Moduli spaces of algebraic curves and automorphic forms

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Abstract

In this series of three lectures, we will review some results on arithmetic geometry of algebraic curves and their moduli space. In particular, we will explain how Schottky uniformization of Riemann surfaces is extended in arithmetic geometry, and is applied to studying Teichmüller modular forms which are defined as automorphic forms on the moduli space of curves.

In the first lecture, we consider the arithmetic Schottky uniformization theory which constructs generalized Tate curves, and show that their multiplicative periods, called universal periods, are computable integral power series. In the second lecture, using the evaluation theory on the generalized Tate curves we study arithmetic properties of Teichmüller modular forms, and apply our result to the geometry of the moduli space of curves via Mumford's isomorphism and Klein's amazing formula. In the third lecture, by Teichmüller modular forms and nonarchimedean theta functions, we consider the Schottky problem characterizing the Jacobian locus and Jacobian varieties, and give algebraic and rigid analytic versions of results of Shiota and Krichever.

Contents

Introduction

In this note, we will review some results about algebraic curves and their moduli spaces which are focused on the author's interest. In particular, we will explain arithmetic Schottky uniformization theory, and its applications to Teichmüller modular forms and to the Schottky problem. Here we do not review the application of this theory to Galois and monodromy representations associated with Teichmüller modular groups (cf. [IN] and [I7, 8]).

In Section 1, we first recall the Schottky uniformization theory [S] which is useful to study nearly degenerate Riemann surfaces and their 1-forms and periods. Then we show that this classical theory and its nonarchimedean version given by Mumford [Mu2] are unified as the arithmetic Schottky uniformization theory. By this theory, we obtain a generalized Tate curve, i.e., a higher genus version of the Tate curve which is a uniformized stable curve as a universal deformation of degenerate curves with fixed dual graph. The multiplicative periods of this curve are computable integral power series which we call universal periods.

In Section 2, we study Teichmüller modular forms which are defined as global sections of line bundles on the moduli space of curves of fixed genus > 1. Note that this moduli space is not an arithmetic quotient of a symmetric domain, and hence Teichmüller modular forms are not ordinary modular forms arising from algebraic groups in general. Our main tool is their expansion theory based on the arithmetic Schottky uniformization. We consider the $\mathbb{Z}$-module of integral Teichmüller modular forms of fixed weight and the ring of these forms of all weights, and show that the finiteness of this rank and the number of generators respectively. Furthermore, we construct an integral and primitive Teichmüller modular form proportional to a square root of the product of even theta constants. This form is seen to be connected with the geometry of the moduli of curves via Mumford's isomorphism [Mu4] and Klein's amazing formula [K].

In Section 3, we consider the Schottky problem in the two formulations; to characterize Jacobian varieties among abelian varieties, and to characterize the Jacobian locus in the moduli of abelian varieties. First, we discuss the latter formulation since the universal periods are directly applied to characterize Siegel modular forms vanishing on the Jacobian locus. Further, following [LRZ] we use the above Teichmüller modular form to answer Serre's question of characterizing Jacobian varieties of dimension 3 over a subfield of $\mathbb{C}$. Second, we study the former formulation by showing that there are algebraic and rigid analytic versions of results of Shiota [Sh] and Krichever [Kr]. In particular, we construct solutions of new type to the KP equation from nonarchimedean theta functions, and characterize Jacobian varieties via tangential trisecant conditions of finite order over fields of characteristic $\neq 2$.

Keywords and their relations

§1. Arithmetic of Schottky uniformization

Generalized Tate curves with universal periods

complex geometry ↙ ↘ rigid geometry

Schottky uniformized Riemann surfaces

Mumford curves over local fields

§2. Teichmüller modular forms

Teichmüller modular forms $\xrightarrow{\text{universal periods}}$ Power series representations

roots of $\Downarrow$ theta constants

Geometry of moduli of curves (Mumford's iso., Klein's formula)

§3. Schottky problem

Universal periods
Teichmüller modular forms
(Nonarch.) theta functions

↙ ↘

Characterizing Jacobian locus

Characterizing Jacobian varieties

§1. Arithmetic of Schottky uniformization

1.1. Schottky uniformization and periods

Schottky uniformization is to construct Riemann surfaces of genus g from a $2g$ holed Riemann sphere by identifying these holes in pairs. More precisely, put

$$PGL_2(\mathbb{C}) \overset{\text{def}}{=} GL_2(\mathbb{C})/\,\mathbb{C}^\times$$

which acts on $\mathbb{P}^1(\mathbb{C})$ by the Möbius transformation:

$$\gamma(z) = \frac{az+b}{cz+d} \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathrm{mod}(\mathbb{C}^\times) \in PGL_2(\mathbb{C}),\ z \in \mathbb{P}^1(\mathbb{C})\right),$$

and let

$$\begin{aligned}
&D_{\pm 1}, ..., D_{\pm g} \subset \mathbb{P}^1(\mathbb{C}) :\ \text{disjoint closed domains bounded by Jordan curves } \partial D_i,\\
&\gamma_1, ..., \gamma_g \in PGL_2(\mathbb{C}) \text{ such that } \gamma_i(\mathbb{P}^1(\mathbb{C}) - D_{-i}) = \text{the interior } D_i^\circ \text{ of } D_i,\\
&\Gamma \overset{\text{def}}{=} \langle \gamma_1, ..., \gamma_g \rangle :\ \text{the subgroup of } PGL_2(\mathbb{C}) \text{ generated by } \gamma_1, ..., \gamma_g,\\
&\Omega_\Gamma \overset{\text{def}}{=} \bigcup_{\gamma \in \Gamma} \gamma\left(\mathbb{P}^1(\mathbb{C}) - \bigcup_{i=1}^{g} (D_i^\circ \cup D_{-i}^\circ)\right).
\end{aligned}$$

Then the Riemann surface

$$\begin{aligned}
R_\Gamma &\overset{\text{def}}{=} \left(\mathbb{P}^1(\mathbb{C}) - \bigcup_{i=1}^{g} (D_i^\circ \cup D_{-i}^\circ)\right) \Big/ \partial D_i \overset{\gamma_i}{\sim} \partial D_{-i} \ (:\ \text{gluing by } \gamma_i)\\
&= \Omega_\Gamma/\Gamma
\end{aligned}$$

is called (Schottky) uniformized by the **Schottky group** Γ. It is known that any Riemann surface can be Schottky uniformized. The counterclockwise oriented boundaries ∂D_i and oriented paths from $w_i \in \partial D_{-i}$ to $\gamma_i(w_i) \in \partial D_i$ $(1 \le i \le g)$ give symplectic basis of $H_1(R_\Gamma, \mathbb{Z})$, which we denote them by α_i, β_i respectively.

Remark. Γ is a free group with generators $\gamma_1, ..., \gamma_g$, and the action of Γ on Ω_Γ is free and properly discontinuous. Further, each γ_i $(1 \le i \le g)$ is uniquely represented by

$$\gamma_i = \begin{pmatrix} t_i & t_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & s_i \end{pmatrix} \begin{pmatrix} t_i & t_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \mathrm{mod}(\mathbb{C}^\times),$$

where $t_i \in D_i^\circ$, $t_{-i} \in D_{-i}^\circ$ and $|s_i| < 1$ (hence γ_i is hyperbolic (or loxodromic)), and

$$t_{\pm i} = \lim_{n \to \infty} \gamma_i^{\pm n}(z) \ (z \in \Omega_\Gamma).$$

t_i, t_{-i} are called the **attractive**, **repulsive** fixed point of γ_i respectively, and s_i is called the **multiplier** of γ_i.

Theorem 1.1. (Schottky [S]) *Assume that* $\infty \in \Omega_\Gamma$ *and that* $\sum_{\gamma\in\Gamma} |\gamma'(z)|$ *converges uniformly on any compact subset of*

$$\Omega_\Gamma - \bigcup_{\gamma\in\Gamma} \gamma(\infty).$$

Then we have

(1) *For* $n \geq 1$ *and a point* $p \in \Omega_\Gamma - \bigcup_{\gamma\in\Gamma} \gamma(\infty)$,

$$w_{n,p}(z) \overset{\text{def}}{=} \sum_{\gamma\in\Gamma} \frac{d\gamma(z)}{(\gamma(z)-p)^n} = \sum_{\gamma\in\Gamma} \frac{\gamma'(z)}{(\gamma(z)-p)^n} dz$$

becomes a meromorphic 1*-form on* R_Γ*. If* $n > 1$*, then* $w_{n,p}$ *is of the* 2*nd kind, more precisely, it has only poles of order* n *at the point* $\overline{p}$ *on* R_Γ *given by* p*, and if* $n = 1$*, then* $w_{n,p}$ *is of the* 3*rd kind, more precisely, it has only simple poles at* $\overline{p}, \overline{\infty}$*. Furthermore, for* $n \geq 0$,

$$\sum_{\gamma\in\Gamma} \gamma(z)^n d\gamma(z) = \sum_{\gamma\in\Gamma} \gamma(z)^n \cdot \gamma'(z) dz$$

becomes a meromorphic 1*-form on* R_Γ *which has only pole of order* $n+2$ *at* $\overline{\infty}$.

(2) *For* $i = 1, ..., g$,

$$\omega_i(z) = \frac{1}{2\pi\sqrt{-1}} \sum_{\gamma\in\Gamma/\langle\gamma_i\rangle} \left(\frac{1}{z-\gamma(t_i)} - \frac{1}{z-\gamma(t_{-i})} \right) dz$$

give a base of $H^0(R_\Gamma, \Omega_{R_\Gamma})$ *satisfying that* $\displaystyle\int_{\alpha_i} \omega_j = \delta_{ij}$.

(3) *For* $1 \leq i, j \leq g$ *and* $\gamma \in \Gamma$*, put*

$$\psi_{ij}(\gamma) = \begin{cases} s_i & (\textit{if } i = j \textit{ and } \gamma \in \langle\gamma_i\rangle), \\ \dfrac{(t_i - \gamma(t_j))(t_{-i} - \gamma(t_{-j}))}{(t_i - \gamma(t_{-j}))(t_{-i} - \gamma(t_j))} & (\textit{otherwise}). \end{cases}$$

Then we have

$$\exp\left(2\pi\sqrt{-1} z_{ij}\right) = \prod_{\gamma\in\langle\gamma_i\rangle\backslash\Gamma/\langle\gamma_j\rangle} \psi_{ij}(\gamma),$$

where $Z = (z_{ij})_{i,j}$ *is the period matrix of* $(R_\Gamma; (\alpha_i, \beta_i)_{1\leq i\leq g})$.

Sketch of Proof. The assertion (1) is evident except the convergence of $w_{n,p}(z)$ which follows from the assumption and that the action of Γ on Ω_Γ is properly discontinuous.

