Practical Basis for Collider Physics Phenomenology

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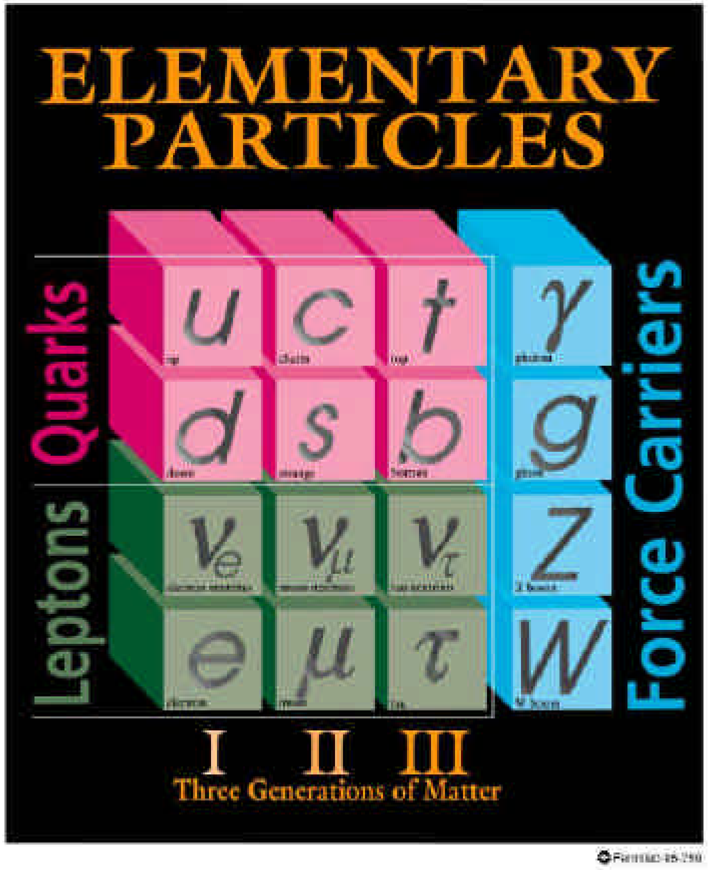
Content

Open KIAS Winter School on Collider Physics

Elementary Particles



Spin-1 gauge bosons : γ, gluons, $W^{\pm}$ and Z

Spin-1/2 matter fermions : quarks and leptons

Spin-0 elusive Higgs boson(s)

Fundamental Framework

Baselines

4-d flat Minkowski spacetime (with gravitation ignored)

$$c = 3 \times 10^8\,\mathrm{m/s} \equiv 1$$

$$\text{Metric}: \; g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

$$x^\mu = (t,\; \vec{r}), \quad x_\mu = g_{\mu\nu} x^\nu = (t, -\vec{r})$$

$$p^\mu = (E, \vec{p}), \quad p_\mu = g_{\mu\nu} p^\nu = (E, -\vec{p})$$

$$\text{Invariant interval}: \; dx^2 \equiv dx^\mu dx_\mu = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{r}^{\,2}$$

Description

(Renormalizable) relativistic quantum field theory

$$\hbar = 6.6 \times 10^{-25}\,\mathrm{GeV\,s} \equiv 1$$

$$[\mathrm{L}] = [\mathrm{T}], \quad [\mathrm{E}] = [\vec{p}] = [\mathrm{M}] = [\mathrm{L}]^{-1}$$

Key concepts

Symmetries $\Leftrightarrow$ Mass, Spin and Charges

Basic strategy

Perturbation theory

Poincaré Group

The 10-parameter Poincaré group is the fundamental spacetime symmetry group of 4-parameter spacetime translations and 6-parameter Lorentz transformations consisting of 3-parameter rotations and 3-parameter boosts.

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu \quad \Rightarrow \quad \Delta x'^2 = \Delta x^2 : \text{ invariant}$$

Every physical object that lives in the Minkowski space of the 4-dimensional spacetime must belong to some representations of the Poincaré group.

Covariant classical mechanics

A 4-vector velocity of a massive particle can be constructed by differentiation of the spacetime coordinate 4-vector x^μ with respect to the Lorentz invariant proper time τ:

$$u^\mu = \frac{dx^\mu}{d\tau} = (\gamma, \gamma\vec{v}) \qquad \text{with} \;\; d\tau = \sqrt{dt^2 - d\vec{r}^2} = dt\sqrt{1-\beta^2} = dt/\gamma$$

satisfying the relation $u^\mu u_\mu = 1$ with the particle speed $\vec{v} = d\vec{r}/dt = \vec{\beta}$ and the boost factor γ. Naturally, the 4-momentum of a free particle of mass m, transforming as a 4-vector, is defined to be

$$p^\mu = mu^\mu = (E, \vec{p}) = (\gamma m, \gamma m\vec{v}) \quad \Rightarrow \quad p^2 = p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2$$

Note that a massless particle with $m = 0$ always travels at the speed of light $|\beta| = 1$. Massless particles with different energies move at the same speed, and thus cannot be distinguished by their speed. Remarkably this puzzle is resolved in the framework of quantum mechanics.

Covariant classical electrodynamics

Let us introduce the partial derivative operators with respect to x^μ and x_μ:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \vec{\nabla}) \quad \text{and} \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = (\frac{\partial}{\partial t}, -\vec{\nabla})$$

$$\text{Invariant 4-d Laplacian operator } : \Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$$

The invariance of electric charge forces the charge density $\rho(x)$ and current density $\vec{j}(x)$ to satisfy the continuity equation, requiring ρ and $\vec{j}$ to form a 4-vector j^μ:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \quad \Rightarrow \quad \partial_\mu j^\mu = 0 \ \text{ with } \ j^\mu = (\rho, \vec{j})$$

In the Lorentz family of gauges the electromagnetic wave equations for the scalar potential ϕ and the vector potential $\vec{A}$ are

$$\Box\,\phi = \rho \quad \oplus \quad \Box\,\vec{A} = \vec{j} \quad \text{with} \quad \frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$$

Q2

Obviously, Lorentz covariance requires that the potentials ϕ and $\vec{A}$ form a 4-vector potential A^μ, casting the wave equations and the Lorentz condition into the manifestly covariant forms,

$$A^\mu = (\phi, \vec{A}) \quad \Rightarrow \quad \Box A^\mu = j^\mu \quad \oplus \quad \partial_\mu A^\mu = 0$$

Quantum mechanically the wave equation corresponds to the massless spin-1 field equation and the massive spin-1 free field equation is given by

$$(\Box + m^2)A^\mu = 0 \quad \oplus \quad \partial_\mu A^\mu = 0 \quad \Rightarrow \quad A^\mu = \epsilon^\mu(p)\, \mathrm{e}^{-ip\cdot x} \;:\; \text{plane wave}$$

with $p^2 = m^2$ and the constraint $p\cdot\epsilon(p) = 0$.

Lorentz transformations

The Lorentz transformation to the spacetime coordinate 4-vector x^μ

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$$

leaving the length of the four vector x^μ invariant as

$$x'^2 = g_{\mu\nu} \, x'^\mu x'^\nu = x^2$$

should satisfy the relation:

$$g_{\mu\nu} \, \Lambda^\mu{}_\rho \Lambda^\nu{}_\tau = [\, \Lambda^T \, g \, \Lambda \,]_{\rho\tau} = g_{\rho\tau}$$

The six (anti-symmetric) generators $M_{\rho\sigma}$ of the Lorentz group defined by

$$\Lambda^{\mu}{}_{\nu} = \left[\exp\left(-\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right)\right]^{\mu}{}_{\nu} \quad \text{with} \quad [M_{\rho\sigma}]^{\mu}{}_{\nu} = i\left(g^{\mu}_{\rho}g_{\sigma\nu} - g^{\mu}_{\sigma}g_{\rho\nu}\right)$$

satisfy the Lie algebra

$$[M_{\mu\nu},\, M_{\rho\sigma}] = i\,[g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} - g_{\nu\sigma}M_{\mu\rho}]$$

Q3

which can be expressed as

$$\begin{aligned} [J_i,\; J_j] &= +i\,\epsilon_{ijk}\,J_k \\ [J_i,\; K_j] &= +i\,\epsilon_{ijk}\,K_k \\ [K_i,\, K_j] &= -i\,\epsilon_{ijk}\,J_k \end{aligned}$$

in terms of the generators of rotations and boosts

$$J_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk}, \qquad K_i \equiv M^{0i} = -M_{0i}$$

The mixed algebra of $\{J_i, K_i\}$ can be diagonalized as

$$[J_{Li},\ J_{Rj}] = 0$$
$$[J_{Li},\ J_{Lj}] = i\,\epsilon_{ijk}\,J_{Lk}$$
$$[J_{Ri},\ J_{Rj}] = i\,\epsilon_{ijk}\,J_{Rk}$$

by introducing two independent sets of generators

$$\mathcal{L}\ :\ \ J_{Li}\ \equiv \frac{1}{2}\,(J_i + i\,K_i)$$
$$\mathcal{R}\ :\ \ J_{Ri} \equiv \frac{1}{2}\,(J_i - i\,K_i)$$

In terms of J_{Li} and J_{Ri}, we can construct SU(2)$_L \times$SU(2)$_R$ representations of the Lorentz group

$$(j_L, j_R) \quad : \quad \begin{cases} J_L^2 \,|(j_L, j_R)\rangle = j_L(j_L+1)\,|(j_L, j_R)\rangle \\ J_R^2 \,|(j_L, j_R)\rangle = j_R(j_R+1)\,|(j_L, j_R)\rangle \end{cases}$$

with $J_{L,R} = 0, 1/2, \cdots$. The two SU(2) generators are related by Parity P

$$P \,:\, \begin{cases} J_i \;\to\; +J_i \\ K_i \;\to\; -K_i \end{cases} \quad \Rightarrow \quad \vec{J}_L \;\leftrightarrow\; \vec{J}_R$$

The handedness is a convention. In the following we shall discuss these two spinor representations in some detail so as to construct the Lorentz covariant Dirac equation for a massive spin-1/2 particle.

Spinors

We introduce 2–component spinors transforming under the SU$(2)_L \times$SU$(2)_R$ group as

$$\left(\frac{1}{2}, 0\right) \; : \quad \psi_L(x) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \{\eta_\alpha\} \quad \Rightarrow \quad \psi'_L(x') = \Lambda_L \psi_L(x)$$

$$\left(0, \frac{1}{2}\right) \; : \quad \psi_R(x) = \begin{pmatrix} \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix} = \{\bar{\chi}^{\dot{a}}\} \quad \Rightarrow \quad \psi'_R(x') = \Lambda_R \psi_R(x)$$

The rotation/boost generators in the L/R-spinor representation read

$$J_k = \frac{1}{2}\,\sigma^k \quad \text{and} \quad K_k = \mp\frac{i}{2}\,\sigma^k \quad \text{for} \quad L/R$$

in terms of the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma^i \sigma^j = \delta_{ij} + i\,\epsilon_{ijk}\sigma^k$$

For the rotation of $\vec{\theta}$ and the boost of $\vec{\eta}$, the Lorentz transformations of the L and R spinor fields are described by the operators

$$\Lambda_L = \exp\left\{-i(\vec{J}\cdot\vec{\theta} + \vec{K}\cdot\vec{\eta})\right\} = \exp\left\{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta} - i\vec{\eta})\right\}$$
$$\Lambda_R = \exp\left\{-i(\vec{J}\cdot\vec{\theta} + \vec{K}\cdot\vec{\eta})\right\} = \exp\left\{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta} + i\vec{\eta})\right\}$$

The magnitude $\theta = |\vec{\theta}|$ is the rotation angle while the magnitude of the boost vector $\vec{\eta}$ (which is parallel to the velocity vector $\vec{\beta}$) is called the rapidity related to the velocity parameter β as

$$e^{\eta} = \gamma(1+\beta) = \sqrt{(1+\beta)/(1-\beta)}\,; \quad \cosh\eta = \gamma,\ \sinh\eta = \gamma\beta$$

with the boost factor $\gamma = 1/\sqrt{1-\beta^2}$.

