# On the $(29,5)$-arcs in $P G(2,7)$ and linear codes 

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## Projective Geometry

Let $\mathbb{F}_{q}$ be the finite field of $q$ elements.
Let $\mathbb{F}_{q}^{3}$ be the 3-dimensional vector space over $\mathbb{F}_{q}$.
Let $P G(2, q)$ be the set of all lines through the origin of $\mathbb{F}_{q}^{3}$ over $\mathbb{F}_{q}$, that is,

$$
\mathbb{F}_{q}^{3} \backslash\{(0,0,0)\} / \sim
$$

where $(a, b, c) \sim(x, y, z) \Longleftrightarrow(a, b, c)=\lambda(x, y, z)$, for some $\lambda \neq 0$ and $\lambda \in \mathbb{F}_{q}$.
$P G(2, q)$ is called a projective plane over $\mathbb{F}_{q}$.
Let $\theta_{2}$ denote the number of all points in $\operatorname{PG}(2, q)$, i.e.,

$$
\theta_{2}:=\frac{q^{3}-1}{q-1}=q^{2}+q+1
$$

We note that the linear equation

$$
a x+b y+c z=0, \quad a, b, c \in \mathbb{F}_{q}, \quad(a, b, c) \neq(0,0,0)
$$

corresponds to a line $\ell$ in $P G(2, q)$, simply denoted $\ell=[a, b, c]$.

## Lemma

The following are known basic properties of $\operatorname{PG}(2, q)$;

- The number of points in $P G(2, q)=\theta_{2}=q^{2}+q+1$.
- The number of lines in $P G(2, q)=\theta_{2}=q^{2}+q+1$.
- There are $q+1$ points on any line in $P G(2, q)$.
- There are $q+1$ lines through a point in $P G(2, q)$.


## Arcs

Definition. An $(n, r)_{q}$-arc
An $(n, r)_{q}$-arc is a set $\mathcal{K}$ of $n$ points of $P G(2, q)$ such that some $r$ but no $r+1$ of them, are collinear, i.e., $|\mathcal{K} \cap \ell| \leq r$ for any line $\ell$ and $|\mathcal{K} \cap \ell|=r$ for some $\ell$ in $P G(2, q)$.

## Example

(1) Let $C$ be a conic in $P G(2, q)$. Then $|C|=q+1$ and for any line $\ell$, we have $|C \cap \ell|=0$ or 1 or 2 . Thus $C$ is a $(q+1,2)_{q}$-arc in

(2) Let $\mathcal{T}=P G(2, q) \backslash \ell_{0}$, where $\ell_{0}$ is a line in $P G(2, q)$. For any line $\ell$ in $P G(2, q)$, we have $|\mathcal{T} \cap \ell|=0$ or $q$.


Thus $\mathcal{T}$ is a $\left(q^{2}, q\right)_{q}$-arc in $P G(2, q)$.

For an $(n, r)_{q}$-arc $\mathcal{K}, i$-line or $i$-secant is a line meeting $\mathcal{K}$ in exactly $i$ points. Define $a_{i}$ as the number of $i$-lines to $\mathcal{K}$.

Note that $a_{i}=0$ for $i \geq r+1$.
The $(r+1)$-tuple $\left(a_{0}, a_{1}, \ldots, a_{r}\right)$ is called the spectrum of the arc $\mathcal{K}$.

Note that an arc can be considered as a blocking set.
Definition. A $t$-fold blocking set
A $t$-fold blocking set of size $m$ is the complement of
a $\left(\theta_{2}-m, q+1-t\right)_{q}$-arc in $P G(2, q)$.

## The value of $m_{r}(2, q)$

Let $m_{r}(2, q)$ denote the largest $n$ for which there exists an $(n, r)_{q}$-arc for given $r$ and $q$.

An interesting problem in the projective geometry is to determine the exact values of $m_{r}(2, q)$. Obviously, we have the bound for $m_{r}(2, q)$;

$$
m_{r}(2, q) \leq(r-1) q+r .
$$

We can easily see the following;
(1) For $r=1$, the value $m_{1}(2, q)=1$ and the arc is a point set.
(2) For $r=q$, the value $m_{q}(2, q)=q^{2}$ and the arc is the complement of a line $\ell_{0}$, i.e., $P G(2, q) \backslash \ell_{0}$.
(3) For $r=q+1$, the value $m_{q+1}(2, q)=q^{2}+q+1$ and the arc is the entire projective plane.

A few values of $m_{r}(2, q),(2 \leq r \leq q-1)$ are known in general $q$.