Further, $w_{1,p}(z)$ has simple poles at $\overline{p}, \overline{\infty}$ with residues $1, -1$ respectively, and satisfies that $\int_{\alpha_i} w_{1,p} = 0$ $(1 \leq i \leq g)$. Then by Riemann's period relation,

$$
\begin{aligned}
2\pi\sqrt{-1}\omega_i(z) &= d\left(\int_{\zeta_i}^{\gamma_i(\zeta_i)} \sum_{\gamma\in\Gamma} \frac{d\gamma(\zeta)}{\gamma(\zeta)-z}\right);\ \zeta_i \text{ is a point on } \partial D_{-i} \\
&= \left(\sum_{\gamma\in\Gamma} \log\left(\frac{(\gamma\gamma_i)(\zeta_i)-z}{\gamma(\zeta_i)-z}\right)\right) dz \\
&= \sum_{\gamma\in\Gamma}\left(\frac{1}{z-(\gamma\gamma_i)(w_i)} - \frac{1}{z-\gamma(w_i)}\right) dz \\
&= \sum_{\gamma\in\Gamma/\langle\gamma_i\rangle} \sum_{n\in\mathbb{Z}} \left(\frac{1}{z-(\gamma\gamma_i^{n+1})(w_i)} - \frac{1}{z-(\gamma\gamma_i^{n})(w_i)}\right) dz,
\end{aligned}
$$

and since $t_{\pm i} = \lim_{n\to\infty} \gamma_i^{\pm n}(w_i) \in D_{\pm i}^{\circ}$, we have

$$
\omega_i(z) = \frac{1}{2\pi\sqrt{-1}} \sum_{\gamma\in\Gamma/\langle\gamma_i\rangle} \left(\frac{1}{z-\gamma(t_i)} - \frac{1}{z-\gamma(t_{-i})}\right) dz,
$$

which proves (2). Finally, one has (3) by similar calculation. □

Remark. Under that $\Omega_\Gamma \ni \infty$ and that $t_{\pm i}$ are fixed and s_i are sufficiently small, one can show that the assumption in Theorem 1.1 is satisfied as follows.

For 2 disks $D_i, D_j \subset \mathbb{C}$ with radius r_i, r_j respectively, put

$$
\begin{aligned}
\rho_{i,j} &: \text{ the distance between the centers of } D_i \text{ and } D_j, \\
K_{i,j} &= \frac{(r_i^2+r_j^2-\rho_{i,j}^2)^2}{4r_i^2 r_j^2} - 1 \geq 0, \\
L_{i,j} &= \frac{1}{\sqrt{1+K_{i,j}}+\sqrt{K_{i,j}}} \leq 1.
\end{aligned}
$$

Then $K_{i,j}$ and $L_{i,j}$ are invariant under any Möbius transformation, and $r_i \leq L_{i,j} \cdot r_j$ if $D_i \subset D_j$. Under the assumption, one can take disks $D_{\pm 1}, ..., D_{\pm g}$ such that the sum of $L_{i,j}$ $(i, j \in \{\pm 1, ..., \pm g\}, i \neq j)$ is smaller than 1. Hence by the above, there is a positive constant C such that if $\gamma = \prod_{s=1}^{l} \gamma_{k(s)} \in \Gamma$ is expressed as

$$
\begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \bmod(\mathbb{C}^\times); \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in SL_2(\mathbb{C}),
$$

then

$$
\frac{1}{|c_\gamma|^2} \leq C \cdot \prod_{s=1}^{l-1} L_{-k(s),k(s+1)}.
$$

Therefore,

$$\sum_{\gamma\in\Gamma-\{1\}} \frac{1}{|c_\gamma|^2} \leq C\cdot\sum_{m=0}^{\infty}\left(\sum_{i\neq j} L_{i,j}\right)^m < \infty,$$

and hence

$$\sum_{\gamma\in\Gamma} |\gamma'(z)| \leq 1 + \frac{1}{d(z)^2}\sum_{\gamma\in\Gamma-\{1\}}\frac{1}{|c_\gamma|^2}$$

satisfies the condition since $d(z) \stackrel{\text{def}}{=} \min\{|z-\gamma^{-1}(\infty)|; \gamma\in\Gamma\} > 0$ is bounded on any compact subset outside $\bigcup_{\gamma\in\Gamma}\gamma(\infty)$.

Schottky [S] gives a (more geometric) convergence condition on $\sum_{\gamma\in\Gamma}|\gamma'(z)|$ as follows: all $\partial D_{\pm i}$ can be taken as circles (in this case, Γ is called classical) and there are $2g-3$ circles $C_1, ..., C_{2g-3}$ in $F = \mathbb{P}^1(\mathbb{C}) - \bigcup_{i=1}^{g}\left(D_i^\circ \cup D_{-i}^\circ\right)$ satisfying that

- $C_1, ..., C_{2g-3}$, $\partial D_{\pm 1}, ..., \partial D_{\pm g}$ are mutually disjoint;
- $C_1, ..., C_{2g-3}$ divide F into $2g-2$ domains $R_1, ..., R_{2g-2}$;
- each R_i has exactly three boundary circles.

Variation of forms and periods. Let $\Gamma = \langle\gamma_1, ..., \gamma_g\rangle$ be a Schottky group of rank g as above, and put $\Gamma' = \langle\gamma_1, ..., \gamma_{g-1}\rangle$ which is a Schottky group of rank $g-1$. If the multiplier

$$\begin{aligned} s_g &= \frac{\gamma_g(z)-t_g}{z-t_g}\cdot\frac{z-t_{-g}}{\gamma_g(z)-t_{-g}} \\ &: \ \text{the product of local coordinates around } t_g, t_{-g} \text{ respectively} \end{aligned}$$

of γ_g tends to 0, then by Theorem 1.1,

- $R_\Gamma \longrightarrow \begin{cases} \text{the singular curve } R'_{\Gamma'} \text{ with unique singular (ordinary double) point} \\ \text{obtained by identifying 2 points } t_g, t_{-g} \in R_{\Gamma'}; \end{cases}$
- $$2\pi\sqrt{-1}\,\omega_i(z) = \sum_{\gamma\in\Gamma/\langle\gamma_i\rangle}\left(\frac{1}{z-\gamma(t_i)} - \frac{1}{z-\gamma(t_{-i})}\right)dz \in H^0\left(R_\Gamma, \Omega_{R_\Gamma}\right)$$
  $$\longrightarrow \begin{cases} \displaystyle\sum_{\gamma\in\Gamma'/\langle\gamma_i\rangle}\left(\frac{1}{z-\gamma(t_i)} - \frac{1}{z-\gamma(t_{-i})}\right)dz & (i<g), \\ \displaystyle\left(\frac{1}{z-t_g} - \frac{1}{z-t_{-g}}\right)dz + \cdots & (i=g) \end{cases}$$
  which has a pole at the ordinary double point $t_g = t_{-g}$ on $R'_{\Gamma'}$ if $i=g$;
- (Fay's formula [F]) $p_{ij} \longrightarrow \begin{cases} \text{the multiplicative periods of } R_{\Gamma'} & (i,j<g), \\ 0 & (i=j=g). \end{cases}$

Remark. We can obtain variational formula under other degenerations (see [I5]).

Mumford curves. Mumford [Mu2] gave a higher genus version of the Tate curve over complete local domains as an analogy of Schottky uniformization theory, i.e., for a complete integrally closed noetherian local ring R with quotient field K, and a Schottky group $\Gamma \subset PGL_2(K)$ over K which is *flat* over R, he constructed a **Mumford curve** C_Γ over K. By definition, a Mumford curve over K is a proper smooth curve over K which has a model as a stable curve over R such that the normalization of each irreducible component of its special fiber has genus 0, and that each singular point of the special fiber is rational over the residue field of R. Furthermore, he showed that $\Gamma \mapsto C_\Gamma$ gives rise to the following bijection:

$$\left\{ \begin{array}{c} \text{Conjugacy classes of flat} \\ \text{Schottky groups over } K \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{Mumford curves over } K \end{array} \right\}.$$

Assume that K is a complete valuation field. Then any Schottky group Γ over K is flat over its valuation ring, and using rigid analytic geometry, Gerritzen and van der Put [GP] constructed C_Γ as the quotient by Γ of its region of discontinuity Ω_Γ in $\mathbb{P}^1(K)$:

$$C_\Gamma = \Omega_\Gamma/\Gamma; \ \Omega_\Gamma \overset{\text{def}}{=} \mathbb{P}^1(K) - \overline{\{\text{fixed points of } \Gamma - \{1\}\}}.$$

Further, Manin and Drinfeld [MD] showed that the Jacobian variety $\mathrm{Jac}\,(C_\Gamma)$ of C_Γ is K-isomorphic to the Mumford's abelian variety [Mu3] associated with

$$\left(K^\times\right)^g \big/ \left\langle (p_{ij})_{1 \le i \le g} \ \middle| \ 1 \le j \le g \right\rangle,$$

where the multiplicative periods p_{ij} of C_Γ are given by

$$\prod_{\gamma \in \langle \gamma_i \rangle \backslash \Gamma / \langle \gamma_j \rangle} \psi_{ij}(\gamma)$$

as in Theorem 1.1 (3).

Remark. In [I4], Schottky uniformization theory is extended for Riemann surfaces and Mumford curves of infinite genus.

1.2. Generalized Tate curves and universal periods

Degenerate curves and dual graphs. A **degenerate curve** over a field is a stable curve such that the normalization of each irreducible component has genus 0. For a degenerate curve, by the correspondence:

$$\begin{array}{rcl} \text{its irreducible components} & \longleftrightarrow & \text{vertices} \\ \text{its singular points} & \longleftrightarrow & \text{edges} \end{array}$$

(an irreducible component contains a singular point if and only if the corresponding vertex is contained in (or adjacent to) the corresponding edge), we have its **dual graph** which becomes a **stable graph**, i.e., a connected and finite graph whose vertices have at least 3 branches. For a degenerate curve C with dual graph Δ,

$$\begin{aligned}\text{the genus of } C &= \text{rank}_{\mathbb{Z}} H_1(\Delta, \mathbb{Z}) \\ &= \text{the number of generators of the free group } \pi_1(\Delta).\end{aligned}$$

Since any triplet of distinct points on $\mathbb{P}^1$ is uniquely translated to $(0, 1, \infty)$ by the action of PGL_2, for a stable graph Δ, the moduli space of degenerate curves with dual graph Δ has dimension

$$\sum_{v:\text{ vertices of }\Delta} (\deg(v) - 3),$$

where $\deg(v)$ denotes the number of branches ($\neq$ edges) starting from v. In particular, a stable graph is **trivalent**, i.e., all the vertices have just 3 branches if and only if the corresponding curves are **maximally degenerate**, in which case their moduli space consists of only one point.