Vectors

Polarization vectors of a massive vector boson in the frame where its momentum is chosen as 3–axis can be obtained by boosting the polarization vectors in the rest frame along the 3-axis:

$$p^\mu = (m,0,0,0) \quad \stackrel{\text{boost}}{\longrightarrow} \quad (E,0,0,p) \;\; \text{with} \;\; E = \gamma m,\; p = \gamma\beta m$$

$$\begin{array}{llll} \epsilon^\mu(p,1) = (0,1,0,0) & & (0,1,0,0) & : \text{transverse} \\ \epsilon^\mu(p,2) = (0,0,1,0) & & (0,0,1,0) & : \text{transverse} \\ \epsilon^\mu(p,3) = (0,0,0,1) & \stackrel{\text{boost}}{\longrightarrow} & (p/m,0,0,E/m) & : \text{longitudinal} \end{array}$$

These polarization vectors satisfy the orthogonal conditions

$$p_\mu \epsilon^\mu(p,a) = 0, \quad \epsilon_\mu(p,a)\,\epsilon^\mu(p,b) = -\delta_{ab}\,; \quad a,b = 1,2,3$$

and the helicity spin-1 states are determined from the above polarization vectors by

$$\epsilon^\mu(p,\pm) = \frac{1}{\sqrt{2}}\,[\mp\,\epsilon^\mu(p,1) - i\,\epsilon^\mu(p,2)]$$
$$\epsilon^\mu(p,0) = \epsilon^\mu(p,3)$$

Together with p^μ/m the tree polarization vectors $\epsilon^\mu(p,\lambda)$ form a complete orthonormal basis.

Dirac Equation

For a free particle, the only operator appearing in the wave equation is the 4–momentum operator $p_\mu = i\,\partial_\mu$ or in the spinor notation with $\sigma_\pm^\mu = (1, \pm\vec{\sigma})$

$$p_\mu\;\sigma^\mu_{+\alpha\dot{\alpha}} = \begin{pmatrix} p_0 - p_z & -p_x + ip_y \\ -p_x - ip_y & p_0 + p_z \end{pmatrix} \quad \Leftarrow \quad \Lambda_L^{-1}\sigma_+^\mu\,\Lambda_R = \Lambda^\mu{}_\nu\,\sigma_+^\nu$$

Q4

$$p_\mu\,\sigma_-^{\mu\,\dot{\alpha}\alpha} = \begin{pmatrix} p_0 + p_z & p_x - ip_y \\ p_x + ip_y & p_0 - p_z \end{pmatrix} \quad \Leftarrow \quad \Lambda_R^{-1}\sigma_-^\mu\,\Lambda_L = \Lambda^\mu{}_\nu\,\sigma_-^\nu$$

One simple wave equation can be a linear differential relation between the components of spinors, expressed by the operators $p_\mu\sigma^\mu_{+\alpha\dot{\alpha}}$ and $p_\mu\sigma_-^{\mu\dot{\alpha}\alpha}$

$$p_\mu\,\sigma^\mu_{+\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = m\,\eta_\alpha \quad \text{and} \quad p_\mu\,\sigma_-^{\mu\,\dot{\alpha}\alpha}\,\eta_\alpha = m\bar{\chi}^{\dot{\alpha}}$$

The need to use the mass in the wave equation implies the simultaneous consideration of two L- and R-spinors (η_α and $\bar{\chi}^{\dot{\alpha}}$); with only one of these, it would not be possible to construct a relativistically invariant equation containing a dimensional parameter. The above relativistic wave equation is called the Dirac equation having been first derived by Dirac in 1928.

The spinor form of the Dirac equation is the most natural one, in the sense that its relativistic invariance is immediately apparent. In applications of the equation, however, other forms of the wave equation may be more convenient. We denote a four–component Dirac spinor by the symbol ψ. In the spinor representation, it is a bi-spinor consisting of two 2-component spinors

$$\psi = \begin{pmatrix} \eta_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

The Dirac equation is put in terms of the 4–component spinor in the form

$$p_\mu \gamma^\mu \psi \equiv \not{p}\psi = m\,\psi \quad \Rightarrow \quad (\not{p} - m)\,\psi = 0$$

The spinor form of the wave equation with the components of the above bispinor corresponds to the 4×4 matrices γ^μ :

$$\text{Weyl/Chiral :} \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma_+^\mu \\ \sigma_-^\mu & 0 \end{pmatrix}$$

We introduce an additional gamma matrix γ_5 and two chiral projection operators :

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_\lambda = \frac{1+\lambda\gamma_5}{2} \qquad \lambda = \pm = \mathsf{R/L}$$

In the general case, the matrices γ^μ need to satisfy only the conditions ensuring that $p^2 = m^2$. To find these conditions, we multiply the Dirac equation by $\not{p}$

$$\not{p}\,\not{p}\psi = \frac{1}{2}p_\mu p_\nu\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\psi = m\,\not{p}\psi = m^2\psi$$

and we must therefore have

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}$$

which can be checked with the Weyl representations of the gamma matrices given above.

Free–particle Solutions

Let us solve the Dirac equation. We take into account a plane wave solution

$$\psi(x) = u(p)\,\mathrm{e}^{-ip\cdot x} \quad p^2 = m^2 \oplus p^0 > 0 \quad \Rightarrow \; (\gamma^\mu p_\mu - m)u(p) = 0$$

It is easiest to analyze this equation in the rest frame, where $p_0^\mu = (m, \vec{0})$; the solution for a general p can then be found by boosting the rest–frame solution

$$(m\gamma^0 - m)u(p_0) = m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p_0) = 0 \quad \Rightarrow \; u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

for any 2–component spinor ξ normalized to be $\xi^\dagger \xi = 1$.

Now that we have the general form of $u(p)$ in the rest frame, we can obtain $u(p)$ in any other frame by boosting. Consider a boost along the 3–direction. Then with the rapidity η the boosted spinor is given by

$$
\begin{aligned}
u(p) &= \exp\left[-\frac{1}{2}\eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\
&= \left[\cosh\frac{\eta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sinh\frac{\eta}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\
&= \begin{bmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right] \xi \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right] \xi \end{bmatrix}
\end{aligned}
$$

The last line can be simplified to give

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma_+}\,\xi \\ \sqrt{p \cdot \sigma_-}\,\xi \end{pmatrix}$$

where it is understood that in taking the square root of a matrix we take the positive root of each eigenvalue. This expression for $u(p)$ is not only more compact, but is also valid for any arbitrary direction of $\vec{p}$.

The amplitude of the plane wave contains one arbitrary two–component quantity ξ. Thus, for a given momentum, there are two different independent states, corresponding to the two possible values of the spin component. But, in the relativistic theory the orbital angular momentum $\vec{l}$ and the spin $\vec{s}$ of a moving particle are not separately conserved. Only the total angular momentum $\vec{j} = \vec{l} + \vec{s}$ is conserved. The component of the spin in any fixed direction is therefore also not conserved. However, the component of the spin in the direction of the momentum is conserved; since $\vec{l} = \vec{r} \times \vec{p}$ the product $\hat{p} \cdot \vec{s}$ is equal to the conserved product $\hat{p} \cdot \vec{j}$. This quantity is called the helicity. Helicity states correspond to plane waves in which $\xi = \xi_\lambda$ is an eigenvalue of the operator $\hat{p} \cdot \hat{\sigma}$:

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}\, \xi_\lambda(p) = \lambda\, \xi_\lambda(p) \qquad \lambda = \pm$$

If we write

$$\frac{\vec{p}}{|\vec{p}|} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

then the normalized helicity eigenstates can be expressed as

$$\xi_+(p) = \frac{1}{\sqrt{2(1+n_z)}}\begin{pmatrix} 1+n_z \\ n_x + i n_y \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\,\mathrm{e}^{i\phi} \end{pmatrix}$$

$$\xi_-(p) = \frac{1}{\sqrt{2(1+n_z)}}\begin{pmatrix} -n_x + i n_y \\ 1+n_z \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2}\,\mathrm{e}^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

Q5

apart from arbitrary phases. In the helicity basis we can express the Dirac spinor as

$$u(p,\lambda) = \begin{bmatrix} (E-\lambda p)^{1/2}\,\xi_\lambda(p) \\ (E+\lambda p)^{1/2}\,\xi_\lambda(p) \end{bmatrix} \equiv \begin{pmatrix} u_-(p,\lambda) \\ u_+(p,\lambda) \end{pmatrix}$$

The subscripts are so chosen as to satisfy

$$P_+u(p,\lambda) = \frac{1+\gamma_5}{2}u(p,\lambda) = \begin{pmatrix} 0 \\ u_+(p,\lambda) \end{pmatrix}$$

$$P_-(p,\lambda) = \frac{1-\gamma_5}{2}u(p,\lambda) = \begin{pmatrix} u_-(p,\lambda) \\ 0 \end{pmatrix}$$

In addition to the positive–frequency plane wave solution $u(p)$, there exists a negative–frequency plane wave solution

$$\psi(x) = v(p)\,\mathrm{e}^{+ip\cdot x}$$

satisfying the same Dirac equation. The easiest way to find this solution is to consider charge conjugation represented by a unitary matrix C:

$$C\gamma_\mu^T C^\dagger = -\gamma_\mu \qquad CC^\dagger = 1 \qquad \Rightarrow \qquad C = -i\gamma^2\gamma^0$$

$$v(p,\lambda) = C\bar{u}^T(p,\lambda) = (-i\gamma^2\gamma^0)\,\gamma^0 u^*(p,\lambda) = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} u^*(p,\lambda)$$

which can be derived through the following procedure

$$\begin{aligned}
&(i\gamma^\mu\partial_\mu - m)\,\psi = 0\\
&\psi^\dagger\left(-i\,\overleftarrow{\partial_\mu}\,\gamma^{\mu\dagger} - m\right) = 0\\
&\psi^\dagger\gamma^0\left(-i\,\overleftarrow{\partial_\mu}\,\gamma^0\gamma^{\mu\dagger}\gamma^0 - m\right) = 0\\
&\bar{\psi}\left(-i\,\overleftarrow{\partial_\mu}\,\gamma^\mu - m\right) = 0\\
&(-i\gamma^{\mu\,T}\partial_\mu - m)\,\bar{\psi}^T = 0\\
&(-iC\gamma^{\mu\,T}C^\dagger\partial_\mu - m)\,C\bar{\psi}^T = 0
\end{aligned}$$

Noting that $i\,\sigma^2\,\xi^*_\lambda = -\lambda\xi_{-\lambda}$ $[\lambda = \pm]$ we can obtain the negative–frequency solution in the helicity basis as

$$v(p,\lambda) = \lambda\begin{bmatrix} -(E+\lambda p)^{1/2}\,\xi_{-\lambda}(p) \\ (E-\lambda p)^{1/2}\,\xi_{-\lambda}(p) \end{bmatrix} \equiv \begin{pmatrix} v_-(p,\lambda) \\ v_+(p,\lambda) \end{pmatrix}$$

It is an easy exercise to check that the helicity spinors $u(p,\lambda)$ and $v(p,\lambda)$ satisfy the following relations

$$\bar{u}(p,\lambda)u(p,\lambda') = +2m\delta_{\lambda\lambda'} \quad \text{and} \quad \bar{v}(p,\lambda)v(p,\lambda') = -2m$$

For a massive spin-1/2 particle with 4-momentum $p^\mu = (E,\vec{p})$, the spin 4-vector is defined as

$$s^\mu = \lambda\left(|\vec{p}|, E\hat{p}\right)/m$$

where $\lambda = \pm 1$ is twice the spin-1/2 particle helicity. Note that in the rest frame, $s = \lambda(0,\hat{p})$, while in the high energy limit with $E \gg m$, $s = \lambda p/m$. The helicity spinors satisfying the Dirac equation are eigenstates of $\gamma_5\ \not{s}$ with unit eigenvalue. From these results, we can derive the helicity projection operators for a massive spin-1/2 particle.

Q6

$$u(p,\lambda)\bar{u}(p,\lambda) = \frac{1}{2}(1+\gamma_5\ \not{s})(\not{p}+m) \quad \Rightarrow \quad \sum_{\lambda=\pm} u(p,\lambda)\bar{u}(p,\lambda) = \not{p}+m$$

$$v(p,\lambda)\bar{v}(p,\lambda) = \frac{1}{2}(1+\gamma_5\ \not{s})(\not{p}-m) \quad \Rightarrow \quad \sum_{\lambda=\pm} v(p,\lambda)\bar{v}(p,\lambda) = \not{p}-m$$

We have seen that the necessity of two spinors (η, χ) to describe a particle with spin $1/2$ is due to the mass of the particle. This necessity disappears if the mass is zero. The wave equation which describes such a particle can be derived from a single spinor, say the undotted spinor

$$p_\mu \, \sigma_-^{\mu \, \dot{\alpha}\alpha} \eta_\alpha = 0 \quad \Rightarrow \quad (E + \vec{p} \cdot \vec{\sigma}) \, \eta = 0$$

The energy and momentum of a particle with $m = 0$ are related by $E = |\vec{p}|$ so that we have

$$(\hat{p} \cdot \vec{\sigma}) \eta(p) = -\eta(p) \ : \quad \text{helicity } \lambda = -1/2 = -$$

On the other hand the dotted spinor $\bar{\chi}$ satisfies

$$(\hat{p} \cdot \vec{\sigma}) \bar{\chi}(p) = +\bar{\chi}(p) \ : \quad \text{helicity } \lambda = +1/2 = +$$

Consequently, states of the massless particle with a definite momentum are necessarily helicity states, for which the spin component in the direction of motion has a definite value. If the particle spin is opposite to the momentum (helicity $-1/2$), the antiparticle spin is along the momentum (helicity $+1/2$). The neutrinos in the Standard Model were supposed to be such particles possessing these properties.