Theorem A. Bose (1947): On the values of $m_{2}(2, q)$
We have

$$
m_{2}(2, q)= \begin{cases}q+1, & q \text { odd } \\ q+2, & q \text { even }\end{cases}
$$

Theorem B. Barlotti (1965) and Ball(1996)
For $q$ odd prime, we have

$$
m_{r}(2, q)=(r-1) q+1 \quad \text { for } \quad r=\frac{q+1}{2} \quad \text { or } \quad r=\frac{q+3}{2} .
$$

Theorem C. Denniston (1969)
For $q$ even, we have

$$
m_{r}(2, q)=(r-1) q+r \quad \text { for } \quad r=2^{e} \leq q
$$

## The values of $m_{r}(2, q)$

The value of $m_{r}(2, q)$ is known for $3 \leq q \leq 9$ but it is still open for $q \geq 11$.

Values of $m_{r}(2, q)$ for $q \leq 13$.

| $r / q$ | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 13 | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 4 | 6 | 6 | 8 | 10 | 10 | 12 | 14 | $\cdots$ |
| 3 |  | 9 | 11 | 15 | 15 | 17 | 21 | 23 |  |
| 4 |  |  | 16 | 22 | 28 | 28 | 32 | $38-40$ |  |
| 5 |  |  |  | 29 | 33 | 37 | $43-45$ | $49-53$ |  |
| 6 |  |  |  | 36 | 42 | 48 | 56 | $64-66$ |  |
| 7 |  |  |  |  | 49 | 55 | 67 | 79 |  |
| 8 |  |  |  |  |  | 65 | 78 | 92 |  |
| 9 |  |  |  |  |  |  | $89-90$ | 105 |  |
| 10 |  |  |  |  |  |  | $100-102$ | $118-119$ |  |

## On $\left(m_{r}(2, q), r\right)_{q}$-arcs

Consider the classification of $\left(m_{r}(2, q), r\right)_{q}$-arcs.
Segre (1950's) : The largest arcs

- For $q$ odd, every $(q+1,2)_{q}$-arc is a conic.
- For $q$ even, $(q+2,2)_{q}$-arc is a conic and its nucleus with $q=2,4,8$.

For $q \geq 2^{n}(n \geq 4)$, there are non-equivalent $(q+2,2)_{q}$-arcs other than a conic and its nucleus.

## Conic

A conic in $P G(2, q)$ is a curve $C$ with homogeneous quadratic equation in 3 variables, for example $x^{2}=y z$ or $x^{2}+y^{2}=z^{2}$. Note that

$$
|C|=q+1
$$

For any line $\ell$, we have one of the following;

$$
|C \cap \ell|=0 \quad \text { or } 1 \quad \text { or } 2 .
$$

Here we call the line $\ell$ external or tangent or secant to the $C$.
A point $P$ in $P G(2, q)$ is called external or internal point to the $C$ if $P$ lies on two or no tangent lines of $C$.
Let $\mathcal{I}(C)$ denote the set of all internal points of $C$ and $\mathcal{E}(C)$ the set of all external points of $C$. Then we have

$$
\begin{aligned}
|\mathcal{I}(C)| & =\frac{q(q-1)}{2} \\
|\mathcal{E}(C)| & =\frac{q(q+1)}{2}
\end{aligned}
$$

## Construction of largest arcs in $\operatorname{PG}(2, q)$

There are some largest arcs with nice geometric descriptions which are given by Barlotti (1965);

Theorem 1. Barlotti Construction (1965)
For $q$ odd, let $C$ be a conic in $P G(2, q)$.
(1) The following set $\mathcal{K}$ is a $\left(\frac{q^{2}-q+2}{2}, \frac{q+1}{2}\right)_{q}$-arc;

$$
\mathcal{K}=\mathcal{I}(C) \cup\{P\}, \quad \text { where } \quad P \in C
$$

(2) The following set $\mathcal{K}$ is a $\left(\frac{q^{2}+q+2}{2}, \frac{q+3}{2}\right)_{q}$-arc;

$$
\mathcal{K}=\mathcal{I}(C) \cup C
$$

## The numbers of non-equivalent largest arcs

The following shows the number of non-equivalent largest arcs.

| $r / q$ | 3 |  | 4 |  | 5 |  | 7 |  | 8 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 4 | 1 | 6 | 1 | 6 | 1 | 8 | 1 | 10 | 1 |$)$

## Classification of the largest arcs in $P G(2,7)$

For $q=7$, we note that $m_{2}(2,7)=8, \quad m_{3}(2,7)=15$,

$$
m_{4}(2,7)=22, \quad m_{5}(2,7)=29, \quad m_{6}(2,7)=36
$$

Lemma. The classification of the largest arcs in $P G(2,7)$
(1) For $r=2$, the $(8,2)_{7}$-arc is unique and it is a conic.
(2) For $r=3$, the $(15,3)_{7}$-arc is unique and it is proved by Marcugini, Milani and Pambianco in 2004.

Classification of the $(n, 3)$-arcs in $P G(2,7)$.
(3) For $r=4$, there are three non-equivalent $(22,4)_{7}$-arcs which are classified by Hill and Love in 2003.

On the $(22,4)_{7}$-arcs in $P G(2,7)$ and related codes.

Now we consider when $r=5$, that is, $(29,5)_{7}$-arcs.