General degenerating process. (Ihara and Nakamura [IN]). For a stable graph Δ with orientation on each edge,

$$\begin{aligned} g &\stackrel{\text{def}}{=} \text{rank}_{\mathbb{Z}} H_1(\Delta, \mathbb{Z}), \\ P_v &\stackrel{\text{def}}{=} \mathbb{P}^1(\mathbb{C}) \quad (v : \text{vertices of } \Delta). \end{aligned}$$

and for each oriented edge e ($v_{-e} \stackrel{e}{\to} v_e$) of Δ, let

$$\begin{aligned} v_e &\stackrel{\text{def}}{=} \text{the end point of } e, \\ v_{-e} &\stackrel{\text{def}}{=} \text{the starting point of } e, \\ \gamma_e &: \text{a hyperbolic element of } PGL_2(\mathbb{C}) \text{ which gives } \gamma_e : P_{v_{-e}} \stackrel{\sim}{\to} P_{v_e}, \\ t_e &: \text{the attractive fixed point of } \gamma_e \text{ on } P_{v_e}, \\ t_{-e} &: \text{the repulsive fixed point of } \gamma_e \text{ on } P_{v_{-e}}, \\ s_e &: \text{the multiplier of } \gamma_e \\ &\Rightarrow \gamma_e = \begin{pmatrix} t_e & t_{-e} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & s_e \end{pmatrix} \begin{pmatrix} t_e & t_{-e} \\ 1 & 1 \end{pmatrix}^{-1} \mod(\mathbb{C}^\times). \end{aligned}$$

Fix a vertex v_0 of Δ, and put

$$\Gamma \stackrel{\text{def}}{=} \left\{ \gamma_{e_1}^{i_1} \cdots \gamma_{e_n}^{i_n} \;\middle|\; e_k : \text{edges, } i_k \in \{\pm 1\} \text{ such that } e_n^{i_n} \cdots e_1^{i_1} \in \pi_1(\Delta; v_0) \right\}.$$

Then under the assumption that the multipliers s_e of all γ_e are sufficiently small,

- Γ is a Schottky group of rank g;
- If $\infty \in \Omega_\Gamma$, then $\sum_{\gamma \in \Gamma} |\gamma'(z)|$ converges uniformly on any compact subset of $\Omega_\Gamma - \bigcup_{\gamma \in \Gamma} \gamma(\infty)$;
- $R_\Gamma = \Omega_\Gamma / \Gamma$ is a Riemann surface of genus g obtained from holed Riemann spheres P_v (v : vertices of Δ) gluing by γ_e (e : edges of Δ);

and hence

$$\begin{aligned} & s_e \to 0 \ (e : \text{ edges of } \Delta) \\ \Rightarrow \quad & R_\Gamma \to \text{the degenerate curve } C_0 = \left(\bigcup_v P_v \right) \Big/ \begin{array}{l} t_e = t_{-e} \\ (e : \text{edges of } \Delta) \end{array} \quad \text{with dual graph } \Delta. \end{aligned}$$

Since $\mathbb{P}^1$ has only trivial deformation, R_Γ gives a universal deformation of C_0, and hence varying $t_{\pm e}$ as the **moduli parameters** $x_{\pm e}$, s_e as the **deformation parameters** y_e, R_Γ make an open subset (of dimension $3g-3$) of the moduli space of curves of genus g.

Arithmetic Schottky uniformization. This theory is an extension of the above process in terms of arithmetic geometry unifying complex geometry, rigid geometry and formal geometry over $\mathbb{Z}$. In this theory, we give a higher genus version of the Tate curve, and hence by moduli theory, its base ring denoted by B_Δ below is the local coordinate ring of the moduli space of curves around a degenerate curve. These coordinates will be seen to be useful to study arithmetic geometry of this moduli space.

Theorem 1.2. ([I5], (1)–(3) were already proved in [IN] for maximally degenerate case without singular components). *Let*

$$\begin{aligned} A_0 &\stackrel{\text{def}}{=} \textit{the affine coordinate ring (of moduli parameters } x_{\pm e}\textit{) over } \mathbb{Z} \\ &\quad\ \textit{of the moduli space of degenerate curves with dual graph } \Delta, \\ A_\Delta &\stackrel{\text{def}}{=} A_0[[y_e \ (e : \textit{edges of } \Delta)]]; \ y_e : \textit{deformation parameters}, \\ B_\Delta &\stackrel{\text{def}}{=} A_\Delta \left[1/y_e \ (e : \textit{edges of } \Delta) \right]. \end{aligned}$$

Then there exists a stable curve C_Δ (called the ***generalized Tate curve****) over A_Δ of genus $g \stackrel{\text{def}}{=} \mathrm{rank}_\mathbb{Z} H_1(\Delta, \mathbb{Z})$ satisfying:*

(1) *C_Δ is a universal deformation of the universal degenerate curve with dual graph Δ.*

(2) *By substituting $t_{\pm e} \in \mathbb{C}$ to $x_{\pm e}$ and $s_e \in \mathbb{C}^\times$ to y_e (e are edges of Δ), C_Δ becomes a Schottky uniformized Riemann surface if s_e are sufficiently small.*

(3) *C_Δ is smooth over B_Δ, and is Mumford uniformized by a Schottky group over B_Δ. Furthermore, for a complete integrally closed noetherian local ring R with quotient field K and a Mumford curve C over K such that Δ is the dual graph of its degenerate reduction, there is a ring homomorphism $A_\Delta \to R$ which gives rise to $C_\Delta \otimes_{A_\Delta} K \cong C$.*

(4) *Using Mumford's theory [Mu3] on degenerating abelian varieties, the (generalized) Jacobian of C_Δ can be expressed as*

$$\mathbb{G}_m^g / \left\langle (p_{ij})_{1 \le i \le g} \;\middle|\; 1 \le j \le g \right\rangle ; \quad \mathbb{G}_m \overset{\text{def}}{=} \textit{the multiplicative algebraic group,}$$

where the multiplicative periods p_{ij} of C_Δ (called ***universal periods****) are given as computable elements of B_Δ.*

Generalized Tate curves with universal periods

complex geometry $\swarrow$		$\searrow$ rigid geometry
Schottky uniformized Riemann surfaces	Arithmetic uniformization	Mumford curves over local fields

Sketch of Proof.

- Step 1 of constructing C_Δ is to give a Schottky group Γ_Δ over B_Δ as in the above *general degenerating process* by replacing $t_{\pm e}, s_e$ with $x_{\pm e}, y_e$ respectively, and show that Γ_Δ is flat over A_Δ.
- Step 2 is, following argument in [Mu2], to show that the collection of sets consisting of 3 fixed points in $\mathbb{P}^1$ of $\Gamma_\Delta - \{1\}$ gives a tree which is the universal cover of Δ, and to construct C_Δ as the quotient by $\pi_1(\Delta) \cong \Gamma_\Delta$ of the glued scheme of $\mathbb{P}^1_{A_\Delta}$ associated with this tree using Grothendieck's formal existence theorem.
- In order to give a power series expansion of p_{ij}, use the infinite product presentation by Schottky [S], Manin and Drinfeld [MD] of the multiplicative periods given in Theorem 1.1 (3). $\square$

Example 1.3. ([I1]). When Δ consists of one vertex and g loops, degenerate curves with dual graph Δ are obtained from $\mathbb{P}^1$ with $2g$ points $x_{\pm 1}, ..., x_{\pm g}$ by identifying $x_i = x_{-i}$ $(1 \le i \le g)$. Then

$$\begin{aligned} A_0 &= \mathbb{Z}\left[\frac{(x_i - x_j)(x_k - x_l)}{(x_i - x_l)(x_k - x_j)} \left(\begin{array}{ccc} i, j, k, l & \in & \{\pm 1, ..., \pm g\} \\ & : & \text{mutually different} \end{array} \right) \right], \\ A_\Delta &= A_0[[y_1, ..., y_g]], \end{aligned}$$

and C_Δ is uniformized by

$$\Gamma_\Delta \overset{\text{def}}{=} \left\langle \phi_i \overset{\text{def}}{=} \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y_i \end{pmatrix} \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \bmod(\mathbb{G}_m) \;\middle|\; 1 \le i \le g \right\rangle.$$

Hence by Theorem 1.1 (3),

$$p_{ij} \;=\; \prod_{\phi\in\langle\phi_i\rangle\backslash\Gamma_\Delta/\langle\phi_j\rangle} \psi_{ij}(\phi),$$

where

$$\psi_{ij}(\phi) = \begin{cases} y_i & (\text{if } i=j \text{ and } \phi\in\langle\phi_i\rangle), \\ \dfrac{(x_i-\phi(x_j))(x_{-i}-\phi(x_{-j}))}{(x_i-\phi(x_{-j}))(x_{-i}-\phi(x_j))} & (\text{otherwise}). \end{cases}$$

Let I_Δ be the ideal of A_Δ generated by $y_1, ..., y_g$, and put $\phi_{-i} \overset{\text{def}}{=} \phi_i^{-1}$ $(1 \le i \le g)$. Then

$$\Phi_{ij} = \left\{ \phi = \phi_{\sigma(1)}\cdots\phi_{\sigma(n)} \;\middle|\; \begin{array}{l} \sigma(1)\neq \pm i,\ \sigma(n)\neq\pm j, \\ \sigma(k)\neq -\sigma(k+1)\ (1\le k\le n-1) \end{array} \right\}$$

gives a set of complete representatives of $\langle\phi_i\rangle\backslash\Gamma_\Delta/\langle\phi_j\rangle$. For $\phi = \phi_{\sigma(1)}\cdots\phi_{\sigma(n)} \in \Phi_{ij}$, $\phi(x_{\pm j}) \in x_{\sigma(1)} + I_\Delta$, and hence by putting $\phi' = \phi_{\sigma(2)}\cdots\phi_{\sigma(n)}$

$$\begin{aligned} & \phi(x_j)-\phi(x_{-j}) \\ = \;& \frac{(x_{\sigma(1)}-x_{-\sigma(1)})^2(\phi'(x_j)-\phi'(x_{-j}))y_{\sigma(1)}}{(\phi'(x_j)-x_{-\sigma(1)}-y_{\sigma(1)}(\phi'(x_j)-x_{\sigma(1)}))(\phi'(x_{-j})-x_{-\sigma(1)}-y_{\sigma(1)}(\phi'(x_{-j})-x_{\sigma(1)}))} \\ = \;& \cdots \in I_\Delta^n. \end{aligned}$$

by inductive calculus, and hence

$$\frac{(x_i-\phi(x_j))(x_{-i}-\phi(x_{-j}))}{(x_i-\phi(x_{-j}))(x_{-i}-\phi(x_j))} = 1 + \frac{(x_i-x_{-i})(\phi(x_j)-\phi(x_{-j}))}{(x_i-\phi(x_{-j}))(x_{-i}-\phi(x_j))} \;\in\; 1+I_\Delta^n.$$

Therefore,

$$p_{ij} \;=\; c_{ij}\left(1+\sum_{|k|\neq i,j} \frac{(x_i-x_{-i})(x_j-x_{-j})(x_k-x_{-k})^2}{(x_i-x_k)(x_{-i}-x_k)(x_j-x_{-k})(x_{-j}-x_{-k})} y_{|k|} + \cdots\right),$$

where

$$c_{ij} \;\overset{\text{def}}{=}\; \begin{cases} y_i & (\text{if } i=j), \\ \dfrac{(x_i-x_j)(x_{-i}-x_{-j})}{(x_i-x_{-j})(x_{-i}-x_j)} & (\text{if } i\neq j). \end{cases}$$

Remark. Denote by

T_g	:	the Teichmüller space of degree g,
S_g	:	the Schottky space of degree g (the moduli space of Schottky groups with free g generators),
M_g	:	the moduli space of Riemann surfaces of genus g
H_g	:	the Siegel upper half space of degree g,
A_g	:	the moduli space of principally polarized complex abelian varieties of dimension g.