2-component Spinor Technique

For the contraction of a four–vector a^μ and γ^μ we write

$$\not{a} = \begin{pmatrix} 0 & \not{a}_+ \\ \not{a}_- & 0 \end{pmatrix} \qquad \not{a}_\pm = a_\mu \sigma^\mu_\pm$$

For the Pauli–adjoint of the four–component spinors we have

$$\bar{u}(p,\lambda) = u^\dagger(p,\lambda)\gamma^0 = \left(u^\dagger_+(p,\lambda), u^\dagger_-(p,\lambda)\right)$$

$$\bar{v}(p,\lambda) = v^\dagger(p,\lambda)\gamma^0 = \left(v^\dagger_+(p,\lambda), v^\dagger_-(p,\lambda)\right)$$

The contraction of the helicity polarization vectors and γ^μ is

$$\not{\epsilon}_\pm(p,+) = \pm\sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \not{\epsilon}_\pm(p,-) = \mp\sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\not{\epsilon}_\pm(p,\ 0) = \begin{pmatrix} \frac{1}{m}(p \mp E) & 0 \\ 0 & \frac{1}{m}(p \pm E) \end{pmatrix}$$

which will be useful in evaluating the transition amplitudes involving massive vector bosons.

Strings with even and odd numbers of γ–matrices are expressed, respectively, as follows:

$$\begin{aligned}
\bar{u}(\bar{p},\bar{\lambda})P_{\pm}u(p,\lambda) &= u^{\dagger}_{\mp}(\bar{p},\bar{\lambda})u_{\pm}(p,\lambda)\\
\bar{u}(\bar{p},\bar{\lambda})\not{a}P_{\pm}u(p,\lambda) &= u^{\dagger}_{\pm}(\bar{p},\bar{\lambda})\not{a}_{\pm}u_{\pm}(p,\lambda)\\
\bar{u}(\bar{p},\bar{\lambda})\not{a}\not{b}P_{\pm}u(p,\lambda) &= u^{\dagger}_{\mp}(\bar{p},\bar{\lambda})\not{a}_{\mp}\not{b}_{\pm}u_{\pm}(p,\lambda)
\end{aligned}$$

and the similar relations hold for strings with v's and with u and v.

An Example

As a first example, let us calculate the helicity amplitude for the process

$$e^-(k,\sigma) + e^+(\bar{k},\bar{\sigma}) \rightarrow \mu^-(p,\lambda) + \mu^+(\bar{p},\bar{\lambda})$$

in the lowest order. We choose the e^- momentum direction as the positive z-axis and assume that the muon pair is produced on the x-z plane with the μ^- scattering angle θ:

$$k = \frac{\sqrt{s}}{2}(1,0,0,+1) \qquad \bar{k} = \frac{\sqrt{s}}{2}(1,0,0,-1)$$

$$p = \frac{\sqrt{s}}{2}(1,\beta\sin\theta,0,\beta\cos\theta) \quad \bar{p} = \frac{\sqrt{s}}{2}(1,-\beta\sin\theta,0,-\beta\cos\theta)$$

$$q = k + \bar{k} = p + \bar{p} \qquad s = q^2 \qquad \beta = \sqrt{1-4m^2/s}$$

where m is the muon mass and the electron mass is neglected. In this coordinate system the electron and muon 2-component spinors are given by

$$u(k,+)_a = \delta_{a+} s^{1/4} \begin{pmatrix} +1 \\ 0 \end{pmatrix} \qquad u(k,-)_a = \delta_{a-} s^{1/4} \begin{pmatrix} 0 \\ +1 \end{pmatrix}$$

$$v(\bar{k},-)_a = \delta_{a+} s^{1/4} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad v(\bar{k},+)_a = \delta_{a-} s^{1/4} \begin{pmatrix} +1 \\ 0 \end{pmatrix}$$

$$u(p,+)_b = \omega_b \begin{pmatrix} c_h \\ s_h \end{pmatrix} \qquad u(p,-)_b = \omega_{-b} \begin{pmatrix} -s_h \\ c_h \end{pmatrix}$$

$$v(\bar{p},-)_b = -b\omega_b \begin{pmatrix} s_h \\ -c_h \end{pmatrix} \qquad v(\bar{p},+)_b = b\omega_{-b} \begin{pmatrix} c_h \\ s_h \end{pmatrix}$$

where $\omega_b = (E + b\,p)^{1/2}$, $c_h = \cos\theta/2$ and $s_h = \sin\theta/2$.

The scattering amplitude due to the γ and Z exchanges is written as

$$\mathcal{M}(\sigma\bar{\sigma} : \lambda\bar{\lambda}) = \frac{e^2}{s} Q_{ab} \left[\bar{v}(\bar{k}, \bar{\sigma})\gamma_\mu P_a u(k, \sigma)\right] \left[\bar{u}(p, \lambda)\gamma^\mu P_b v(\bar{p}, \bar{\lambda})\right]$$

$$Q_{ab} = 1 + \frac{s}{s - m_Z^2 + i m_Z \Gamma_Z} a_a^e a_b^\mu$$

$$a_L^f = \frac{T_3^f - Q_f \sin^2\theta_W}{\cos\theta_W \sin\theta_W} \qquad a_R^f = -\frac{Q_f \sin^2\theta_W}{\cos\theta_W \sin\theta_W}$$

It is quite straightforward to evaluate the electron current (although you need to do a little exercise to get familiar with the technique)

$$
\begin{aligned}
j^{\mu}_{+}(\sigma\bar{\sigma}) &= \bar{v}(\bar{k},\bar{\sigma})\gamma^{\mu}P_{+}u(k,\sigma) = v(\bar{k},\bar{\sigma})^{\dagger}_{+}\sigma^{\mu}_{+}u(k,\sigma)_{+} \\
&= \delta_{\sigma+}\,\delta_{\bar{\sigma}-}s^{1/2}(0,-1)\,[1,\,\vec{\sigma}]\begin{pmatrix}1\\0\end{pmatrix} \\
&= \delta_{\sigma+}\,\delta_{\bar{\sigma}-}\sqrt{s}\,(0,-1,-i,0) = \delta_{\sigma+}\,\delta_{\bar{\sigma}-}\sqrt{2s}\,\epsilon^{\mu}(q,+) \\
j^{\mu}_{-}(\sigma\bar{\sigma}) &= \bar{v}(\bar{k},\bar{\sigma})\gamma^{\mu}P_{-}u(k,\sigma) = v(\bar{k},\bar{\sigma})^{\dagger}_{-}\sigma^{\mu}_{-}u(k,\sigma)_{-} \\
&= \delta_{\sigma-}\,\delta_{\bar{\sigma}+}s^{1/2}(1,0)\,[1,-\vec{\sigma}]\begin{pmatrix}0\\1\end{pmatrix} \\
&= \delta_{\sigma-}\,\delta_{\bar{\sigma}+}\sqrt{s}\,(0,-1,+i,0) = -\delta_{\sigma-}\,\delta_{\bar{\sigma}+}\sqrt{2s}\,\epsilon^{\mu}(q,-)
\end{aligned}
$$

Note that the positron helicity is always opposite to the electron helicity. On the other hand, it is a little demanding to evaluate the muon current.

For opposite helicities,

$$\begin{aligned}
J_{+}^{\mu}(+,-) &= \bar{u}(p,+)\gamma^{\mu}P_{+}v(\bar{p},-) = -\omega_{+}^{2}(c_h, s_h)\,[1,\ \vec{\sigma}]\begin{pmatrix} s_h \\ -c_h \end{pmatrix} \\
&= \frac{\sqrt{s}}{2}(1+\beta)\,(0, -\cos\theta, i, \sin\theta) \\
J_{-}^{\mu}(-,+) &= \bar{u}(p,-)\gamma^{\mu}P_{-}v(\bar{p},+) = -\omega_{+}^{2}(-s_h, c_h)\,[1,\ -\vec{\sigma}]\begin{pmatrix} c_h \\ -s_h \end{pmatrix} \\
&= \frac{\sqrt{s}}{2}(1+\beta)\,(0, -\cos\theta, -i, \sin\theta)
\end{aligned}$$

and for same helicities,

$$
\begin{aligned}
J^{\mu}_{+}(+,+) &= \bar{u}(p,+)\gamma^{\mu}P_{+}v(\bar{p},+) = \omega_{+}\omega_{-}(c_h, s_h)\,[1,\ \vec{\sigma}] \begin{pmatrix} c_h \\ s_h \end{pmatrix} \\
&= m\,(1, \sin\theta, 0, \cos\theta) \\
J^{\mu}_{-}(-,-) &= \bar{u}(p,-)\gamma^{\mu}P_{-}v(\bar{p},-) = \omega_{-}\omega_{+}(-s_h, c_h)\,[1,\ -\vec{\sigma}] \begin{pmatrix} s_h \\ -c_h \end{pmatrix} \\
&= m\,(-1, \sin\theta, 0, \cos\theta)
\end{aligned}
$$

With these expressions we obtain the helicity amplitudes:

$$\mathcal{M}(\sigma\bar{\sigma} : \lambda\bar{\lambda}) = \frac{e^2}{s} Q_{ab}\, j_a(\sigma, \bar{\sigma}) \cdot J_b(\lambda, \bar{\lambda})$$
$$\equiv e^2\, \delta_{\sigma,-\bar{\sigma}}\, \delta_{\sigma a}\, Q_{ab}\, \langle \sigma : \lambda\bar{\lambda} \rangle$$

$$\langle \pm : ++ \rangle = \frac{m}{\sqrt{s}} \sin\theta$$
$$\langle \pm : +- \rangle = \frac{1}{2}(1+\beta)(\mp 1 - \cos\theta)$$
$$\langle \pm : -+ \rangle = \frac{1}{2}(1+\beta)(\pm 1 - \cos\theta)$$
$$\langle \pm : -- \rangle = \frac{m}{\sqrt{s}} \sin\theta$$

At high energies the amplitude is greatly simplified because of chirality conservation. The non-vanishing helicity amplitudes can be written in a compact form as

$$\mathcal{M}(\sigma, -\sigma : \lambda, -\lambda) = -e^2 Q_{\sigma\lambda} \left(\sigma\lambda + \cos\theta\right)$$

Averaging over the four initial-state spin orientations and summing over the four final-state spin orientations, we find for the differential cross section of the QED process $e^+e^- \to \gamma^* \to \mu^+\mu^-$ through a virtual photon

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \overline{\sum} |\mathcal{M}|^2 = \frac{\alpha^2}{4s} \left(1 + \cos^2\theta\right)$$

where $\alpha = e^2/4\pi$ and $s = 4E_{cm}^2$.

