# Shuffle Product Formula of Multiple Zeta Values

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**Abstract**

A multiple zeta values or $r$-fold Euler sum defined by

$$\zeta(\alpha_1, \alpha_2, \cdots, \alpha_r) = \sum_{1 \le n_1 < n_2 < \cdots < n_r} n_1^{-\alpha_1} n_2^{-\alpha_2} \cdots n_r^{-\alpha_r}$$

with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ an $r$-tuple of positive integers and $\alpha_r \ge 2$, is a natural generalization of the classical Euler sum

$$S_{p,q} = \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{k} \frac{1}{j^p}.$$

It is a problem proposed by Goldbach to Euler in an attempt to evaluate $S_{p,q}$ in terms of the special values at positive integers of the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Res > 1.$$

The shuffle product formula of two multiple zeta values then express the product of two multiple zeta values as a linear combination of multiple zeta values with integral coefficients.