Then

$$\begin{array}{cccl} T_g & \xrightarrow{p} & H_g & \text{: the period map (transcendental)} \\ \downarrow & & \downarrow \exp(2\pi\sqrt{-1}\cdot) & \\ S_g & \longrightarrow & H_g/\mathbb{Z}^{g(g+1)/2} & \text{: computable as power series} \\ \downarrow & & \downarrow & \\ M_g & \xrightarrow{\tau} & A_g & \text{: the Torelli map (algebraic).} \end{array}$$

Problem. When any vertex of Δ has just 3 branches (i.e., the corresponding degenerate curve is maximally degenerate), the moduli space of degenerate curves with dual graph Δ consists of one point, and hence $A_0 = \mathbb{Z}$. Then express integral coefficients of

$$p_{ij} \ \in \ A_\Delta = \mathbb{Z}[[y_e \ (e : \text{edges of } \Delta)]]$$

by using some arithmetic functions.

§2. Teichmüller modular forms

2.1. Basic properties

Teichmüller modular forms (TMFs) are defined as

$$\begin{array}{rcl} \text{analytically} & : & \text{automorphic functions on the Teichmüller space} \\ & = & \text{automorphic forms on the moduli space of Riemann surfaces,} \\ \text{algebraically} & : & \text{global sections of line bundles on the moduli of curves.} \end{array}$$

This naming is an analogy of

$$\begin{array}{rl} & \text{Siegel modular forms (SMFs)} \\ = & \text{automorphic functions on the Siegel upper half space} \\ = & \text{global sections of line bundles} \\ & \text{on the moduli of principally polarized abelian varieties.} \end{array}$$

Definition of TMFs. In what follows, put

$$\begin{array}{rcl} \mathcal{M}_g & \stackrel{\text{def}}{=} & \text{the moduli stack over } \mathbb{Z} \text{ of proper smooth curves of genus } g, \\ \mathcal{A}_g & \stackrel{\text{def}}{=} & \text{the moduli stack over } \mathbb{Z} \text{ of principally polarized abelian schemes} \\ & & \text{of relative dimension } g. \end{array}$$

Let $\pi : \mathcal{C} \to \mathcal{M}_g$ be the universal curve, and let $\lambda \stackrel{\text{def}}{=} \bigwedge^g \pi_* \left(\Omega_{\mathcal{C}/\mathcal{M}_g} \right)$ be the **Hodge line bundle**. Then for a $\mathbb{Z}$-module M, we call elements of

$$T_{g,h}(M) \stackrel{\text{def}}{=} H^0 \left(\mathcal{M}_g, \lambda^{\otimes h} \otimes_{\mathbb{Z}} M \right)$$

Teichmüller modular forms of degree g and weight h with coefficients in M. By the pullback of the Torelli map $\tau : \mathcal{M}_g \to \mathcal{A}_g$ sending curves to their Jacobian varieties with canonical polarization, we have a linear map τ^* from the space

$$S_{g,h}(M) \stackrel{\text{def}}{=} H^0 \left(\mathcal{A}_g, \lambda^{\otimes h} \otimes_{\mathbb{Z}} M \right); \ \lambda\text{: the Hodge line bundle on } \mathcal{A}_g$$

of Siegel modular forms into $T_{g,h}(M)$ for $\mathbb{Z}$-modules M. If $g = 2, 3$, then the image of the Torelli map is Zariski dense, and hence τ^* is injective.

Analytic TMFs. If $n \geq 3$, then

$$\begin{array}{rcl} M_{g,n} & \stackrel{\text{def}}{=} & \text{the moduli space of proper smooth curves over } \mathbb{C} \\ & & \text{of genus } g \text{ with symplectic level } n \text{ structure,} \\ A_{g,n} & \stackrel{\text{def}}{=} & \text{the moduli space of principally polarized abelian varieties over } \mathbb{C} \\ & & \text{of dimension } g \text{ with symplectic level } n \text{ structure} \end{array}$$

are given as fine moduli schemes over $\mathbb{C}$. Let $M_{g,n}^*$ be the **Satake-type** compactification, i.e., the normalization of the Zariski closure of

$$(\iota \circ \tau)(M_{g,n}) \subset A_{g,n}^*,$$

where $\tau : M_{g,n} \to A_{g,n}$ denote the Torelli map, and $\iota : A_{g,n} \to A_{g,n}^*$ denote the natural inclusion to the Satake compactification. Then each point of $M_{g,n}^* - M_{g,n}$ corresponds to the product $J_1 \times \cdots \times J_m$ of Jacobian varieties over $\mathbb{C}$ with canonical polarization and symplectic level n structure such that $\sum_{i=1}^m \dim(J_i) \leq g$ and that $(m, g) \neq (1, \dim(J_1))$. Therefore, if $g \geq 3$, then $M_{g,n}^* - M_{g,n}$ has codimension 2 in $M_{g,n}^*$, and hence by applying Hartogs' theorem to $M_{g,n} \subset M_{g,n}^*$ and GAGA's principle to $M_{g,n}^*$, one can see that analytic TMFs become algebraic TMFs. Hence

$$T_{g,h}(\mathbb{C}) \cong \left\{ \begin{array}{l} \text{holomorphic functions on the Teichmüller space } T_g \\ \text{of degree } g \text{ with automorphy condition of weight } h \\ \text{for the action of the Teichmüller modular group} \end{array} \right\},$$

and this space is finite dimensional over $\mathbb{C}$ by the properness of $M_{g,n}^*$ over $\mathbb{C}$.

Expansion of TMFs. Let C_Δ be the generalized Tate curve given in Theorem 1.2 which is smooth over the ring B_Δ, and let p_{ij} $(1 \leq i, j \leq g)$ be its multiplicative periods. Then the Jacobian $\mathrm{Jac}(C_\Delta)$ of C_Δ is represented as

$$\mathbb{G}_m^g / \langle (p_{ij})_{1 \leq i \leq g} \mid 1 \leq j \leq g \rangle,$$

and hence $H^0(C_\Delta, \Omega_{C_\Delta}) \cong H^0\left(\mathrm{Jac}(C_\Delta), \Omega_{\mathrm{Jac}(C_\Delta)}\right)$ has a canonical basis dz_i/z_i $(1 \leq i \leq g)$, where z_i are the natural coordinates on $\mathbb{G}_m^g$. Therefore, as in the Siegel modular case, the evaluation on

$$\left(C_\Delta, ((dz_1/z_1) \wedge \cdots \wedge (dz_g/z_g))^{\otimes h} \right)$$

gives rise to a homomorphism

$$\kappa_\Delta : T_{g,h}(M) \longrightarrow B_\Delta \otimes_{\mathbb{Z}} M$$

which is the expansion map by the corresponding local coordinates on $\mathcal{M}_g$ under the trivialization of λ via $(dz_1/z_1) \wedge \cdots \wedge (dz_g/z_g)$.

Theorem 2.1. ([I5]). *Fix $g > 1$ and $h \in \mathbb{Z}$.*

(1) *κ_Δ is injective, and for a Teichmüller modular form $f \in T_{g,h}(M)$ and a $\mathbb{Z}$-submodule N of M,*

$$f \in T_{g,h}(N) \iff \kappa_\Delta(f) \in B_\Delta \otimes_{\mathbb{Z}} N.$$

(2) *For a Siegel modular form $\varphi \in S_{g,h}(M)$ with Fourier expansion $F(\varphi)$,*

$$\kappa_\Delta(\tau^*(\varphi)) \;=\; F(\varphi)|_{q_{ij} = p_{ij}},$$

where p_{ij} are the multiplicative periods of C_Δ given in Theorem 1.2 (4).

Sketch of Proof. The assertion (1) follows from the fact that C_Δ corresponds to the generic point on $\mathcal{M}_g$, and that $\mathcal{M}_g$ is smooth over $\mathbb{Z}$ with geometrically irreducible fibers. (2) follows from Theorem 1.2 (4). $\square$

$$\boxed{\boldsymbol{p_{ij}} \text{ are computable, hence } \boldsymbol{\kappa_\Delta} \text{ are computable}}$$

Schottky problem. As an application of Theorem 2.1, we can give a solution to the Schottky problem characterizing Siegel modular forms vanishing on the Jacobian locus:

$$\tau^*(\varphi) = 0 \iff F(\varphi)|_{q_{ij}=p_{ij}} = 0.$$

This applications and examples are given in §3.

2.2. Relation to geometry of moduli

Theta constants and ring structure.

For $g \geq 2$, let

$$\theta_g(Z) \stackrel{\text{def}}{=} \prod_{\substack{a,b \in \{0,1/2\}^g \\ 2a^tb \in \mathbb{Z}}} \sum_{n \in \mathbb{Z}^g} \exp\left(2\pi\sqrt{-1}\left[\frac{1}{2}(n+a)Z^t(n+a) + (n+a)^t b\right]\right)$$

be the product of even **theta constants (theta null-values)** of degree g. If $g \geq 3$, then θ_g is an integral Siegel modular form of degree g and weight $2^{g-2}(2^g+1)$.

Theorem 2.2. ([I3, 5]). *Assume that $g \geq 3$.*

(1) *$T_{g,h}(\mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank satisfying that $T_{g,h}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = T_{g,h}(\mathbb{C})$, and that $T_{g,0}(\mathbb{Z}) = \mathbb{Z}$, $T_{g,h}(\mathbb{Z}) = \{0\}$ if $h < 0$. Furthermore, the ring of integral Teichmüller modular forms of degree g:*

$$T_g^*(\mathbb{Z}) \stackrel{\text{def}}{=} \bigoplus_{h \geq 0} T_{g,h}(\mathbb{Z})$$

becomes a normal ring which is finitely generated over $\mathbb{Z}$.

(2) *Put*

$$N_g \stackrel{\text{def}}{=} \begin{cases} -2^{28} & (g=3), \\ 2^{2^{g-1}(2^g-1)} & (g \geq 4). \end{cases}$$

*Then $\mu_g = \sqrt{\tau^*(\theta_g)/N_g}$ is a **primitive** element of $T_{g,2^{g-3}(2^g+1)}(\mathbb{Z})$, i.e., not congruent to 0 modulo any prime.*

(3) *$T_3^*(\mathbb{Z})$ is generated by Siegel modular forms of degree 3 over $\mathbb{Z}$, and by μ_3 which is of weight 9, hence is not a Siegel modular form.*

Remark. The ring structure of Siegel modular forms of degrees 2 and 3 are described by Igusa [Ig1, 3] and by Tsuyumine [T1] respectively.

Sketch of Proof. (1) First, using κ_Δ in Theorem 2.1 it is shown that integral Teichmüller modular forms can be extended to global sections on Deligne-Mumford's compactification $\overline{\mathcal{M}}_g$ which is a proper smooth stack over $\mathbb{Z}$. Then by the same way to the proof in the Siegel modular case [FC], one has the finiteness of $\mathrm{rank}_{\mathbb{Z}} T_{g,h}(\mathbb{Z})$ and of generators of $T_g^*(\mathbb{Z})$.