Conventional Trace Technique

Typically, deriving the analytic expressions for (unpolarized) decay widths and cross sections involving at least two fermions requires us to calculate

$$\sum_{\lambda,\bar{\lambda}} \left|\bar{u}(p,\lambda)\Gamma v(\bar{p},\bar{\lambda})\right|^2 = \mathrm{Tr}[(\not{p}+m)\Gamma(\not{\bar{p}}-m')\overline{\Gamma})]$$

where $\overline{\Gamma}$ is the Dirac conjugate of a matrix Γ defined as

$$\overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0 \quad \Rightarrow \quad \begin{array}{ll} \overline{1} = 1\,, & \overline{\gamma_5} = -\gamma_5 \\ \overline{\gamma_\mu} = \gamma_\mu\,, & \overline{\gamma_5\gamma_\mu} = \gamma_5\gamma_\mu \end{array}$$

The traces can be efficiently calculated by exploiting several representation-independent identities derived from the (anti-)commutation relations of the gamma matrices

$$\mathrm{Tr}(\text{odd \# of } \gamma \text{ matrices}) = 0$$

$$\mathrm{Tr}(1) = 4$$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4\, g_{\mu\nu}$$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho})$$

$$\mathrm{Tr}(\gamma_5) = 0$$

$$\mathrm{Tr}(\gamma_5 \gamma_\mu \gamma_\nu) = 0$$

$$\mathrm{Tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4i\epsilon_{\mu\nu\rho\sigma} \quad \text{with} \quad \epsilon_{0123} = +1$$

$$\mathrm{Tr}(\gamma_{\mu_1} \gamma_{\mu_2} \cdots \gamma_{\mu_{2n}}) = \sum_{i=2} (-1)^i g_{\mu_1 \mu_i} \mathrm{Tr}(\gamma_{\mu_2} \cdots \gamma_{\mu_{i-1}} \gamma_{\mu_{i+1}} \cdots \gamma_{\mu_{2n}})$$

In conjunction with the trace formulas the contraction formulas can be used:

$$\gamma_\mu \gamma^\mu = 4\,, \qquad \gamma_\mu \gamma_\alpha \gamma^\mu = -2\gamma_\alpha$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4\, g_{\alpha\beta}\,, \qquad \gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma^\mu = -2\gamma_\gamma \gamma_\beta \gamma_\alpha$$

Q7

The proof is a straightforward exercise to be done.

Renormalizable (Gauge) Interactions

Spin	0	1/2	1
1	Gauge $\vert D_\mu \varphi\vert^2$	Gauge $\bar{\psi} \not{D} \psi$	Gauge $F^{\mu\nu} F_{\mu\nu}$
1/2	Yukawa $\bar{\psi}\psi\varphi$	No	No
0	Scalar φ^3, φ^4	No	No

- Obtained by the replacement

$$\partial_\mu \quad \Rightarrow \quad D_\mu = \partial_\mu + igT^a A^a_\mu$$

- Universality : there is only one coupling constant for each simple group. The gauge interaction of a particle is totally determined by knowing the representation of the particle.

- Conserves fermion chirality

$$\bar{\psi}\gamma_\mu(v - a\gamma_5)\,\psi Z^\mu = \left[(v+a)\bar{\psi}_L\gamma_\mu\,\psi_L + (v-a)\bar{\psi}_R\gamma_\mu\,\psi_R\right] Z^\mu$$

  Breaking of fermion chirality is entirely due to a mass term or a Yukawa interaction [apart from anomalies].

- Non–renormalizable effective interaction of gauge bosons can be constructed from D_μ and $F_{\mu\nu}$ such as

$$\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu} \qquad \bar{\psi}\gamma_\mu D_\nu \psi F^{\mu\nu}$$

Prescription of Model Building

- Fix the gauge group
  - Gauge bosons are determined
  - Parameters : gauge couplings

- Fix the representations of fermions and scalars
  - Gauge interactions of matter particles are fixed (no new parameters)
  - The total fermion representation must be anomaly–free

- Impose global symmetries if needed
- Write down all possible mass terms and interactions compatible with the symmetries
  - scalar potential parameters ($\varphi^2, \varphi^3, \varphi^4$)
  - fermion masses
  - Yukawa couplings

Lagrangian to Observables

- Particle states in the Hilbert space
  - The vacuum

$$\langle 0|0\rangle = 1 \qquad |0\rangle \ : \text{dim} = 0$$

  - One–particle state

$$\langle p|p'\rangle = 2p^0(2\pi)^3\delta^3(\vec{p}-\vec{p'}) \qquad |p\rangle \ : \text{dim} = -1$$

$$1 = |0\rangle\langle 0| + \sum_p |p\rangle\langle p| + \cdots \qquad \sum_p = \frac{d^3p}{(2\pi)^3 2p^0}$$

- Quantum fields

$$\varphi(x) = \sum_p \left[a(p)\,\mathrm{e}^{-ip\cdot x} + a^\dagger(p)\,\mathrm{e}^{ip\cdot x}\right]$$

$$\psi(x) = \sum_p \sum_{\lambda=\pm} \left[a_\lambda(p) u(p,\lambda)\,\mathrm{e}^{-ip\cdot x} + b_\lambda^\dagger(p) v(p,\lambda)\,\mathrm{e}^{ip\cdot x}\right]$$

$$A^\mu(x) = \sum_p \sum_{\lambda=\pm,0} \left[a_\lambda(p)\,\epsilon^\mu(p,\lambda)\,\mathrm{e}^{-ip\cdot x} + a_\lambda^\dagger(p)\,\epsilon^{\mu *}(p,\lambda)\,\mathrm{e}^{ip\cdot x}\right]$$

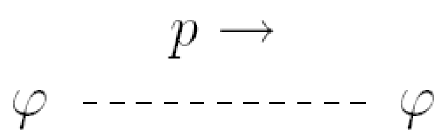
- S matrix

$$S_{fi} \equiv \langle f\ \mathrm{out}|i\ \mathrm{in}\rangle = 1_{fi} + i\,(2\pi)^4\,\delta^4(P_f - P_i)\,T_{fi}$$

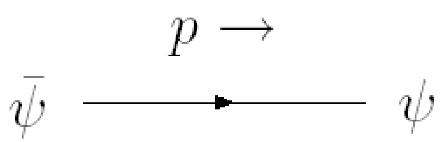
$$\text{Unitarity}\ :\ SS^\dagger = S^\dagger S = 1 \quad\Rightarrow\quad T^\dagger T = -i\left(T - T^\dagger\right)$$

Lagrangian to Feynman Rules

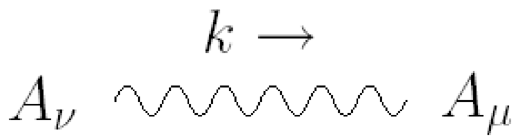
- Free parts (or kinetic terms) $\Rightarrow$ Propagators; Interactions $\Rightarrow$ Vertices



$$i\,\frac{1}{p^2-m^2+i\varepsilon}$$



$$i\,\frac{(\not{p}+m)}{p^2-m^2+i\varepsilon}$$



$$i \, \frac{-g_{\mu\nu} + k_\mu k_\nu / m^2}{k^2 - m^2 + i\varepsilon}$$

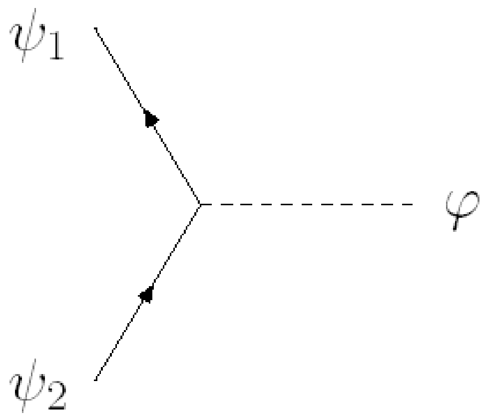
Massless : $$i \, \frac{-g_{\mu\nu} + (1-\xi) \, k_\mu k_\nu / k^2}{k^2 + i\varepsilon}$$

$\xi = 1$: Feynman gauge

$\xi = 0$: Landau gauge

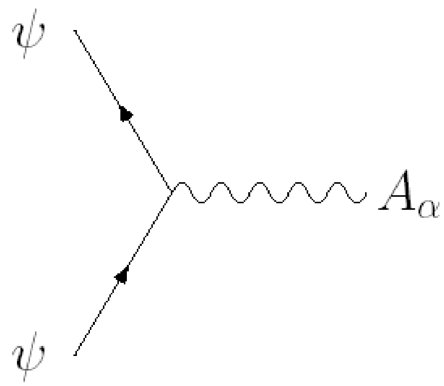
- Vertices : from $i\,\mathcal{L}_{\text{int}}$
  - No derivatives

$$\boxed{\mathcal{L} = i\,f\,\bar{\psi}_1\gamma_5\psi_2\varphi}$$



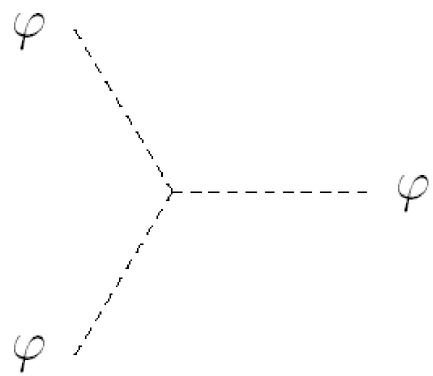
$$i \cdot i\,f\gamma_5 = -f\,\gamma_5$$

$$\mathcal{L} = -e\bar{\psi}\gamma_\mu\psi\, A^\mu$$



$$i \cdot (-e)\gamma^\mu g_{\mu\alpha} = -i\, e\, \gamma_\alpha$$

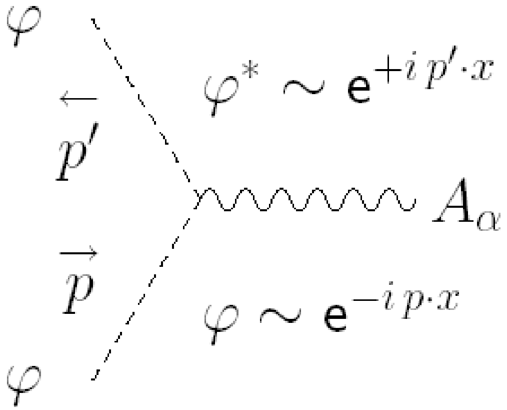
$$\mathcal{L} = \mu\varphi^3 = 6\,\mu\left(\frac{1}{3!}\varphi^3\right)$$



$$i \cdot 6\,\mu = 6\, i\mu$$

– With derivatives

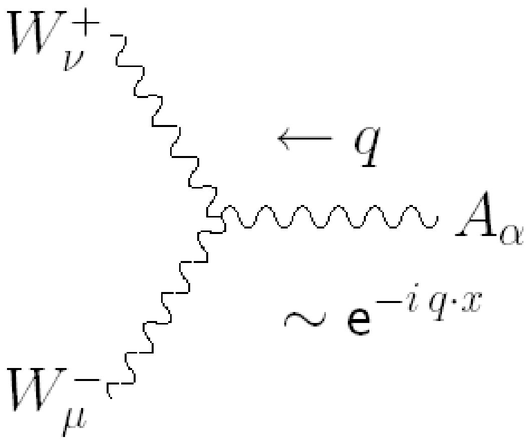
$$\mathcal{L} = -ie(\varphi^* \partial_\mu \varphi - \partial_\mu \varphi^* \, \varphi) A^\mu$$



$$i \cdot (-ie) \cdot [(-ip_\alpha) - (ip'_\alpha)]$$

$$= -i\,e\,(p + p')_\alpha$$

$$\mathcal{L} = -i\, e\kappa\, W_\mu^+ W_\nu \left(\partial^\mu g^{\nu\alpha} - \partial^\nu g^{\mu\alpha}\right) A_\alpha$$



$$i \cdot (-i\, e\, \kappa)\, [(-iq_\mu) g_{\nu\alpha} - (-iq_\nu) g_{\mu\alpha}]$$

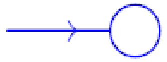
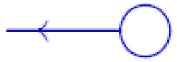
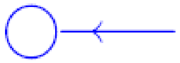
$$= -i\, e\, \kappa \left(q_\mu\, g_{\nu\alpha} - q_\nu\, g_{\mu\alpha}\right)$$

Feynman Rules to Scattering Amplitude

A Feynman graph is a sum of all possible graphs for a given process using the vertices and propagators of the model.

- Vertex $\longrightarrow$ vertex factor
- Internal line $\longrightarrow$ Propagator
- Loop $\longrightarrow \int \frac{d^4k}{(2\pi)^4}$

- External line $\longrightarrow$ wave function
  - scalar : 1
  - fermion :

• initial	particle	$u(p)$	
	antiparticle	$\bar{v}(p)$	
• final	particle	$\bar{u}(p)$	
	antiparticle	$v(p)$	

  - vector :
    - initial $\epsilon_\mu(p)$ final $\epsilon^*_\mu(p)$

- Closed fermion loop $\longrightarrow$ factor (-1)
- A graph with an exchanged fermion pair $\longrightarrow$ factor (-1)

Following the above prescription one can obtain $iT_{fi} \equiv i\mathcal{M}$ where $\mathcal{M}$ is called the scattering or transition amplitude.

Scattering Amplitude to Cross Section/Width

- Decay rate (in the rest frame)

$$d\Gamma(p \to k_1 + \cdots + k_n) = \frac{1}{2M} \overline{\sum} |\mathcal{M}|^2 \, d\Phi_n$$

- Scattering cross section

$$d\sigma(p_1 + p_2 \to k_1 + \cdots + k_n) = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - M_1^2 M_2^2}} \overline{\sum} |\mathcal{M}|^2 \, d\Phi_n$$

where $\overline{\sum}$ denotes the average for the initial states and the sum for the final states.