## The $(29,5)_{7}$-arcs

We give some constructions of the $(29,5)_{7}$-arcs by giving their geometrical descriptions.

## Case 1.

Let $\mathcal{K}$ be the union of tangent lines of a conic $C$ except $C$. Then $\mathcal{K}$ is a $(29,5)_{7}$-arc with the spectrum $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=$ ( $0,8,0,0,21,28$ ).

Case 1 is same as the arc consisting of a conic and its internal set (Barlotti construction)


## Case 2.

Let $C$ be a conic with the equation; $x^{2}=y z$.

$$
\begin{aligned}
\mathcal{K} & =\mathcal{I}(C) \cup(C \backslash\{(4,2,1),(2,4,1),(6,1,1)\}) \\
& \cup\{(1,0,0),(4,0,1),(4,1,0)\}
\end{aligned}
$$

Then $\mathcal{K}$ is a $(29,5)_{7}$-arc with the spectrum $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(3,2,0,6,18,28)$. Furthermore, the set $\{(4,2,1),(2,4,1),(6,1,1),(1,0,0),(4,0,1),(4,1,0)\} \quad$ is $\quad$ a complete $(6,2)_{7}$-arc in $P G(2,7)$.

## Case 3.

Let $C$ be a conic with the equation; $x^{2}=y z$.

$$
\begin{aligned}
\mathcal{K} & =C \cup(\mathcal{I}(C) \backslash\{(1,2,1),(0,1,1),(4,4,1)\}) \\
& \cup\{(1,0,0),(4,0,1),(4,1,0)\}
\end{aligned}
$$

Then $\mathcal{K}$ is a $(29,5)_{7}$-arc with the spectrum $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(0,5,3,9,6,34)$. Furthermore, the set $\{(1,2,1),(0,1,1),(4,4,1),(1,0,0),(4,0,1),(4,1,0)\} \quad$ is $\quad$ a complete $(6,2)_{7}$-arc in $P G(2,7)$.


Next four arcs are constructed with the following four lines which are not concurrent. Let $\ell_{1}=[0,0,1], \ell_{2}=[0,1,0], \ell_{3}=[1,1,1]$ and $\ell_{4}=[2,4,1]$.


Let $\mathcal{L}=\bigcup_{i=1}^{4} \ell_{i}$. Then we have $|\mathcal{L}|=26$.
Note that a $(29,5)_{7}$-arcs is a 3 -fold blocking set of size 28.

Case 4.
Let $B=\bigcup_{i=1}^{4} \ell_{i} \cup\{(5,3,1),(3,6,1)\}$ and $\mathcal{K}=P G(2,7)-B$. Then $\mathcal{K}$ is a $(29,5)_{7}$-arc with the spectrum $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=$ $(4,0,0,8,17,28)$.


## Case 5.

Let $B=\bigcup_{i=1}^{4} \ell_{i} \cup\{(5,3,1),(1,3,1)\}$ and $\mathcal{K}=P G(2,7)-B$. Then $\mathcal{K}$ is a $(29,5)_{7}$-arc with the spectrum $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=$ $(4,0,1,5,20,27)$.


## Case 6.

Let $B=\bigcup_{i=1}^{4} \ell_{i} \backslash\{(1,1,1)\} \cup\{(5,3,1),(5,2,1),(2,2,1)\}$ with $(1,1,1) \in \ell_{4}$ and $\mathcal{K}=P G(2,7)-B$. Then $\mathcal{K}$ is a $(29,5)_{7}$-arc with the spectrum $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(3,1,2,6,16,29)$.


Next we give seven other $(29,5)_{7}$-arcs in $P G(2,7)$.

## Seven other cases

There are $(29,5)_{7}$-arcs with the spectrum
(1) $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(3,1,3,3,19,28)$,
(2) $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(3,1,1,9,13,30)$,
(3) $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(3,0,3,9,11,31)$,
(4) $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(2,3,2,4,17,29)$,
(5) $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(2,2,4,4,15,30)$,
(6) $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(2,2,2,10,9,32)$,
(7) $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)=(2,1,5,7,10,32)$.

## Arcs and Linear Codes

A projective $[n, 3, d]_{q}$ linear code is equivalent to an $(n, n-d)_{q}$-arc in $P G(2, q)$. For an $[n, 3, d]_{q}$ linear code, we have the following;

$$
n \geq d+\left\lceil\frac{d}{q}\right\rceil+\left\lceil\frac{d}{q^{2}}\right\rceil . \quad(\text { Griesmer bound })
$$

A code meeting the Griesmer bound is a length-optimal code.
We have the following;
(1) There are at least 13 non-equivalent $(29,5)_{7}$-arcs in $P G(2,7)$.
(2) $(29,5)_{7}$-arcs are $[29,3,24]_{7}$ linear codes which meet the Griesmer bound.
(3) $\left(a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right)$ gives weight enumerator and there are at least thirteen $[29,3,24]_{7}$ length optimal codes with different weight enumerators.

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