(2) Let D be the divisor of $\mathcal{M}_g \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}$ consisting of curves C which have a line bundle L such that $L^{\otimes 2} \cong \Omega_C$ and that $\dim H^0(C,L)$ is positive and even. Then as is shown in [T2], $2D$ gives the divisor of $\tau^*(\theta_g)$, and hence a Teichmüller modular form with divisor D, which exists and is uniquely determined up to constant, is a root of $\tau^*(\theta_g)$ up to constant (see below). Since D is stable under any Galois action over $\mathbb{Q}$, a root of $\tau^*(\theta_g)$ times a certain number is defined and primitive over $\mathbb{Z}$. To determine this number, κ_Δ is used as follows: Let A_0, A_Δ, p_{ij} be as in Example 1.3. Then

$$\theta_g(Z) \;=\; 2^{2^{g-1}(2^g-1)} \left(\prod_{\substack{(b_1,\dots,b_g)\,\in\,\{0,1/2\}^g \\ \sum_i b_i \,\in\, \mathbb{Z}}} (-1)^{\sum_i b_i} \right) P \cdot \alpha^2,$$

where

$$\begin{aligned} \alpha \quad &: \quad \text{a primitive element of } \mathbb{Z}\left[q_{ij}^{\pm 1}\ (i \neq j)\right][[q_{11},\dots,q_{gg}]]\,, \\ P \quad &= \prod_{\substack{(b_1,\dots,b_g)\,\in\,\{0,1/2\}^g \\ \sum_i b_i \,\in\, \mathbb{Z}}} \frac{1}{2} \sum_{S \subset \{1,\dots,g\}} (-1)^{\sharp\{k \in S \mid b_k \neq 0\}} \prod_{i \in S, j \notin S} q_{ij}^{-1/2} \\ &\Rightarrow \left(\text{the constant term of } P|_{q_{ij}=p_{ij}} \in A_\Delta\right)\Big|_{x_1=x_{-2},\dots,x_g=x_{-1}} = 1. \end{aligned}$$

Therefore,

$$\left(\prod_{\substack{(b_1,\dots,b_g)\,\in\,\{0,1/2\}^g \\ \sum_i b_i \,\in\, \mathbb{Z}}} (-1)^{\sum_i b_i} \right) = \begin{cases} 1 & (g=3), \\ -1 & (g \geq 4), \end{cases}$$

and hence we have

$$\begin{aligned} &\sqrt{\text{the constant term of } P|_{q_{ij}=p_{ij}}} \in A_0 \\ \Rightarrow\ &\sqrt{\theta_g|_{q_{ij}=p_{ij}}} \in \begin{cases} \sqrt{-1} \cdot 2^{27} \cdot A_\Delta & (g=3), \\ 2^{2^{g-1}(2^g-1)-1} \cdot A_\Delta & (g \geq 4). \end{cases} \end{aligned}$$

(3) Recall the result of Igusa [Ig2] that the ideal of $S_3^*(\mathbb{C})$ vanishing on the hyperelliptic locus is generated by θ_3. Since the Torelli map $\mathcal{M}_3 \to \mathcal{A}_3$ is dominant and of degree 2, if ι denotes the multiplication by -1 on abelian varieties, then

$$\begin{aligned}\bigoplus_{h:\ \text{even}} T_{3,h}(\mathbb{C}) &= \{ f \in T_3^*(\mathbb{C}) \mid \iota(f) = f \} = S_3^*(\mathbb{C}), \\ \bigoplus_{h:\ \text{odd}} T_{3,h}(\mathbb{C}) &= \{ f \in T_3^*(\mathbb{C}) \mid \iota(f) = -f \}.\end{aligned}$$

Assume that f has odd weight. Since $\iota(f) = -f$, $f = 0$ on the hyperelliptic locus, and hence by Igusa's result, f^2/θ_3 becomes a Siegel modular form. Therefore, one can see that $T_3^*(\mathbb{C})$ is generated by $S_3^*(\mathbb{C})$ and $\sqrt{\tau^*(\theta_3)}$ which implies (3) because μ_3 is integral and primitive. $\square$

TMFs of degree 2. Let k be an algebraically closed field of characteristic $\neq 2$. Then any proper smooth curve C of genus 2 over k is hyperelliptic, more precisely a base of $H^0(C, \Omega_C)$ gives rise to a morphism $C \to \mathbb{P}_k^1$ of degree 2 ramified at 6 points, and hence

$$\mathcal{M}_2 \otimes_{\mathbb{Z}} k \ \cong \ \left\{ (x_1, x_2, x_3) \in \mathbb{P}_k^1 - \{0, 1, \infty\} \ \middle| \ x_i \neq x_j \ (i \neq j) \right\} \big/ S_6,$$

where each element σ of the symmetric group S_6 degree 6 acts on (x_1, x_2, x_3)'s such as

$$(\sigma(x_1), \sigma(x_2), \sigma(x_3), 0, 1, \infty)$$

is obtained from $\sigma(x_1, x_2, x_3, 0, 1, \infty)$ by a certain Möbius transformation. Therefore, $\mathcal{M}_2 \otimes_{\mathbb{Z}} k$ becomes an affine variety, and $T_{2,h}(k) = H^0(\mathcal{M}_2, \lambda^{\otimes h} \otimes_{\mathbb{Z}} k)$ is infinite dimensional. In fact, it is proved in [I5] that the ring of integral Teichmüller modular forms of degree 2:

$$T_2^*(\mathbb{Z}) \overset{\text{def}}{=} \bigoplus_{h \in \mathbb{Z}} T_{2,h}(\mathbb{Z})$$

is generated by $\tau^*(S_2^*(\mathbb{Z}))$ and by $2^{12}/\left(\tau^*(\theta_2)\right)^2$ which is of weight -10.

Construction of TMFs. Assume that $g \geq 3$. Then by results of Mumford [Mu1] and Harer [H], the **Picard group** of $\mathcal{M}_g$:

$$\mathrm{Pic}(\mathcal{M}_g) \ \overset{\text{def}}{=} \ \text{the group of linear equivalence classes of line bundles on } \mathcal{M}_g.$$

is isomorphic to $H^2(\mathcal{M}_g(\mathbb{C}), \mathbb{Z}) \cong H^2(\Pi_g, \mathbb{Z})$ (Π_g denotes the Teichmüller modular group of degree g), and this is free of rank 1 generated by the Hodge line bundle λ. Therefore,

- $D \neq 0$ is an effective divisor on $\mathcal{M}_g$ over a subfield K of $\mathbb{C}$
  $\Rightarrow$ there are $n, h \in \mathbb{N}$ such that $\mathcal{O}_{\mathcal{M}_g}(D)^{\otimes n} \cong \lambda^{\otimes h}$
  $\Rightarrow$ there is $f \in T_{g,h}(K)$ such that $(f) = n \cdot D$

(for the application, see the proof of Theorem 2.2 (2)),

- $\mathcal{L}$ is a line bundle on $\mathcal{M}_g \otimes_{\mathbb{Z}} K$
  $\Rightarrow$ there are $n, h \in \mathbb{Z}$ such that $\mathcal{L}^{\otimes n} \cong \lambda^{\otimes h}$
  $\Rightarrow$ there is $g \in H^0(\mathcal{M}_g \otimes_{\mathbb{Z}} K, \lambda^{\otimes h} \otimes \mathcal{L}^{\otimes -n})$ giving $\mathcal{O}_{\mathcal{M}_g} \xrightarrow{\sim} \lambda^{\otimes h} \otimes \mathcal{L}^{\otimes -n}$,

and f, g are uniquely determined by the existence of the Satake-type compactification of $\mathcal{M}_g$. From this method, one can construct Teichmüller modular forms and study their rationality using κ_Δ.

Mumford's isomorphism. By applying Grothendieck-Riemann-Roch's theorem to the universal stable curve $\pi : \mathcal{C} \to \overline{\mathcal{M}}_g$ over Deligne-Mumford's compactification [DM], we have Mumford's isomorphism [Mu4]:

$$\bigwedge^{3g-3} \pi_* \left(\mathcal{T}^\vee_{\overline{\mathcal{M}}_g} \right) \cong \bigwedge^{3g-3} \pi_* \left(\Omega_{\mathcal{C}/\overline{\mathcal{M}}_g} \otimes \omega_{\mathcal{C}/\overline{\mathcal{M}}_g} \right) \cong \lambda^{\otimes 13} \otimes \mathcal{O}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{M}}_g - \mathcal{M}_g)^{\otimes(-2)},$$

and hence

$$\lambda_2 \overset{\text{def}}{=} \bigwedge^{3g-3} \pi_* \left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_g} \right) \cong \lambda^{\otimes 13}$$

which is connected with the string amplitude in **String Theory**.

In order to express this isomorphism, we consider the homomorphism

$$\rho_g : S^2 \left(\pi_* \left(\Omega_{\mathcal{C}/\mathcal{M}_g} \right) \right) \ni (s, s') \mapsto s \cdot s' \in \pi_* \left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_g} \right)$$

between vector bundles on $\mathcal{M}_g$.

Theorem 2.3. ([I3]).

(1) *When $g = 2$, ρ_2 is an isomorphism and gives*

$$\lambda^{\otimes 3} \overset{\det(\rho_2)}{\longrightarrow} \lambda_2 \cong \lambda^{\otimes 13} \;\Rightarrow\; \mathcal{O}_{\mathcal{M}_2} \ni 1 \mapsto \pm \left(\tau^*(\theta_2)/2^6 \right)^2 \in \lambda^{\otimes 10}.$$

(2) *When $g = 3$, ρ_3 is an isomorphism generically and vanishes on the hyperelliptic locus, hence this gives*

$$\lambda^{\otimes 4} \overset{\det(\rho_3)}{\longrightarrow} \lambda_2 \cong \lambda^{\otimes 13} \;\Rightarrow\; \mathcal{O}_{\mathcal{M}_3} \ni 1 \mapsto \pm \mu_3 = \pm \sqrt{-\tau^*(\theta_3)/2^{28}} \in \lambda^{\otimes 9}.$$

Sketch of Proof. Up to constants, the assertions were known and easily seen from the properties of even theta constants. The constants are determined by Theorem 2.2. $\square$

Problem. For $g > 1$, the pullback of the Prym map:

$$\begin{array}{ccc} \{\text{curves of genus } g \text{ with unramified double cover}\} & \longrightarrow & \mathcal{A}_{g-1} \\ C' \to C & \longmapsto & \mathrm{Jac}(C')/\mathrm{Jac}(C). \end{array}$$

gives a weight-preserving ring homomorphism:

$$\{\text{SMFs of degree } g-1\} \longrightarrow \{\text{TMFs of degree } g \text{ with level 2 structure}\}$$

by a result of Faber and van der Geer [FG]. Describe this lift map using κ_Δ.