- Final–state phase space

$$d\Phi_n \; = \; (2\pi)^4\,\delta^4\left(p - \sum_{i=1}^{n} k_i\right)\prod_{i=1}^{n}\frac{d^3k_i}{(2\pi)^3 2k_i^0}$$

The evaluation of the phase space integrals is facilitated by the identity

$$\frac{d^3k_i}{2k_i^0} = d^4p_i\,\delta(k_i^2 - m_i^2)\,\theta(k_i^0)$$

  – 1-body phase space

$$d\Phi_1 = 2\pi\delta(p^2 - m_1^2)$$

  – 2-body phase space

$$d\Phi_2 = \frac{\lambda_f^{1/2}}{32\pi^2}\,d\Omega_1 \quad \text{with} \quad \lambda_f = \frac{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2}{s^2}$$

  with $d\Omega_1 = \sin\theta_1 d\theta_1 d\phi_1$ and $s = p^2$.

– 3-body phase space

$$d\Phi_3 = \frac{p^2}{128\pi^3}\, dz_1 dz_2 \quad \text{with} \quad z_1 = \frac{2p\cdot k_1}{p^2} \quad \text{and} \quad z_2 = \frac{2p\cdot k_2}{p^2}$$

Q8

For all massless final states, the allowed region of the variables z_1 and z_2 is given by the relations $0 \leq z_1, z_2 \leq 1$ and $z_1 + z_2 \geq 1$.

Concrete examples

The interaction Lagrangian of the charged electroweak gauge bosons $W^{\pm}$ and leptons are given by

$$\mathcal{L} = -\frac{g}{\sqrt{2}}\,\bar{\ell}\gamma^{\mu}P_L\nu_{\ell}\,W_{\mu}^{-} + \mathrm{h.c.} \quad \text{for} \quad \ell = e, \mu, \tau$$

Based on the Lagrangian, calculate the widths of the 2-body decay process $W^{-} \to e\nu_e$ and the 3-body decay process $\mu^{-} \to e^{-}\bar{\nu}_e\nu_{\mu}$ step by step.

Useful Formulae

- $\vec{\nabla}\cdot\vec{E} = \rho, \qquad \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}$

  $\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{0}$

- $\Lambda^\mu{}_\nu \approx \delta^\mu{}_\nu + \omega^\mu{}_\nu$

  $\approx \delta^\mu{}_\nu - \frac{i}{2}\omega^{\rho\sigma}(M_{\rho\sigma})^\mu{}_\nu$

  $\Rightarrow \boxed{\omega^\mu{}_\nu = -\frac{i}{2}\omega^{\rho\sigma}(M_{\rho\sigma})^\mu{}_\nu}$

  rotation : $\omega^{ij} = \epsilon_{ijk}\theta_k$

  boost : $\omega^{0i} = -\eta_i$

- $e^{\mp\frac{1}{2}\vec{\eta}\cdot\vec{\sigma}} = \sqrt{\frac{p\cdot\sigma_\pm}{m}}$

  $= \frac{E+m \mp \vec{p}\cdot\vec{\sigma}}{\sqrt{2(E+m)}}$

- $\lambda\,\vec{\sigma}\cdot\hat{p} = \frac{1}{m}\sqrt{p\cdot\sigma_+}\; S\cdot\sigma_-\sqrt{p\cdot\sigma_+}$

  $= \frac{-1}{m}\sqrt{p\cdot\sigma_-}\; S\cdot\sigma_+\sqrt{p\cdot\sigma_-}$

- $\sigma_+^\mu\sigma_-^\nu + \sigma_+^\nu\sigma_-^\mu = 2g^{\mu\nu}$

  $\sigma_-^\mu\sigma_+^\nu + \sigma_-^\nu\sigma_+^\mu = 2g^{\mu\nu}$

Problem Set #1

LEPI과 SLC에서 Z보존 생성과 붕괴를 통한 전기약작용 이론의 검증

유럽 CERN의 LEP과 미국 SLAC의 SLD실험은 전자와 양전자의 충돌을 통한 Z 보존의 생성과 두 페르미온 입자쌍 $f\bar{f}$ $(f = b, c, s, u, d, e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau)$로의 붕괴과정

$$e^-(k,\sigma) + e^+(\bar{k},\bar{\sigma}) \to Z \to f(p,\lambda) + \bar{f}(\bar{p},\bar{\lambda}) \tag{1}$$

을 정밀하게 연구하였다. [$\sigma, \bar{\sigma}, \lambda, \bar{\lambda}$는 각 입자의 1/2단위의 헤리시티(helicity)다.] 전자와 페르미온 질량은 Z 보존 질량에 해당하는 약 91 GeV의 충돌에너지에 비해 매우 작으므로 무시한다. 전기약작용이론에 따르면 Z 보존과 f의 상호작용은

$$\mathcal{L}_{Zff} = -g_Z \bar{f}\gamma_\mu(C_L^f P_L + C_R^f P_R) f Z^\mu \quad \text{with} \quad C_L^f = I_3^f - Q_f \sin^2\theta_W, C_R^f = -Q_f \sin^2\theta_W \tag{2}$$

로 기술된다. Q_f 는 양성자 전하을 단위로 한 페르미온의 전하(예를 들어 전자는 -1)이고 I_3^f는 $u, c, \nu_e, \nu_\mu, \nu_\tau$ 대해서는 1/2이고 d, s, b, e, μ, τ에 대해서는 $-1/2$로 주어진다. 또한 $P_{L,R}$는 각각 $(1 \mp \gamma_5)/2$로 카이랄 투사 연산자라 불린다.

1. 각 페르미온 입자의 C_L^f 와 C_R^f 값을 $\sin^2\theta_W = 0.23$을 택하여 계산하라.

2. 전자를 왼쪽/오른쪽 편극시킨 상태로 Z 보존을 생성하여 얻은 편극 비대칭량

$$\mathcal{A}_{LR}^e = \frac{\sigma(e_L^- e^+ \to Z) - \sigma(e_R^- e^+ \to Z)}{\sigma(e_L^- e^+ \to Z) + \sigma(e_R^- e^+ \to Z)} \tag{3}$$

   을 측정할 수 있다. 이 편극 비대칭량 $\mathcal{A}_{LR}^e$ 을 $C_{L,R}^e$ 로 표현하고 그 값을 구체적으로 구하라.

3. 생성된 페르미온의 전자의 운동량 방향에 대한 산란각을 고려하여 앞쪽과 뒤쪽으로 산란되는 비율의 비대칭성 $\mathcal{A}_{FB}$을 측정할 수 있다. 이 앞/뒤 비대칭량(FB asymmetry)을 구하고 각각의 페르미온에 대해 그 값을 계산하라.

4. Z 보존 붕괴에 의해서 생성된 입자가 τ 경입자이면 이 τ 입자의 붕괴과정을 통하여 τ 입자의 편극을 측정할 수 있다. 이를 통하여 측정할 수 있는 편극 비대칭성을 정의하고 계산하라.

5. 위의 다양한 독립적인 방법에 의해 전자기약작용의 중요한 물리량인 $\sin^2\theta_W$를 측정할 수 있다. 각각의 측정실험의 CERN과 SLAC에서 진행 과정에 대해 좀 더 상세하게 알아 보아라.

Problem Set #2

W 보존 질량 : 횡질량(transverse mass)과 야코비안 피크(Jacobian peak)

전기약작용이론(electroweak theory)가 예측한 $W^\pm$ 와 Z 보존의 존재는 1983년 CERN에서 양성자-반양성자충돌을 이용한 UA1 과 UA2 실험에서 발견된 후 (1984년 노벨 물리학상: 루비아와 반데어미어), W 보존의 질량을 측정하기 위해 쿼크와 반쿼크의 충돌을 통한 W^+ 보존의 생성과 붕괴과정

$$q + \bar{q}' \quad \rightarrow \quad W^+ \quad \rightarrow \quad e^+ + \nu_e \tag{1}$$

이 정밀하게 연구되어 왔다.

1. 위와 같이 W 보존이 렙톤으로 2체 붕괴할 때 전자의 횡운동량 크기 P_{eT} 에 대한 미분단면적이 다음과 같음을 보이고 처음 나타나는 야코비 항의 특이성(singularity)을 고려하여 운동량에 대한 사건분포를 개략적으로 예측해 보라.

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}P_{eT}} = \frac{4P_{eT}}{s\sqrt{1-4P_{eT}^2/s}} \cdot \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\cos\theta^*} \tag{2}$$

여기서 $P_{eT}(\equiv |\vec{P}_e|\sin\theta^*)$ 와 θ^* 는 각각 두 경입자의 질량중심계에서의 잰 전자의 횡운동량(transverse momentum) 크기와 극각(polar angle)을 나타낸다.

2. W 보존 붕괴 시에 보이는 두 경입자 운동량 중에 실제로 관측 가능한 것이 무엇인 지를 밝히고, Z 보존의 질량 측정방법과 비교하여 W 보존의 질량 측정에는 어떠한 차이점이 있는 지를 논하라.

3. 실제 W 보존이 붕괴할 때 다음과 같이 정의된 두 경입자 횡질량(transverse mass)의 최대값이 W 보존 질량과 같음을 보여라.

$$m_{e\nu T}^2 = (E_{eT} + E_{\nu T})^2 - (\vec{P}_{eT} + \vec{P}_{\nu T})^2 \leq m_W^2 \tag{3}$$

4. 반응과정 (1)에 대한 경입자 횡질량 분포가 다음과 같이 주어짐을 보여라.

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}m_{e\nu T}^2} = \frac{|V_{qq'}|^2 G_F^2}{8\pi} \cdot \frac{m_W^4}{(s-m_W^2)^2 + m_W^2\Gamma_W^2} \cdot \frac{(2-m_{e\nu T}^2/s)}{\sqrt{1-m_{e\nu T}^2/s}} \tag{4}$$

여기서 $V_{qq'}$ 와 G_F는 각각 CKM 행렬의 $[qq']$ 성분과 페르미 상수를 나타낸다.

CHAPTER 3

Muon decay

The muon decay $\mu \to e\nu\bar{\nu}$ is a process which traditionally opens calculations of weak decays in textbooks. The reason is two-fold. First, this is a pure leptonic process not involving hadrons, and therefore is easily calculated to the end. Second, experimentally this is one of the most thoroughly investigated decays of elementary particles. In this chapter we calculate the electron spectrum, the total decay probability, and finally the angular and spin correlations in polarized muon decays.

3.1. Decay amplitude

Let us consider the decay $\mu^-(p) \to e^-(k)\bar{\nu}_e(q_1)\nu_\mu(q_2)$ (quantities in parentheses denote 4-momenta of the particles). The Feynman graph of this process given in fig. 3.1a is equivalent to that of fig. 3.1b. In this second case the matrix element is of the form

$$M = \sqrt{\tfrac{1}{2}}\, G\bar{\nu}_\mu O^\alpha \mu \bar{e} O_\alpha \nu_e,$$

where $O_\alpha \equiv O_\alpha^{\mathrm{L}} = \gamma_\alpha(1+\gamma_5)$, and particle symbols stand for their wave functions. By using the Fierz transformation (see appendix) this matrix element can be recast in the form

$$M = -\sqrt{\tfrac{1}{2}}\, G\bar{e} O^\beta \mu \bar{\nu}_\mu O_\beta \nu_e.$$

The complex conjugate of this last expression is

$$M^* = M^+ = -\sqrt{\tfrac{1}{2}}\, G\nu_e^+ O^{\beta+} \gamma_0 \nu_\mu \mu^+ O_\beta^+ \gamma_0 e$$

$$= -\sqrt{\tfrac{1}{2}}\, G\nu_e^+ \gamma_0\gamma_0 O^{\beta+} \gamma_0 \nu_\mu \mu^+ \gamma_0\gamma_0 O_\beta^+ \gamma_0 e = -\sqrt{\tfrac{1}{2}}\, \bar{\nu}_e O^\beta \nu_\mu \bar{\mu} O_\beta e.$$

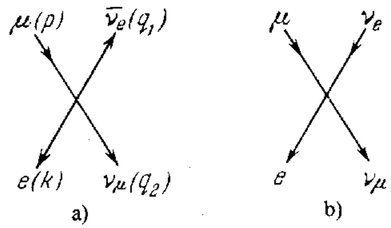


Fig. 3.1.

