}_n(t))$ in $T\widetilde{M}$ are actually contained in $\widetilde{E}^{-1}(c - \delta, c + \delta)$. Then we get

$$\begin{aligned} \frac{1}{T_n} A_{L+c}(\gamma_n) &= \frac{1}{T_n} A_{L+k_n}(\gamma_n) + (c - k_n) \\ &= \frac{1}{T_n} \int_{\widetilde{\Gamma}_n} \Theta + (c - k_n) \\ &\geq \frac{\epsilon}{2} + (c - k_n) \to \frac{\epsilon}{2}, \end{aligned} \tag{4.2}$$

which contradicts the fact that

$$\frac{1}{T_n} A_{L+c}(\gamma_n) \to 0 \qquad \text{as } n \to \infty.$$

□

5 Helicity

Let N be a closed oriented 3-manifold with volume form Ω. A volume preserving vector field F is said to be *exact* if the closed 2-form $\iota_F \Omega$ is exact. Given a volume preserving exact vector field F, the *helicity* $\mathcal{H}(F)$ is defined by

$$\mathcal{H}(F) = \int_N \tau \wedge d\tau = \int_N \tau(F)\Omega,$$

where τ is any primitive 1-form of $\iota_F \Omega$. One can check that the helicity is well-defined by using the Stokes' theorem. Note that $(N, \iota_F \Omega)$ is an exact Hamiltonian structure. If $\mathcal{H}(F) = 0$, then the Hamiltonian structure is not contact. This can be shown by applying Theorem 3.10.

Let M be a closed oriented surface of genus ≥ 2 with Riemannian metric g. The unit circle bundle SM determined by g is a closed 3-manifold with volume form $\Omega = \alpha \wedge d\alpha$, where α is the contact 1-form of the geodesic flow of g. Let X denote the geodesic vector field of g, V be the infinitesimal generator of the circle action on the fibers of SM, and $H = [X, V]$. Then $\{X, H, V\}$ forms a frame on SM. A basic result in 2-dimensional Riemannian geometry tells us that the coframe $\{\alpha, \gamma, \psi\}$ of $\{X, H, V\}$, satisfies *Cartan's structure equation*

$$\begin{cases} d\alpha = \psi \wedge \gamma \\ d\gamma = -\psi \wedge \alpha \\ d\psi = -K\alpha \wedge \gamma, \end{cases} \tag{5.1}$$

where $K : M \to \mathbb{R}$ is the Gaussian curvature. Given a 2-form σ on M, we may write $\sigma = f\sigma_g$, where σ_g is the area form of g and $f : M \to \mathbb{R}$. Let us consider the vector field $F = X + fV$. Then we obtain

$$\begin{aligned}\iota_F\Omega &= \iota_{X+fV}(\alpha \wedge d\alpha)\\ &= d\alpha + f\iota_v(\alpha \wedge d\alpha)\\ &= d\alpha + f(-\alpha \wedge \iota_V d\alpha)\\ &= d\alpha + f(-\alpha \wedge \gamma)\\ &= d\alpha - f\pi^*\sigma_g\\ &= d\alpha - \pi^*\sigma,\end{aligned} \tag{5.2}$$

where $\pi : SM \to M$ is the canonical foot-point projection. Since $\mathrm{H}^2(M, \mathbb{R}) = \mathbb{R}$, we can set $\sigma = -aK\sigma_g + d\beta$ on M to obtain

$$\begin{aligned}\pi^*\sigma &= -aK\pi^*\sigma_g + d\pi^*\beta\\ &= -aK\alpha \wedge \gamma + d\pi^*\beta\\ &= ad\psi + d\pi^*\beta\\ &= d(a\psi + \pi^*\beta).\end{aligned} \tag{5.3}$$

Hence we conclude that F is a volume preserving exact vector field satisfying

$$\iota_F\Omega = d\tau, \qquad \tau = \alpha - a\psi - \pi^*\beta,$$

$$\tau(F)(x, v) = 1 - af(x) - \beta_x(v).$$

The helicity is computed as follows

$$\begin{aligned}\mathcal{H}(F) &= \int_N \tau(F)\Omega\\ &= 2\pi A - 2\pi a[\sigma]\\ &= 2\pi A + \frac{[\sigma]^2}{\chi},\end{aligned} \tag{5.4}$$

where A is the area of M with respect to the metric g and $[\sigma] = -a\int_M K\sigma_g = -2\pi a\chi$. Let us consider the scaling $\sigma \mapsto s\sigma$, then by the argument above we obtain a primitive $\tau_s := \alpha - as\psi - s\pi^*\beta$ of $\iota_{F_s}\Omega$ such that

$$\tau_s(F_s)(x, v) = 1 - as^2 f(x) - s\beta_x(v).$$

There is a unique positive value of s for which $\mathcal{H}(F_s) = 0$ which is given by

$$s_h^2 := \frac{-2\pi\chi A}{[\sigma]^2}.$$

Theorem 5.1 (Paternain [9])**.** *For an arbitrary pair (g, σ) on a closed surface of genus ≥ 2 with $[\sigma] \neq 0$, we have $s_c \leq s_h$ with equality if and only if g has constant Gaussian curvature and σ is a constant multiple of the area form of g, where $s_c := 1/\sqrt{2c}$.*

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