Problem. Are there Hecke-type operators acting on the space of Teichmüller modular forms? Katsurada pointed that Schottky's J defining the Jacobian locus in $\mathcal{A}_4$ is a Hecke eigenform and is obtained by Ikeda's lift [Ik] from $\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$.

Invariants and TMFs.

Klein [K] proved that θ_3 coincides with the square of the discriminant of quartic forms up to constant, and this constant is determined in [LRZ] as follows:

Theorem 2.4. (Klein's formula [K], see also [LRZ, Theorem 4.1.2]) *Let F be a homogeneous polynomial of x_1, x_2, x_3 of degree 4 over $\mathbb{C}$ such that the associated curve*

$$C_F = \left\{(a_1 : a_2 : a_3) \in \mathbb{P}^2(\mathbb{C}) \mid F(a_1, a_2, a_3) = 0\right\}$$

is smooth (and then C_F is a proper smooth curve of genus 3). Define the discriminant $\mathrm{Disc}(F)$ *of F as*

$$\mathrm{Disc}(F) \stackrel{\text{def}}{=} 2^{-14} \cdot \mathrm{Res}\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}\right) : \ 2^{-14} \times \textit{the multivariate resultant of } \frac{\partial F}{\partial x_i}.$$

Let ω_f $(f = x_1, x_2, x_3)$ be a basis of $H^0(C_F, \Omega_{C_F})$ defined as

$$\omega_f = \frac{f \cdot (x_j dx_k - x_k dx_j)}{\partial F/\partial x_i} \text{ on } \partial F/\partial x_i \neq 0; \ (ijk) \textit{ is even},$$

and let $\gamma_1, ..., \gamma_6$ be a symplectic basis of $H_1(C_F, \mathbb{Z})$ for the intersection pairing. Put

$$\Omega = (\Omega_1, \Omega_2) = \begin{pmatrix} \int_{\gamma_1} \omega_{x_1} & \cdots & \int_{\gamma_6} \omega_{x_1} \\ \int_{\gamma_1} \omega_{x_2} & \cdots & \int_{\gamma_6} \omega_{x_2} \\ \int_{\gamma_1} \omega_{x_3} & \cdots & \int_{\gamma_6} \omega_{x_3} \end{pmatrix},$$

and $Z = \Omega_1^{-1}\Omega_2$ which belongs to the Siegel upper half space H_3 of degree 3. Then

$$\mathrm{Disc}(F)^2 = \frac{(2\pi\sqrt{-1})^{3\cdot 18} \cdot \theta_3(Z)}{(\det \Omega_1)^{18} \cdot N_3} = \frac{(2\pi)^{54} \cdot \theta_3(Z)}{2^{28} \cdot (\det \Omega_1)^{18}}.$$

Sketch of Proof. For a field k of characteristic $\neq 2$, let X_4 be the vector space of k-linear forms on k^3 of degree 4, and X_4° be its nonsingular locus defined as

$$X_4^{\circ} = \{F \in X_4 \mid \mathrm{Disc}(F) \neq 0\}.$$

Then

$$Y_4^{\mathrm{o}} = \left\{ (F, x) \in X_4^{\mathrm{o}} \times \mathbb{P}_k^2 \mid F(x) = 0 \right\}$$

is a family of non-hyperelliptic curves of genus 3 over X_4^{o}, and hence the associated morphism $\phi : X_4^{\mathrm{o}} \to \mathcal{M}_3$ gives rise to a linear map

$$\phi^* : T_{3,h}(k) \to \Gamma\left(X_4^{\mathrm{o}}, \lambda^{\otimes h} \right).$$

By invariant theory, $\phi^*(\theta_3)$ is an invariant of degree 54, and hence is a constant multiple of Disc^2 because Disc is an invariant of degree 27. Furthermore, this constant can be determined in [LR] by huge calculation of theta constants considering Jacobians isogeneous to the products of 3 elliptic curves. $\square$

Remark. We give a simple proof of this fact by using arithmetic of Teichmüller modular forms. By the theory of canonical curves, the morphism $\phi : X_4^{\mathrm{o}} \to \mathcal{M}_3$ can be defined over any ring. Since μ_3 is primitive and its divisor is the hyperelliptic locus, $\phi^*(\mu_3)$ is a nonzero multiple of $\mathrm{Disc} \cdot (\omega_{x_1} \wedge \omega_{x_2} \wedge \omega_{x_3})^{\otimes 9}$, and hence

$$\phi^*(\mu_3) = \pm \mathrm{Disc} \cdot (\omega_{x_1} \wedge \omega_{x_2} \wedge \omega_{x_3})^{\otimes 9}$$

which implies Theorem 2.4 by Theorem 2.2 (2).

Remark. In the Proceedings of Kinosaki Symposium 2010, K. Yoshikawa noticed the following analogy in the genus 1 case, and gave its extension to Borcherds' Φ-function: For $A = (\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3\ \boldsymbol{a}_4) \in M(2, 4; \mathbb{C})$, put $\Delta_{ij}(A) = \det(\boldsymbol{a}_i\ \boldsymbol{a}_j)$ $(1 \leq i, j \leq 4)$. Under that $\prod_{i<j} \Delta_{ij}(A) \neq 0$, let

$$\begin{aligned} E_A \quad &= \quad \left\{ (x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^3(\mathbb{C}) \;\middle|\; \sum_{j=1}^{4} \boldsymbol{a}_j x_j^2 = \boldsymbol{0} \right\} : \text{ elliptic curve over } \mathbb{C}, \\ \omega_A \quad &: \quad \text{canonical basis of } H^0(E_A, \Omega_{E_A}) \text{ given by } \omega|_{E_A}; \\ &\qquad \omega \wedge \bigwedge_{i=1}^{2} \left(\sum_{j=1}^{4} a_{ij} x_j dx_j \right) = \sum_{i=1}^{4} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_4. \end{aligned}$$

Then $q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ corresponds to $c \prod_{i<j} \Delta_{ij}(A)^2 (\omega_A)^{12}$ for some nonzero constant c (may be $\pm 2^n$) which implies that

$$\mathrm{Im}(z)^6 (2\pi)^{12} \left| q \prod_{n=1}^{\infty} (1 - q^n)^{24} \right| = |c| \prod_{i<j} |\Delta_{ij}(A)|^2 \left(\frac{\sqrt{-1}}{2} \int_{E_A} \omega_A \wedge \overline{\omega_A} \right)^6,$$

where $E_A = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}z)$ with $\mathrm{Im}(z) > 0$.

Klein's amazing formula. In the footnote of p.462 in [K], for a non-hyperelliptic curve C of genus 4 given in $\mathbb{P}^3$ as an intersection of a quadric surface Q and a cubic surface E, Klein gives the formula:

$$\frac{\theta_4(Z)}{\det(\Omega_1)^{68}} = c \cdot \Delta(C)^2 \cdot T(C)^8.$$

Here $\Omega = (\Omega_1, \Omega_2)$ is a period matrix of C, $Z = \Omega_1^{-1}\Omega_2$, $\Delta(C)$ and $T(C)$ are the discriminant and the tact invariant respectively which are integral and primitive polynomials of the coefficients of the equations corresponding Q and E. The constant c is determined by arithmetic of Teichmüller modular forms as follows:

Theorem 2.5. ([I10]). *Represent the Jacobian variety of C as a complex torus*

$$\mathbb{C}^4 / \left(\mathbb{Z}^4 \cdot \Omega_1 + \mathbb{Z}^4 \cdot \Omega_2\right),$$

where $Z = \Omega_1^{-1}\Omega_2 \in H_4$. Then

$$\Delta(C)^2 \cdot T(C)^8 = \frac{\left(2\pi\sqrt{-1}\right)^{4\cdot 68} \cdot \theta_4(Z)}{\det(\Omega_1)^{68} \cdot N_4} = \frac{(2\pi)^{272} \cdot \theta_4(Z)}{2^{120} \cdot \det(\Omega_1)^{68}}.$$

Sketch of Proof. For Schottky's J given in 3.1, put

$$S_{ij} = \frac{1 + \delta_{ij}}{2^{16}} q_{ij} \frac{\partial J}{\partial q_{ij}} \quad (1 \leq i, j \leq 4).$$

Then as is shown by Matone and Volpato [MV], $\tau^*\left(\det(S_{ij})\right)$ is a multiple of μ_4 by a nonzero constant. Further, one can show that $\tau^*\left(\det(S_{ij})\right)$ is primitive as an integral Teichmüller modular form, and hence by Theorem 2.2 (2),

$$\Delta \cdot T^4 = \pm\tau^*\left(\det(S_{ij})\right) = \pm\mu_4 = \pm\sqrt{\tau^*(\theta_4)/2^{120}}$$

which implies the assertion. □

Remark. When $g = 4$, from a result in [MV] and the proof of Theorem 2.5, we can describe Mumford's isomorphism $\lambda_2 \cong \lambda^{\otimes 13}$ by $\tau^*(S_{kl})$ for $1 \leq k \leq l \leq 4$:

$$\bigwedge_{\substack{1 \leq i \leq j \leq 4 \\ (i,j) \neq (k,l)}} \omega_i \omega_j = \pm \frac{\tau^*\left(S_{kl}\right)}{1 + \delta_{kl}} \left(\bigwedge_{i=1}^{4} \omega_i\right)^{\otimes 13},$$

where ω_i are the canonical basis dz_i/z_i $(1 \leq i \leq 4)$ of regular 1-forms on a generalized Tate curve.

§3. Schottky problem

3.1. Characterizing the Jacobian locus

First, we recall a corollary of Theorem 2.1 as follows:

Theorem 3.1. ([I5]). *Let p_{ij} $(1 \le i, j \le g)$ be the universal periods of the generalized Tate curve C_Δ. Then for a Siegel modular form $\varphi \in S_{g,h}(M)$ of degree g and weight h with coefficients in a $\mathbb{Z}$-module M, φ vanishes on the Jacobian locus, i.e., $\tau^*(\varphi) = 0$ if and only if its Fourier expansion $F(\varphi)$ satisfies that*

$$F(\varphi)|_{q_{ij}=p_{ij}} = 0 \text{ in } B_\Delta \otimes M.$$

Using the universal periods p_{ij} in Example 1.3, the above implies the following result of Brinkmann and Gerritzen [BG, G]: For the Fourier expansion

$$F(\varphi) = \sum_{T=(t_{ij})} a_T \prod_{1 \le i < j \le g} {q_{ij}}^{2t_{ij}} \prod_{1 \le i \le g} {q_{ii}}^{t_{ii}}$$

of a Siegel modular form φ vanishing on the Jacobian locus,

$$\text{integers } s_1, ..., s_g \ge 0 \text{ satisfy } \sum_{i=1}^{g} s_i = \min\{\mathrm{tr}(T) \mid a_T \ne 0\}$$
$$\Rightarrow \sum_{t_{ii}=s_i} a_T \prod_{i<j} \left(\frac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)} \right)^{2t_{ij}} = 0 \quad \text{in the ring } A_0 \text{ given in 1.3.}$$

Schottky's *J*. For $n \equiv 0 \bmod (4)$, put

$$\begin{aligned}
L_{2n} &\overset{\text{def}}{=} \left\{ (x_1, ..., x_{2n}) \in \mathbb{R}^{2n} \;\middle|\; 2x_i,\ x_i - x_j,\ \frac{1}{2}\sum_i x_i \in \mathbb{Z} \right\} \\
&: \ \text{a lattice in } \mathbb{R}^{2n} \text{ with standard inner product } \langle\ ,\ \rangle, \\
\Theta_n(Z) &\overset{\text{def}}{=} \sum_{(\lambda_1,...,\lambda_4) \in L_{2n}^4} \exp\left(\pi\sqrt{-1} \sum_{i,j=1}^{4} \langle \lambda_i, \lambda_j \rangle z_{ij} \right) \quad (Z = (z_{ij})_{i,j} \in H_4) \\
&: \ \text{a theta series of degree 4 and weight } n, \\
J(Z) &\overset{\text{def}}{=} \frac{2^2}{3^2 \cdot 5 \cdot 7} (\Theta_4(Z)^2 - \Theta_8(Z)) : \ \textbf{Schottky's } \boldsymbol{J} \\
&: \ \text{an integral Siegel modular form of degree 4 and weight 8.}
\end{aligned}$$

Then Schottky and Igusa proved that the Zariski closure of the Jacobian locus in $\mathcal{A}_4 \otimes_{\mathbb{Z}} \mathbb{C}$ is defined by $J = 0$.