[We made use of the fact that $\gamma_0 O^{\beta +}\gamma_0 = O^\beta$, because $\gamma_0[\gamma_0(1+\gamma_5)]^+\gamma_0 = \gamma_0(1+\gamma_5)\gamma_0\gamma_0 = \gamma_0(1+\gamma_5)$, and also

$$\gamma_0[\boldsymbol{\gamma}(1+\gamma_5)]^+\gamma_0 = \gamma_0(1+\gamma_5)(-\boldsymbol{\gamma})\gamma_0 = (1-\gamma_5)\boldsymbol{\gamma} = \boldsymbol{\gamma}(1+\gamma_5)].$$

Similarly, $\gamma_0 O_\beta^+ \gamma_0 = O_\beta$. Therefore,

$$|M|^2 = MM^* = -\tfrac{1}{2}G^2\bar{\nu}_\mu O^\alpha \mu \bar{e} O_\alpha \nu_e \bar{\nu}_e O^\beta \nu_\mu \bar{\mu} O_\beta e$$
$$= -\tfrac{1}{2}G^2\bar{\nu}_\mu O^\alpha \mu \bar{\mu} O_\beta e \bar{e} O_\alpha \nu_e \bar{\nu}_e O^\beta \nu_\mu .$$

As demonstrated in the appendix, the density matrix for an unpolarized Dirac particle has the form

$$\sum_s u^k(s)\bar{u}_i(s) = (\hat{p} + m)_i^k,$$

where the summation is over the polarization states of the particle; p is the particle 4-momentum and m its mass; and i, k are bispinor suffices, $i, k = 1, 2, 3, 4$. Using this we obtain

$$\overline{|M|^2} = -\tfrac{1}{2}G^2 \mathrm{Tr}\,\hat{q}_2 O^\alpha(\hat{p} + m_\mu)O_\beta(\hat{k} + m_e)O_\alpha \hat{q}_1 O^\beta .$$

The bar over $|M|^2$ denotes summation over spin states; Tr denotes the trace of the product of matrices to the right of it. Since $\gamma_5\gamma_\alpha = -\gamma_\alpha\gamma_5$, and since $(1+\gamma_5)^2 = 2(1+\gamma_5)$,

$$\overline{|M|^2} = -8\cdot\tfrac{1}{2}G^2 \mathrm{Tr}\,\hat{q}_2\gamma^\alpha\hat{p}\gamma_\beta\hat{k}\gamma_\alpha\hat{q}_1\gamma^\beta(1+\gamma_5)$$
$$= 8G^2\mathrm{Tr}\,\hat{q}_2\hat{k}\gamma_\beta\hat{p}\hat{q}_1\gamma^\beta(1+\gamma_5) = 32G^2(pq_1)\mathrm{Tr}\,\hat{q}_2\hat{k}(1+\gamma_5)$$
$$= 128G^2(pq_1)(kq_2).$$

(In deriving this equality, we took into account that $\gamma^\alpha\hat{A}\hat{B}\hat{C}\gamma_\alpha = -2\hat{C}\hat{B}\hat{A}$, $\gamma^\alpha\hat{A}\hat{B}\gamma_\alpha = 4AB$, see Chapter 29.)

3.2. Decay probability

It is now possible to calculate the probability of the muon decay by means of the general expression (see Chapter 29, sect. 4)

$$\mathrm{d}\Gamma = \frac{\overline{|M|^2}}{2\cdot 2m}\mathrm{d}\Phi.$$

Here $\mathrm{d}\Gamma$ is the differential probability of decay per unit volume of phase space $\mathrm{d}\Phi$, and the factor $\frac{1}{2}$ appears because of the averaging (and not summation) over muon polarizations; the factor $2m$ (where m is the muon mass) appears because of the chosen normalization of wave functions of particles. As shown in the appendix,

$$\mathrm{d}\Phi = (2\pi)^4\delta^4(p - k_1 - q_1 - q_2)\frac{\mathrm{d}\boldsymbol{k}}{2E(2\pi)^3}\frac{\mathrm{d}\boldsymbol{q}_1}{2\omega_1(2\pi)^3}\frac{\mathrm{d}\boldsymbol{q}_2}{2\omega_2(2\pi)^3},$$

where E, ω_1, ω_2 are the energies of the electron, $\bar{\nu}_e$, and ν_μ; $\boldsymbol{k}$, $\boldsymbol{q}_1$, and $\boldsymbol{q}_2$ stand for their momenta. Neutrinos being not observed, let us integrate over their momenta. This means that we must calculate the integral

$$I_{\alpha\beta} = \int q_{1\alpha}q_{2\beta}\frac{\mathrm{d}\boldsymbol{q}_1}{\omega_1}\frac{\mathrm{d}\boldsymbol{q}_2}{\omega_2}\delta^4(q_1 + q_2 - q),$$

where $q = p - k$. The anticipated result can be written as a sum of two mutually orthogonal terms:

$$I_{\alpha\beta} = A(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta) + B(q^2 g_{\alpha\beta} - 2q_\alpha q_\beta).$$

Here $g_{\alpha\beta}$ is a metric tensor (see Chapter 29), q is the total 4-momentum of two neutrinos, and A and B are dimensionless coefficients that we shall now determine. By multiplying both sides of the equality by $q^2 g^{\alpha\beta} - 2q^\alpha q^\beta$, we obtain

$$B\cdot 4q^4 = \int q_{1\alpha}q_{2\beta}(q^2 g^{\alpha\beta} - 2q^\alpha q^\beta)\cdots$$
$$= \int[q^2(q^1q^2) - 2(qq_1)(qq_2)]\cdots = 0,$$

since $q^2 = (q_1 + q_2)^2 = 2q_1q_2$, $qq_1 = (q_1 + q_2, q_1) = q_1q_2$, $qq_2 = (q_1 + q_2, q_2) = q_1q_2$, $q_1^2 = q_2^2 = 0$ (the neutrino is massless). By multiplying both sides of the tensor equality by $q^2 g^{\alpha\beta} + 2q^\alpha q^\beta$, we obtain

$$A\cdot 12q^4 = q^4\int\frac{\mathrm{d}\boldsymbol{q}_1}{\omega_1}\frac{\mathrm{d}\boldsymbol{q}_2}{\omega_2}\delta^4(q_1 + q_2 - q) = q^4\int\frac{\mathrm{d}\boldsymbol{q}_1}{\omega_1\omega_2}\delta(2\omega_1 - \omega)$$
$$= q^4\cdot 4\pi\cdot\tfrac{1}{2},$$

so that $A = \frac{1}{6}\pi$. (The integral above is taken in the center-of-mass frame of two neutrinos). Finally,

$$I_{\alpha\beta} = \tfrac{1}{6}\pi\left(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta\right),$$

where $q = p - k$. Substitution of this result into the expression for the decay width yields

$$\mathrm{d}\Gamma = \frac{G^2}{2\cdot 2m}\frac{128}{(2\pi)^5 2\cdot 2}\tfrac{1}{6}\pi p^\alpha k^\beta\left[q^2 g_{\alpha\beta} + 2q_\alpha q_\beta\right]\frac{\mathrm{d}\boldsymbol{k}}{2E}$$
$$= \frac{G^2}{48\pi^4 m}\left[q^2(pk) + 2(qp)(qk)\right]\frac{\mathrm{d}\boldsymbol{k}}{E}.$$

As the electron mass is negligibly small compared to its energy, we have $qk = (p-k,k) = pk = mE$, $q^2 = (p-k)^2 = p^2 - 2pk = m^2 - 2mE$. Integration over the electron direction yields 4π, and we obtain

$$\mathrm{d}\Gamma = \frac{G^2}{12\pi^3 m}(pk)(p^2 - 2pk + 2p^2 - 2pk)E\,\mathrm{d}E$$
$$= \frac{G^2}{12\pi^3}(3m^2 - 4mE)E^2\,\mathrm{d}E = \frac{G^2 m^5}{96\pi^3}(3 - 2\varepsilon)\varepsilon^2\,\mathrm{d}\varepsilon,$$

where $\varepsilon = E/E_{\max} = 2E/m$.

The electron spectrum in muon decay, for the most general form of the four-fermion interaction, can be shown to take the form

$$\Gamma(\varepsilon)\,\mathrm{d}\varepsilon = 12\Gamma\left[(1-\varepsilon) - \tfrac{2}{9}\rho(3 - 4\varepsilon)\right]\varepsilon^2\,\mathrm{d}\varepsilon,$$

where the coefficient ρ is called the Michel parameter (the term proportional to ρ gives zero contribution when integrated over the whole range $0 \le \varepsilon \le 1$). Clearly, the spectrum obtained above corresponds to $\rho = 0.75$. Integration over the electron energy yields the total decay width

$$\Gamma = \frac{G^2 m^5}{96\pi^3}\int_0^1 (3 - 2\varepsilon)\varepsilon^2\,\mathrm{d}\varepsilon = \frac{G^2 m^5}{192\pi^3}.$$

After correction for virtual photons, one obtains (see references in Chapter 28)

$$\Gamma = \frac{G^2 m^5}{192\pi^3}\left[1 - \frac{\alpha}{2\pi}\left(\pi^2 - \tfrac{25}{4}\right)\right].$$

Comparison of this expression with the experimentally measured muon lifetime gives the familiar value of the coupling constant G.

3.3. Decay of polarized muon

Let the spin of a muon in its rest frame be directed along the unit vector $\boldsymbol{\eta}$. The muon density matrix then becomes (see appendix)

$$u^k(s)\bar{u}_i(s) = \tfrac{1}{2}\left[(\hat{p} + m)(1 - \gamma_5)\hat{s}\right]_i^k,$$

where a 4-vector s^α has the following properties: $s^2 = -1$, $sp = 0$. In the muon rest frame, $s^0 = 0$, $\boldsymbol{s} = \boldsymbol{\eta}$. In the reference frame in which the muon moves with momentum $\boldsymbol{p}$,

$$s^0 = \frac{\boldsymbol{\eta}\cdot\boldsymbol{p}}{m}, \qquad \boldsymbol{s} = \boldsymbol{\eta} + \frac{\boldsymbol{p}(\boldsymbol{\eta}\cdot\boldsymbol{p})}{m(E+m)}.$$

If we substitute in the above calculations

$$(\hat{p} + m) \to \tfrac{1}{2}(\hat{p} + m)(1 - \gamma_5\hat{s}),$$

we will have to substitute $p - ms$ for p in the earlier result

$$\mathrm{d}\Gamma = \frac{G^2}{48\pi^4 m}\left[q^2(pk) + 2(qp)(qk)\right]\frac{\mathrm{d}\boldsymbol{k}}{E}.$$

[This is readily confirmed by considering that part of the trace which comprises the muon density matrix and its neighbor terms

$$\mathrm{Tr}\cdots(1+\gamma_5)(\hat{p} + m)(1 - \gamma_5\hat{s})\gamma_\beta(1+\gamma_5)\cdots$$
$$= \mathrm{Tr}\cdots(1+\gamma_5)(\hat{p} - m\gamma_5\hat{s} + m + \gamma_5\hat{p}\hat{s})(1-\gamma_5)\gamma_\beta\cdots$$
$$= \mathrm{Tr}\cdots(1+\gamma_5)(\hat{p} - m\gamma_5\hat{s})\gamma_\beta(1+\gamma_5)\cdots$$
$$= \mathrm{Tr}\cdots(1+\gamma_5)(\hat{p} - m\hat{s})\gamma_\beta(1+\gamma_5)\cdots.$$

The terms $m + \gamma_5\hat{p}\hat{s}$ cancelled out since they were multiplied by $(1+\gamma_5) \times (1-\gamma_5)$. As for the factor $\frac{1}{2}$, it was already taken into account in the expression for the probability indicating averaging over muon polarizations, so that we need not do it a second time.] With muon polarization taken into account,

$$\mathrm{d}\Gamma = \frac{G^2}{48\pi^4 m}\left\{\left[q^2(pk) + 2(qp)(qk)\right] - m\left[q^2(sk) + 2(qs)(qk)\right]\frac{\mathrm{d}\boldsymbol{k}}{E}\right\}$$
$$= \frac{G^2}{48\pi^4 m}\Big\{(pk)\left[(p-k)^2 + 2(p^2 - pk)\right] - m(sk)\left[(p-k)^2 - 2pk\right]\Big\}\frac{\mathrm{d}\boldsymbol{k}}{E}$$
$$= \frac{G^2 m^5}{384\pi^4}\left[(3 - 2\varepsilon) + \boldsymbol{\eta}\cdot\boldsymbol{n}(1 - 2\varepsilon)\right]\varepsilon^2\,\mathrm{d}\varepsilon\,\mathrm{d}\Omega,$$

where $\boldsymbol{n} = \boldsymbol{k}/E$ is a unit vector in the direction of the emission of the electron, $\mathrm{d}\Omega = \mathrm{d}\cos\theta\,\mathrm{d}\varphi$ is an element of the solid angle, and $\cos\theta = \boldsymbol{n}\cdot\boldsymbol{\eta}$. The above derivation uses that $qs = (p - k, s) = -ks$ since $ps = 0$; moreover, we take into consideration that in the muon rest frame, $ks = k^0 s_0 - \boldsymbol{k}\cdot\boldsymbol{s} = -E\boldsymbol{n}\cdot\boldsymbol{\eta}$.

The angular distribution integrated over the electron spectrum takes the form

$$\frac{\mathrm{d}\Gamma(\cos\theta)}{\Gamma} = \tfrac{1}{2}\left(1 - \tfrac{1}{3}\cos\theta\right)\mathrm{d}\cos\theta.$$

The above formulas cover the case when the electron polarization is not measured. If we were interested in the decay probability as a function of electron polarization, k in the expression for $|M|^2$ would have to be replaced by $\frac{1}{2}(k - m_e s_e)$ where s_e is a vector characterizing the electron polarization: $s_e^2 = -1$, $s_e k = 0$. Explicit expressions for the components of s_e, $s_e^0 = \boldsymbol{k}\cdot\boldsymbol{\zeta}/m_e$, $\boldsymbol{s}_e = \boldsymbol{\zeta} + (\boldsymbol{k}\cdot\boldsymbol{\zeta})\boldsymbol{k}/m_e(E + m_e)$, demonstrate that the components of s_e normal to $\boldsymbol{n}$ can be neglected, in the ultrarelativistic limit, in comparison to s_e^0 and $\boldsymbol{n}\cdot\boldsymbol{s}_e = (E/m)(\boldsymbol{\zeta}\cdot\boldsymbol{n})$. We obtain therefore that for $v \to 1$,

$$\tfrac{1}{2}(k_\alpha - m_e s_{e\alpha}) \Rightarrow \tfrac{1}{2}k_\alpha(1 - \boldsymbol{\zeta}\cdot\boldsymbol{n}).$$

We have thus reproduced a well-known property of the weak interaction: the left-handed polarization of the emitted relativistic leptons. The probability of the decay $\mu^- \to e^- \nu_\mu \bar{\nu}_e$, with the electron polarization taken into account, is then written as

$$\mathrm{d}\Gamma = \Gamma\tfrac{1}{2}(1 - \boldsymbol{\zeta}\cdot\boldsymbol{n})\left[(3 - 2\varepsilon) + \boldsymbol{\eta}\cdot\boldsymbol{n}(1 - 2\varepsilon)\right]\varepsilon^2\mathrm{d}\varepsilon\,\mathrm{d}\cos\theta.$$

Similar calculations for the decay $\mu^+ \to e^+ \nu_e \bar{\nu}_\mu$ would give

$$\mathrm{d}\Gamma = \Gamma\tfrac{1}{2}(1 + \boldsymbol{\zeta}\cdot\boldsymbol{n})\left[(3 - 2\varepsilon) - \boldsymbol{\eta}\cdot\boldsymbol{n}(1 - 2\varepsilon)\right]\varepsilon^2\mathrm{d}\varepsilon\,\mathrm{d}\cos\theta,$$

where ε is the positron energy divided by its maximum energy, and $\boldsymbol{n}$, $\boldsymbol{\zeta}$, $\boldsymbol{\eta}$ are unit vectors in the directions of momentum of the positron, its spin, and the muon spin, respectively.

3.4. Qualitative discussion

A number of results derived in this chapter are very easy to interpret qualitatively, without any calculations. First of all, the fact that $\Gamma \sim G^2 m^5$ follows from dimension-based arguments, since $[\Gamma] = \mu$, and $[G] = \mu^{-2}$, where μ has the dimension of mass. We only use that the probability is

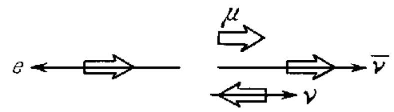


Fig. 3.2.

proportional to G^2 and that the only parameter determining the dynamics of muon decay is the muon mass, since the electron mass can be safely neglected.

Turning to the angular asymmetry, we readily notice that it is neither P- nor C-invariant: its signs are different in the left-handed and right-handed reference frames, as well as for the electron and positron in decays of μ^- and μ^+, respectively. Characteristic features of this asymmetry can also be explained. Let us consider an electron with nearly maximum energy ($\varepsilon \sim 1$). In this situation neutrinos (ν and $\bar{\nu}$) must be emitted in the direction opposite to that of the electron (see fig. 3.2) (the phase-space volume of the configuration in which one of them has low momentum, is small). Recalling that the helicities of ν and $\bar{\nu}$ are of opposite signs, we have to conclude that the angular momentum carried by neutrinos is zero. Consequently, the electron must be emitted with its spin parallel to that of the muon. But, the electron helicity being negative, its momentum must be predominantly directed opposite to the muon spin. This is in agreement with the formula derived above for the angular distribution of electrons, which for $\varepsilon \sim 1$ is proportional to $(1 - \boldsymbol{\eta}\cdot\boldsymbol{n})$.

For $\varepsilon \ll 1$, the neutrino and antineutrino move in the opposite directions, with the total spin equal to unity. This time, conservation of angular momentum demands that electrons be emitted along the muon spin direction (see fig. 3.3). Similar arguments will be valid for decays of positive muons.

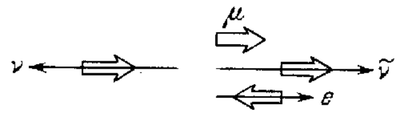


Fig. 3.3.

of c_{SA} appears because $\Delta_{12} = -1$. The negative sign of c_{ST} is easily confirmed if we take into account that the term $\sigma^{10}\sigma^{10}$ enters the sum $\sigma^{\alpha\beta}\sigma_{\alpha\beta}$ with a minus sign (due to the pseudo-euclidean metric).

Pseudoscalar variant. $F = G = \gamma_5$. Since γ^α commutes with γ^5, $c_{PV} = -c_{SV}$, $c_{PA} = -c_{SA}$.

Vector variant. $F = \gamma^\alpha$, $G = \gamma_\alpha$. The following relations determine the coefficients of the second row of the Fierz matrix:

$$
\begin{aligned}
\gamma^\alpha\gamma_A\gamma_\alpha &= 4\gamma_A && \text{for} && \gamma_A = 1,\\
&= -2\gamma_A && \text{for} && \gamma_A = \gamma^\mu,\\
&= 0 && \text{for} && \gamma_A = \sigma^{\mu\nu},\\
&= 2\gamma_A && \text{for} && \gamma_A = \gamma^5\gamma^\mu,\\
&= -4\gamma_A && \text{for} && \gamma_A = \gamma^5.
\end{aligned}
$$

Axial variant. $F = \gamma^5\gamma^\alpha$, $G = \gamma_5\gamma_\alpha$. Calculations are similar to those of the preceding case.

Tensor variant. $F = \sqrt{\tfrac{1}{2}}\,\sigma^{\alpha\beta}$, $G = \sqrt{\tfrac{1}{2}}\,\sigma_{\alpha\beta}$. The following relations are used to calculate the Fierz coefficients:

$$
\begin{aligned}
\sigma^{\alpha\beta}\gamma_A\sigma_{\alpha\beta} &= -12\gamma_A && \text{for} && \gamma_A = 1,\\
&= 0 && \text{for} && \gamma_A = \gamma^\mu,\\
&= -4\gamma_A && \text{for} && \gamma_A = \sigma^{\mu\nu},\\
&= 0 && \text{for} && \gamma_A = \gamma^5\gamma^\mu,\\
&= -12\gamma_A && \text{for} && \gamma_A = \gamma^5.
\end{aligned}
$$

The Fierz relations for longitudinal spinors are obtained by taking $F = \gamma^\alpha(1 + \gamma_5)$, $G = \gamma_\alpha(1 + \gamma_5)$, or $F = \gamma^\alpha(1 + \gamma_5)$, $G = \gamma_\alpha(1 - \gamma_5)$.

29.4. Rules for the calculation of probabilities

29.4.1. S- and T-matrices

Consider a set of physical states which are transformed into one another due to interactions. We shall characterize the transition from state i to state f by a quantity S_{fi}. The set of all values of S_{fi} forms the scattering matrix, also referred to as S-matrix. With all interactions switched off, the S-matrix turns into a unit matrix I: each state is transformed into itself. This means that physical processes occur if the T-matrix is non-zero, with T defined by the relation

$$S = I + iT.$$

Let us introduce the process amplitude M_{fi} as the quantity defined by the relation

$$T_{fi} = (2\pi)^4 \delta^4(p_f - p_i) M_{fi},$$

where p_i and p_f are 4-momenta of the initial and final states, and the δ-function expresses in an explicit form the energy-momentum conservation law:

$$\delta^4(p_f - p_i) = \delta(p_f^x - p_i^x)\delta(p_f^y - p_i^y)\delta(p_f^z - p_i^z)\delta(E_f - E_i).$$

For the sake of brevity the subscripts f and i in M_{fi} are hereafter dropped.

29.4.2. *Probability and cross sections*

The square of the absolute value of T_{fi} determines the transition probability from the initial state i to the final state f:

$$\overline{w_{fi}} = |T_{fi}|^2 = (2\pi)^4 \delta^4(p_f - p_i)(2\pi)^4 \delta(0)\, |M|^2.$$

In order to calculate $\overline{w_{fi}}$, introduce a four-dimensional normalization volume VT, which of course will be eliminated from the final result. As follows from the definition of δ^4, for $V \to \infty$ and $T \to \infty$

$$(2\pi)^4 \delta^4(0) = VT.$$

In order to calculate the transition probability into a group of states instead of a single final state f, we must multiply $\overline{w_{fi}}$ by the element of the phase-space volume $d\overline{\Phi}$ which is written as

$$d\overline{\Phi} = \prod_{l=1}^{n} \frac{d\boldsymbol{k}_l V}{(2\pi)^3},$$

where n is the number of particles in the final state, and $\boldsymbol{k}_l$ is the 3-momentum of the lth particle.

Now we must take care of the correct normalization of the expression for the transition probability. Wave functions of particles will be normalized in such a manner that each unit volume contains $2E$ particles, where E is the energy of a particle. It is clear that this normalization corresponds, in the case of scalar particles, to the wave function $\varphi = \exp(-ikx)$. Indeed, in this case the particle density is

$$i\left(\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}\right) = 2E.$$

The normalized probability is obtained by dividing $\overline{w_{fi}}$ by N:

$$N = \prod_{l=1}^{n} (2E_l V) \prod_{i=1}^{k} (2E_i V),$$

where k is the number of particles in the initial state. A decay corresponds to $k = 1$, and a collision of two particles to $k = 2$.

As a result, the following expression is obtained for the normalized probability transition per unit time:

$$dw_{fi} = \frac{\overline{w_{fi}}}{T} \frac{d\overline{\Phi}}{N} = \frac{V\, |M|^2}{\prod_{i=1}^{k} (2E_i V)}\, d\Phi,$$

where

$$d\Phi = (2\pi)^4 \delta^4(p_f - p_i) \prod_{l=1}^{n} \frac{d\boldsymbol{k}_l}{(2\pi)^3 2E_l}.$$

We obtain for particle decay rate ($k = 1$):

$$d\Gamma = \frac{1}{2E_a} |M|^2 d\Phi,$$

where E_a is the energy of the decaying particle. A collision of two particles ($k = 2$) is usually characterized by the cross section defined by

$$dw_{fi} = d\sigma \cdot j,$$

where j is the flux density. The flux density in the laboratory reference frame where particle a is at rest and particle b impinges on it at a velocity v_b, is

$$j = \frac{v_b}{V}.$$

As a result, the cross section is

$$d\sigma = \frac{dw_{fi}}{j} = \frac{1}{2m_a 2E_b v_b} |M|^2 d\Phi.$$

The quantity $I = m_a E_b v_b = m_a |\boldsymbol{p}_b|$ can be written in an invariant form:

$$I = \sqrt{(p_a p_b)^2 - p_a^2 p_b^2}$$

and finally we obtain:

$$d\sigma = \frac{1}{4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2}} |M|^2 d\Phi.$$

29.4.3. *Particles with non-zero spin*

So far we have been discussing the case of zero-spin particles. The formulas derived above are readily generalized to the case of particles with arbitrary spin. The most typical situation is the calculation of Γ and σ when the polarization states of the initial particles are not fixed, and those of the final particles are not measured. In this case

$$\mathrm{d}\Gamma = \frac{1}{2J_a + 1}\frac{1}{2E_a}\overline{|M|^2}\,\mathrm{d}\Phi,$$

where J_a is the spin of the decaying particle, and

$$\mathrm{d}\sigma = \frac{1}{(2J_a + 1)(2J_b + 1)}\frac{1}{4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2}}\overline{|M|^2}\,\mathrm{d}\Phi,$$

where J_a and J_b are spins of colliding particles. The bar over $|M|^2$ denotes summation over spin states of both the initial and final particles. The factors $1/(2J_a + 1)$ and $1/(2J_a + 1)(2J_b + 1)$ take into account that we actually need to carry out averaging over polarization states of the initial particles and not the summation.