Brinkmann and Gerritzen [BG, G] applied the above their criterion to Schottky's J, and showed that its lowest term is given by

$$F\frac{q_{11}q_{22}q_{33}q_{44}}{\prod_{1\leq i<j\leq 4} q_{ij}},$$

where F is a generator of the ideal of $\mathbb{C}\left[q_{ij}\ (1 \leq i < j \leq 4)\right]$ which is the kernel of the ring homomorphism defined as

$$q_{ij} \mapsto \frac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)} \in A_0.$$

Problem. Let J' be a primitive modular form obtained from J by dividing the GCM (greatest common divisor) of its Fourier coefficients. Then for each prime p,

$$\begin{aligned} &\text{the closed subset of } \mathcal{A}_4 \otimes_{\mathbb{Z}} \mathbb{F}_p \text{ defined by } J' \bmod(p) = 0 \\ \overset{?}{=}\ &\text{the Zariski closure of } \tau(\mathcal{M}_4 \otimes_{\mathbb{Z}} \mathbb{F}_p) \text{ in } \mathcal{A}_4 \otimes_{\mathbb{Z}} \mathbb{F}_p. \end{aligned}$$

Hyperelliptic Schottky problem. ([I6]). Let p_{ij} be the universal periods given in Example 1.3, and put

$$p'_{ij} \overset{\text{def}}{=} p_{ij}|_{x_{-k}=-x_k}\ (1 \leq k \leq g).$$

Theorem 3.2. ([I6]). *For any Siegel modular form φ over a field of characteristic $\neq 2$,*

$$\varphi \text{ vanishes on the locus of hyperelliptic Jacobians} \iff F(\varphi)|_{q_{ij}=p'_{ij}} = 0.$$

Sketch of Proof. Using a result of [GP], one can show that p'_{ij} are the multiplicative periods of the hyperelliptic curve C_{hyp} over

$$\mathbb{Z}\left[\frac{1}{2x_i}, \frac{1}{x_i \pm x_j}(i \neq j)\right][[y_1, ..., y_g]]$$

uniformized by the Schottky group:

$$\left\langle \left(\begin{array}{cc} x_k & -x_k \\ 1 & 1 \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & y_k \end{array}\right)\left(\begin{array}{cc} x_k & -x_k \\ 1 & 1 \end{array}\right)^{-1} \middle|\ k = 1, ..., g \right\rangle.$$

Then the assertion follows from the irreducibility of the moduli space of hyperelliptic curves. $\square$

Problem. Give an explicit lower bound of $n(g) \in \mathbb{N}$ satisfying that for any Siegel modular form of degree g,

$$\varphi \text{ vanishes on the Jacobians locus} \iff F(\varphi)|_{q_{ij}=p_{ij}} \in I^{n(g)},$$

where I is the ideal generated by y_e ($e \in E$) in Theorem 1.2. As for Theorem 3.2, a similar problem is raised and seems more easy to study.

Serre's question.

Serre noticed that geometric Jacobians are not necessarily Jacobians over a fixed field k, and raised a question of giving a function which classifies Jacobians over k. This question is answered in the genus 3 case by Lachaud, Ritzenthaler and Zykin [LRZ] using μ_3:

Theorem 3.3. ([LRZ, Theorem 1.3.3]). *Let (X, Θ) be a principally polarized abelian threefold over a subfield k of $\mathbb{C}$, $(\omega_1, \omega_2, \omega_3)$ be a basis of $H^0(X, \Omega_X)$ and $(\gamma_1, ..., \gamma_6)$ be a symplectic basis of $H_1(X \otimes_k \mathbb{C}, \mathbb{Z})$ for the polarization Θ. Put*

$$\Omega = (\Omega_1, \Omega_2) = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_6} \omega_1 \\ \int_{\gamma_1} \omega_2 & \cdots & \int_{\gamma_6} \omega_2 \\ \int_{\gamma_1} \omega_3 & \cdots & \int_{\gamma_6} \omega_3 \end{pmatrix},$$

and $Z = \Omega_1^{-1}\Omega_2$ which belongs to the Siegel upper half space H_3 of degree 3. Furthermore, assume that (X, Θ) is indecomposable over $\mathbb{C}$. Then

(1) *$\theta_3(Z) = 0$ if and only if there exists a hyperelliptic curve C over k such that (X, Θ) is k-isomorphic to the Jacobian variety $\mathrm{Jac}(C)$ of C with canonical polarization.*

(2)

$$\frac{\left(2\pi\sqrt{-1}\right)^{3 \cdot 18} \cdot \theta_3(Z)}{(\det \Omega_1)^{18} \cdot N_3} = \frac{(2\pi)^{54} \cdot \theta_3(Z)}{2^{28} \cdot (\det \Omega_1)^{18}}$$

is a square in $k^\times$ if and only if there exists a non-hyperelliptic curve C over k such that (X, Θ) is k-isomorphic to $\mathrm{Jac}(C)$ with canonical polarization.

Remark. It is shown in [Ig2] that a principally polarized abelian threefold (X, Θ) is indecomposable over $\mathbb{C}$ if and only if $\Sigma_{3,140}(Z) \neq 0$, where $\Sigma_{3,140}$ a Siegel modular form of degree 3 and weight 140 which is obtained as the symmetric function with degree 35 of the 8th powers of even theta characteristics.

Sketch of Proof. The assertion (1) follows from results of Igusa [Ig2] and Serre, hence we will prove (2). If (X, Θ) is k-isomorphic to $\mathrm{Jac}(C)$ (with canonical polarization) for a non-hyperelliptic curve C over k, then by Theorem 2.2,

$$\frac{\left(2\pi\sqrt{-1}\right)^{54} \cdot \theta_3(Z)}{(\det \Omega_1)^{18} \cdot N_3} = \mu_3(C)^2,$$

where $\mu_3(C)$ is the evaluation on μ_3 on $\left(C, (\omega_1 \wedge \omega_2 \wedge \omega_3)^{\otimes 9}\right)$ under $H^0(C, \Omega_C) \cong H^0(X, \Omega_X)$. Therefore, the above left hand side is a square in $k^\times$. Assume that $\theta_3(Z) \neq 0$.

Then by a result of Serre, there exists a non-hyperelliptic curve C over k and a quadratic character $\varepsilon : \mathrm{Gal}\left(\overline{k}/k\right) \to \{\pm 1\}$ such that $\mathrm{Jac}(C)$ is k-isomorphic to the twist $(X_\varepsilon, \Theta_\varepsilon)$ of (X, θ) by ε. Let $d \in \overline{k}^\times$ be the ratio of basis of $\bigwedge^3 H^0(X, \Omega_X)$ and of $\bigwedge^3 H^0\left(X_\varepsilon, \Omega_{X_\varepsilon}\right)$. Then

$$\sigma(d) = \varepsilon(\sigma)^3 \cdot d \quad \left(\sigma \in \mathrm{Gal}\left(\overline{k}/k\right)\right).$$

If (X, Ω_X) is not k-isomorphic to non-hyperelliptic Jacobians over k, then ε is not trivial by a result of Serre, and hence $(\pm d)^9 \notin k^\times$ because

$$\sigma\left((\pm d)^9\right) = \varepsilon(\sigma)^{27} \cdot (\pm d)^9 \quad \left(\sigma \in \mathrm{Gal}\left(\overline{k}/k\right)\right).$$

Therefore,

$$\frac{\left(2\pi\sqrt{-1}\right)^{54} \cdot \theta_3(Z)}{(\det \Omega_1)^{18} \cdot N_3} = d^{18} \cdot \mu_3(C)^2$$

is not a square in $k^\times$. $\square$

3.2. Characterizing Jacobian theta functions

Review of the complex field case. The Schottky problem in soliton theory is to show the close connection:

Jacobian theta functions $\approx$ solutions to soliton equations

which was established by Novikov, Krichever, Shiota, ...

More precisely, put

$$\begin{aligned}
\Omega &= (\tau_{ij})_{1 \le i,j \le g} \in M_g(\mathbb{C});\ {}^t\Omega = \Omega,\ \mathrm{Im}(\Omega) > 0 \\
&: \text{ period matrix}, \\
X &= \mathbb{C}^g / (\mathbb{Z}^g + \mathbb{Z}^g \cdot \Omega) \\
&\cong \left(\mathbb{C}^\times\right)^g / \langle (q_{ij})_{1 \le i \le g} \mid 1 \le j \le g \rangle;\ q_{ij} = e^{2\pi\sqrt{-1}\tau_{ij}} \\
&: \text{ abelian variety over } \mathbb{C}, \\
\theta_X(z) &= \sum_{n \in \mathbb{Z}^g} \exp\left(\pi\sqrt{-1} n \Omega^t n + 2\pi\sqrt{-1} n z\right)\ (z \in \mathbb{C}^g) \\
&= \sum_{n \in \mathbb{Z}^g} \prod_{i<j} q_{ij}^{n_i n_j} \cdot \prod_i \sqrt{q_{ii}}^{n_i^2} \cdot \prod_i \left(e^{2\pi\sqrt{-1} z_i}\right)^{n_i} \\
&: \textbf{ Riemann's theta function}, \\
\Theta &= \mathrm{div}(\theta_X) : \text{ theta divisor on } X \leftrightarrow \text{ principal polarization.}
\end{aligned}$$

Theorem 3.4. (Novikov's conjecture proved by Shiota [Sh]). *Assume (X, Θ) is indecomposable. Then the following two conditions are equivalent:*