Summation over polarization states of particles with spin $\frac{1}{2}$ is easily realized by means of the relativistic-invariant density matrix:

$$\sum_s u_i(s)\bar{u}^k(s) = (\hat{p} + m)_i^k,$$

where p is the 4-momentum of the particle, and m is its mass. If the spin state, s, of the particle is fixed, then

$$u(s)\bar{u}(s) = \tfrac{1}{2}(\hat{p} + m)(1 - \gamma_5\hat{s})$$

(and $v(s)\bar{v}(s) = \frac{1}{2}(\hat{p} - m)(1 - \gamma_5\hat{s})$ for an antiparticle with 4-momentum p), where

$$s^\mu = \begin{cases} s^0 = \mathbf{p}\cdot\boldsymbol{\xi}/m \\ \mathbf{s} = \boldsymbol{\xi} + (\mathbf{p}\cdot\boldsymbol{\xi})\mathbf{p}/m(m + E) \end{cases}$$

and $\boldsymbol{\xi}$ is a unit vector in the direction of polarization of the particle in its rest frame. It can easily be shown that

$$s^2 = -1, \qquad sp = 0.$$

The relativistic-invariant density matrix, summed over spin states of a massive spin-1 particle, has the form:

$$\sum_s \varphi_\mu(s)\varphi_\nu^*(s) = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right),$$

while for the photon

$$\sum_s e_\mu(s)e_\nu^*(s) = -g_{\mu\nu}.$$

Phase space

$$d\Phi = (2\pi)^4 \delta^4\left(q - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2p_i^0} \qquad \text{dim } -4+2n$$

$q = k_1 + k_2$ for scattering $\qquad p_i^0 = \sqrt{\mathbf{p}_i^2 + m_i^2}$

◇ Lorentz invariant

take the same normalization for fermions and bosons ($\bar{u}u = 2m$)

because

$$\frac{d^3 p_i}{2p_i^0} = d^4 p_i\, \delta(p_i^2 - m_i^2)\, \theta(p_i^0)$$

Note : if there are identical particles, divide by the symmetry factor $n!$ (for total cross section)

1-body phase space

3−4 = −1 variables!

$$d\Phi_1 = 2\pi\, \delta^4(q - p_1) \frac{d^3 p_1}{2p_1^0}$$

$$= 2\pi\, \delta^4(q-p_1)\, d^4 p_1\, \delta(p_1^2 - m_1^2)\, \theta(p_1^0)$$

$$= 2\pi\, \delta(q^2 - m_1^2)$$

resonance formation

Ecm constrained

(例) $e\bar{\nu} \to W$

$$|\mathcal{M}|^2 = g^2 m_W^2 = \frac{4\pi\alpha}{\sin^2\theta_W} m_W^2 \quad (\text{後出})$$

$$\sigma(e\bar{\nu}\to W) = \frac{1}{2s}\, \frac{1}{2} |\mathcal{M}|^2 \cdot 2\pi\,\delta(s - m_W^2) = \frac{2\pi^2 \alpha}{\sin^2\theta_W}\, \delta(s - m_W^2)$$

$$\sigma(u\bar{d} \to W) = \frac{\pi^2 \alpha}{3\sin^2\theta_W}\, \delta(s - m_W^2)$$

Unitarity → Breit-Wigner shape

$$\frac{1}{(s-m_W^2)^2 + m_W^2 \Gamma_W^2} \sim \frac{\pi}{m_W \Gamma_W}\, \delta(s - m_W^2)$$

2-body phase space

6−4 = 2 variables

Constraints for p_i's

$$d\Phi_2 = \frac{1}{(2\pi)^2}\, \delta^4(q - p_1 - p_2) \frac{d^3 p_1}{2p_1^0} \frac{d^3 p_2}{2p_2^0} \qquad p_1^2 = m_1^2,\ p_2^2 = m_2^2$$

$$= \frac{1}{(2\pi)^2}\, \delta^4(q - p_1 - p_2) \frac{d^3 p_1}{2p_1^0}\, d^4 p_2\, \delta(p_2^2 - m_2^2)\, \theta(p_2^0) \qquad p_1^2 = m_1^2$$

$$= \frac{1}{(2\pi)^2} \frac{d^3 p_1}{2p_1^0}\, \delta((q-p_1)^2 - m_2^2)\, \theta(q^0 - p_1^0) \qquad p_1^2 = m_1^2,\ p_2 = q - p_1$$

$d^3 p_1 = \mathrm{p}_1^2\, d\mathrm{p}_1\, d\Omega_1 \qquad (\mathrm{p}_1 = |\mathbf{p}_1|)$

$(p_1^0)^2 = \mathrm{p}_1^2 + m_1^2$

$p_1^0\, dp_1^0 = \mathrm{p}_1\, d\mathrm{p}_1$

$= \mathrm{p}_1\, p_1^0\, dp_1^0\, d\Omega_1$

c.m. frame

$q = (\sqrt{s}, 0, 0, 0)$

$(q - p_1)^2 = q^2 - 2q\cdot p_1 + p_1^2 = s - 2\sqrt{s}\, p_1^0 + m_1^2$

$\delta((q-p_1)^2 - m_2^2) = \delta(s + m_1^2 - m_2^2 - 2\sqrt{s}\, p_1^0)$

$$= \frac{1}{2\sqrt{s}}\, \delta\left(p_1^0 - \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}\right)$$

$$= \frac{1}{(2\pi)^2} \frac{1}{2} \mathrm{p}_1\, dp_1^0\, d\Omega_1\, \frac{1}{2\sqrt{s}}\, \delta(p_1^0 - \cdots)\, \theta(\sqrt{s} - p_1^0)$$

$$\mathrm{p}_1 = \sqrt{(p_1^0)^2 - m_1^2} \equiv \frac{\sqrt{s}}{2}\, \bar{\beta}_f$$

$$= \frac{\bar{\beta}_f}{32\pi^2}\, d\Omega_1$$

$$\left(= \frac{\bar{\beta}_f}{16\pi}\, d\cos\theta_1 \right)$$

→ $\dfrac{dt}{8\pi s \bar{\beta}_i}$

$p_1^0 = \dfrac{s + m_1^2 - m_2^2}{2\sqrt{s}}$

$p_2^0 = \dfrac{s - m_1^2 + m_2^2}{2\sqrt{s}}$

$|\mathbf{p}_1| = |\mathbf{p}_2| = \dfrac{\sqrt{s}}{2} \bar{\beta}_f$

$\cos\theta_2 = -\cos\theta_1$

$\phi_2 = \phi_1 + \pi$

$$\bar{\beta}_f = \frac{1}{s^2}\left[s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2\right]$$

both massless → $\bar{\beta}_f = 1$

equal mass → $\bar{\beta}_f = \beta_f = \sqrt{1 - \frac{4m^2}{s}}$

one massless → $\bar{\beta}_f = 1 - \frac{m^2}{s}$

3-body phase space

$$d\Phi_3 = \frac{1}{(2\pi)^5}\delta^4(q-p_1-p_2-p_3)\frac{d^3p_1}{2p_1^0}\frac{d^3p_2}{2p_2^0}\frac{d^3p_3}{2p_3^0}$$

$$= \frac{1}{(2\pi)^5}\frac{d^3p_1}{2p_1^0}\frac{d^3p_2}{2p_2^0}\delta((q-p_1-p_2)^2-m_3^2)\,\theta(\cdots)$$

$$= \frac{1}{4(2\pi)^5}\, p_1 dp_1^0 d\Omega_1\, p_2 dp_2^0 d\Omega_2\, \delta(q^2-2q\cdot(p_1+p_2)+2p_1\cdot p_2+m_1^2+m_2^2-m_3^2)$$

↙ c.m. frame

$$= \frac{1}{4(2\pi)^5}\, p_1 p_2\, dp_1^0 dp_2^0 d\Omega_1\, \underline{d\cos\theta_{12}}\, d\phi_{12}\, \delta(-2p_1p_2\cos\theta_{12}+\cdots)$$

$$= \frac{1}{8(2\pi)^5}\, dp_1^0 dp_2^0 d\Omega_1 d\phi_{12}$$

$$= \frac{q^2}{1024\pi^5}\, dz_1 dz_2 d\Omega_1 d\phi_{12}$$

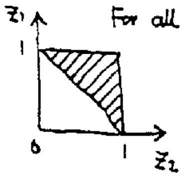
$$\begin{cases} z_1 = \dfrac{2q\cdot p_1}{q^2} = \dfrac{2p_1^0}{\sqrt{q^2}}\Big|_{cm} \\ z_2 = \dfrac{2q\cdot p_2}{q^2} \end{cases}$$

(Normalized energy)

Decay of an unpolarized particle

→ $d\Omega_1 d\phi_{12} = 8\pi^2$

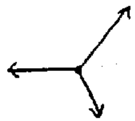
$$= \frac{q^2}{128\pi^3}\, dz_1 dz_2 \quad (*)$$



For all massless final states

$0 \le z_1, z_2 \le 1$

$z_1 + z_2 \ge 1$



3体decayのconfigurationは2粒子のenergyだけで完全に決まる。

$z_3 = \dfrac{2q\cdot p_3}{q^2}$ を定義すると $z_1+z_2+z_3 = 2.$

上の表式(*)で dz_1dz_2 のかわりに dz_1dz_3 または dz_2dz_3 としてもよい。

宿題 1) $p_1^2 = m^2$, $p_2^2 = p_3^2 = 0$ のとき $\{z_i\}$ の boundary を求めよ。

2) 一般の mass のとき 〃 .

Spin sum をとる場合の $\Sigma|M|^2$ の計算法

フェルミオンの取扱 --- Trace technique

$$\sum_{spins}\left|\bar{u}(p)\,\Gamma\, v(\bar{p})\right|^2 = \mathrm{Tr}(\not{p}+m)\,\Gamma\,(\not{\bar{p}}-m')\,\bar{\Gamma}$$

$$\bar{\Gamma} \equiv \gamma_0 \Gamma^\dagger \gamma_0$$

$$[\bar{u}(p)\Gamma v(\bar{p})]^* = v^\dagger(\bar{p})\Gamma^\dagger\gamma_0^\dagger u(p)$$

$$= \bar{v}(\bar{p})\underbrace{\gamma_0\Gamma^\dagger\gamma_0}_{\bar{\Gamma}} u(p)$$

$$\sum\left|\bar{u}(p)\Gamma v(\bar{p})\right|^2 = \sum \bar{u}_\alpha(p)\Gamma_{\alpha\beta} v_\beta(\bar{p})\, \bar{v}_\gamma(\bar{p})\, \bar{\Gamma}_{\gamma\delta}\, u_\delta(p)$$

$$\sum_{spin} u_\alpha(p)\bar{u}_\beta(p) = (\not{p}+m)_{\alpha\beta}$$

$$\sum v_\alpha(p)\bar{v}_\beta(p) = (\not{p}-m)_{\alpha\beta}$$

$$= (\not{p}+m)_{\delta\alpha}\Gamma_{\alpha\beta}(\not{\bar{p}}-\bar{m})_{\beta\gamma}\bar{\Gamma}_{\gamma\delta}$$

$$= \mathrm{Tr}(\not{p}+m)\Gamma(\not{\bar{p}}-\bar{m})\bar{\Gamma}$$

$\bar{\Gamma}$ vs Γ

$$\bar{\gamma}_\mu = \gamma_\mu$$

$$\overline{\gamma_\mu\gamma_\nu\cdots\gamma_\omega} = \gamma_\omega\cdots\gamma_\nu\gamma_\mu$$

$$\overline{\gamma_\mu\gamma_5} = \gamma_\mu\gamma_5$$

$$\bar{\gamma}_5 = -\gamma_5$$