(I) *There is a Riemann surface C such that*

$$(X, \Theta) \cong (\mathrm{Jac}(C), \textit{canonical polarization}).$$

(II) *$\theta_X(z)$ gives solutions to the* ***KP equation (KPE)***:

$$3\frac{\partial^2 u}{\partial t_2^2} + \frac{\partial}{\partial t_1}\left(\frac{\partial^3 u}{\partial t_1^3} + 12u\frac{\partial u}{\partial t_1} - 2\frac{\partial u}{\partial t_3}\right) = 0.$$

<u>*Sketch of Proof.*</u> By Sato's theory on the **KP hierarchy (KPH)** which is a system of nonlinear partial differential equations containing KPE,

$$\begin{array}{ccc}
 & \boxed{\text{Jacobian theta}} & \\
\text{Krichever, Mulase } \swarrow\nearrow & & \searrow\nwarrow \text{ Fay, Krichever} \\
\boxed{\text{Orbits in Sato Grassmann}} & & \boxed{\text{Trisecant conditions}} \\
\text{Sato } \downarrow\uparrow & & \downarrow \text{ taking limit} \\
\boxed{\text{Solutions to the KP hierarchy}} & \overset{\text{Shiota}}{\approx} & \boxed{\text{Solutions to KPE}}
\end{array} \qquad \square$$

Our aim is to consider this rigid analytic analog using nonarchimedean theta functions in order to

$$\begin{cases} \text{construct solutions to KPE of (apparently) new type,} \\ \text{characterize Jacobians over positive characteristic fields.} \end{cases}$$

<u>**Nonarchimedean (NA) field case.**</u> A NA complete valuation field is a field K with valuation $|\cdot|$ satisfying:

$$\begin{cases} |a| \geq 0,\ |a| = 0 \Leftrightarrow a = 0, \\ |ab| = |a||b|, \\ |a+b| \leq \max\{|a|, |b|\} \text{ (NA condition)}, \\ |K| \supsetneqq \{0, 1\} \text{ (Nontriviality)}, \\ K \text{ is complete for the associated metric.} \end{cases}$$

Then for a NA complete valuation field K and

$$q_{ij} \in K^\times \ (1 \leq i, j \leq g) \text{ such that } q_{ij} = q_{ji},\ (\log |q_{ij}|)_{i,j} < 0,$$

Tate and Mumford showed

$$X = \left(K^\times\right)^g / \left\langle (q_{ij})_{1 \leq i \leq g} \mid 1 \leq j \leq g \right\rangle : \text{ abelian variety over } K.$$

Under $\sqrt{q_{ii}} \in K^\times$, Gerritzen and van der Put studied the **nonarchimedean (NA) theta function**:

$$\theta_X(\zeta) = \sum_{n \in \mathbb{Z}^g} \prod_{i<j} q_{ij}^{n_i n_j} \cdot \prod_i \sqrt{q_{ii}}^{n_i^2} \cdot \prod_i \zeta_i^{n_i} \ \left(\zeta \in (K^\times)^g\right).$$

Theorem 3.5. (NA version of Novikov's conjecture [I9]): *Let K be a nonarchimedean complete valuation field of characteristic* 0 *whose residue field is algebraically closed, and assume $(X, \mathrm{div}(\theta_X))$ is indecomposable over $\overline{K}$. Then the following two conditions are equivalent:*

(I) *There is a proper smooth curve C over K such that*

$$(X, \mathrm{div}(\theta_X)) \cong (\mathrm{Jac}(C), \textit{canonical polarization}).$$

(II) *θ_X gives solutions to the KP equation, i.e., there are $a_1 \neq 0, a_2, a_3 \in K^g$, $c \in K$ such that for $\zeta \in (K^\times)^g$ with $\theta_X(\zeta) \neq 0$, the formal power series*

$$u(t_1, t_2, t_3) = \frac{\partial^2}{\partial t_1^2} \log \theta_X \left(\zeta \cdot \exp\left(t_1 a_1 + t_2 a_2 + t_3 a_3\right)\right) + c$$

of t_i is a solution to the KP equation.

Remark. In contrast with the complex field case, if K is NA,

$$\exp(s_1, ..., s_g) = \left(\sum_{i=0}^{\infty} \frac{s_1^i}{i!}, \ ..., \ \sum_{i=0}^{\infty} \frac{s_g^i}{i!} \right)$$

is a K-analytic map defined only locally around $0 \in K^g$.

Proof of (II) $\Rightarrow$ (I). The assertion follows from Shiota's result which is formulated and proved by Marini [M] algebro-geometrically as follows:

X : abelian variety over an algebraically closed field of characteristic 0,

Θ : symmetric divisor on X giving a principal polarization,

θ : nonzero section of $\mathcal{O}_X(\Theta)$ (unique up to constant).

Then

$$\begin{aligned} & (X, \Theta) \text{ is indecomposable and } \theta \text{ solves KPE} \\ \Rightarrow \ & (X, \Theta) \cong \text{a certain Jacobian with canonical polarization.} \ \square \end{aligned}$$

Theta functions for Mumford curves. By results of Deligne and Mumford [DM, M2] and of Gerritzen and van der Put [GP], and that the residue field of K is algebraically closed,

$$\begin{aligned} & \text{a curve } C \text{ satisfies } \mathrm{Jac}(C) \cong \text{the above } X \text{ over NA } K \\ \Rightarrow \ & \begin{cases} C \text{ has stable reduction,} \\ \text{special fiber consists of (smooth or singular) projective lines} \end{cases} \\ \Rightarrow \ & C \text{ is a } \textbf{Mumford curve} \text{ uniformized by a Schottky group } \Gamma \text{ over } K, \\ & \text{i.e., } C \cong \Omega_\Gamma/\Gamma, \text{where} \begin{cases} \Gamma \subset PGL_2(K): \ \text{free, rank = genus of } C = g, \\ \Omega_\Gamma = \mathbb{P}^1(K) - \overline{\{\text{fixed points of } \Gamma - \{1\}\}}. \end{cases} \end{aligned}$$

First proof of (I) $\Rightarrow$ (II). ([I2]). Translate Krichever's result:

Riemann theta functions for Jacobians solve the KP hierarchy (KPH)

to the NA case using **universal periods** given in Theorem 1.2 (4) and **universal 1-forms** which are power series for multiplicative periods and 1-forms of Schottky uniformized Riemann surfaces and Mumford curves:

$$\begin{tikzcd} & \boxed{\text{Universal theta solution}} \arrow[dr, "\text{specialization}"] & \\ \boxed{\text{Riemann theta solutions}} \arrow[ur, "\text{universal periods, 1-forms}"] & \text{Solutions to KPH} & \boxed{\text{NA theta solutions}} \end{tikzcd} \qquad \square$$

Second proof of (I) $\Rightarrow$ (II). Show a rigid analytic version of **Fay's trisecant formula** which claims that under $\mathrm{char}(K) \neq 2$,

$$\begin{aligned} & \theta_C\left(\zeta \cdot \int_a^c \omega\right) \theta_C\left(\zeta \cdot \int_b^d \omega\right) E(c,b)E(a,d) \\ + \quad & \theta_C\left(\zeta \cdot \int_b^c \omega\right) \theta_C\left(\zeta \cdot \int_a^d \omega\right) E(c,a)E(d,b) \\ = \quad & \theta_C\left(\zeta \cdot \int_{a\cdot b}^{c\cdot d} \omega\right) \theta_C(\zeta)\, E(c,d)E(a,b), \end{aligned}$$

where

$$\begin{cases} \theta_C = \theta_{\mathrm{Jac}(C)} : \text{the NA theta function}, \\ a,b,c,d : \text{points on } \Omega_\Gamma, \\ \int^x \omega : \text{the } K\text{-analytic Abel-Jacobi's map } \Omega_\Gamma \to (K^\times)^g, \\ E : \text{the prime form on } \Omega_\Gamma \times \Omega_\Gamma. \end{cases}$$

Then as in the complex field case (cf. Mumford's Tata lectures [Mu5]), taking limits as $d \to b$, $c \to a$, $b \to a$, the NA trisecant formula $\Rightarrow$ KPE. $\square$

Remark. The first proof of (I) $\Rightarrow$ (II) has the merit to can show that $\theta_{\mathrm{Jac}(C)}$ solves the whole KPH not only KPE. The second proof has the following merits:

$$\begin{cases} \text{applicable to the case } \mathrm{char}(K) > 2, \\ \text{not depend on complex analysis}, \\ \text{implies trisecant conditions}. \end{cases}$$

Trisecant criteria. (a discrete version of the KP characterization).

(X, Θ) : abelian variety over a field k with symmetric theta divisor,
$\psi : X \twoheadrightarrow X/\{\pm 1\} \cong$ Kummer variety $\hookrightarrow |2\Theta| \cong \mathbb{P}^{2^g-1}$.

Then we consider the Schottky problem in terms of

$$T_D = \{\alpha \in X \mid \alpha + D \subset \psi^{-1}(l) \text{ for some line } l \subset \mathbb{P}^{2^g-1}/k\}$$

for a k-rational 0-cycle D on X of degree 3 which means an artinian scheme in X of length 3 over k.

Theorem 3.6. (Welters' conjecture proved by Krichever [Kr]). *Assume k is an algebraically closed field of characteristic 0, and (X, Θ) is indecomposable. Then the following two conditions are equivalent:*

(I) *There is a proper smooth curve C over k such that*

$$(X, \Theta) \cong (\mathrm{Jac}(C), \textit{canonical polarization}).$$

(II) *There is the above D such that $T_D \neq \emptyset$.*

Remark. The statement is algebro-geometric, however, Krichever's proof of (II) $\Rightarrow$ (I) depends on complex analysis of difference and differential equations associated with the trisecant condition.

Theorem 3.7. (Trisecant criterion in the positive characteristic case [I9]): *Let k be a nonarchimedean complete valuation field K of characteristic $\neq 2$ whose residue field is algebraically closed, and assume the above*

$$\left(X = \left(K^\times\right)^g / \left\langle (q_{ij})_{1 \le i \le g} \mid 1 \le j \le g \right\rangle, \mathrm{div}(\theta_X)\right)$$

is indecomposable over $\overline{K}$. Then the following two conditions are equivalent:

(I) *There is a proper smooth curve C over K such that*

$$(X, \mathrm{div}(\theta_X)) \cong (\mathrm{Jac}(C), \textit{canonical polarization}).$$

(II) *There is the above D with support $\neq$ {a point} such that T_D contains a 0-cycle of degree $> 12^g g!/2$ whose support is one point.*

Sketch of Proof. (I) $\Rightarrow$ (II) follows from the NA trisecant formula and the quadratic relation, and (II) $\Rightarrow$ (I) follows from results of [AC] and [W]. $\square$

Problem. Relate the NA theta function with Anderson's p-adic soliton theory [A] (cf. Yamazaki's talk [Y] in Mar. 2010).

Furthermore, using an algebraic version of Fay's trisecant formula given by Polishchuk [P], we have:

Theorem 3.8. ([I9]). *Theorem 3.5 can be extended for any abelian variety over an algebraically closed NA complete valuation field of characteristic* 0*. Furthermore, Theorem 3.7 can be extended for any abelian variety over an algebraically closed field of characteristic* $\neq 2$*.*

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