Mathematical Analysis for several inverse problems for fractional diffusion equations

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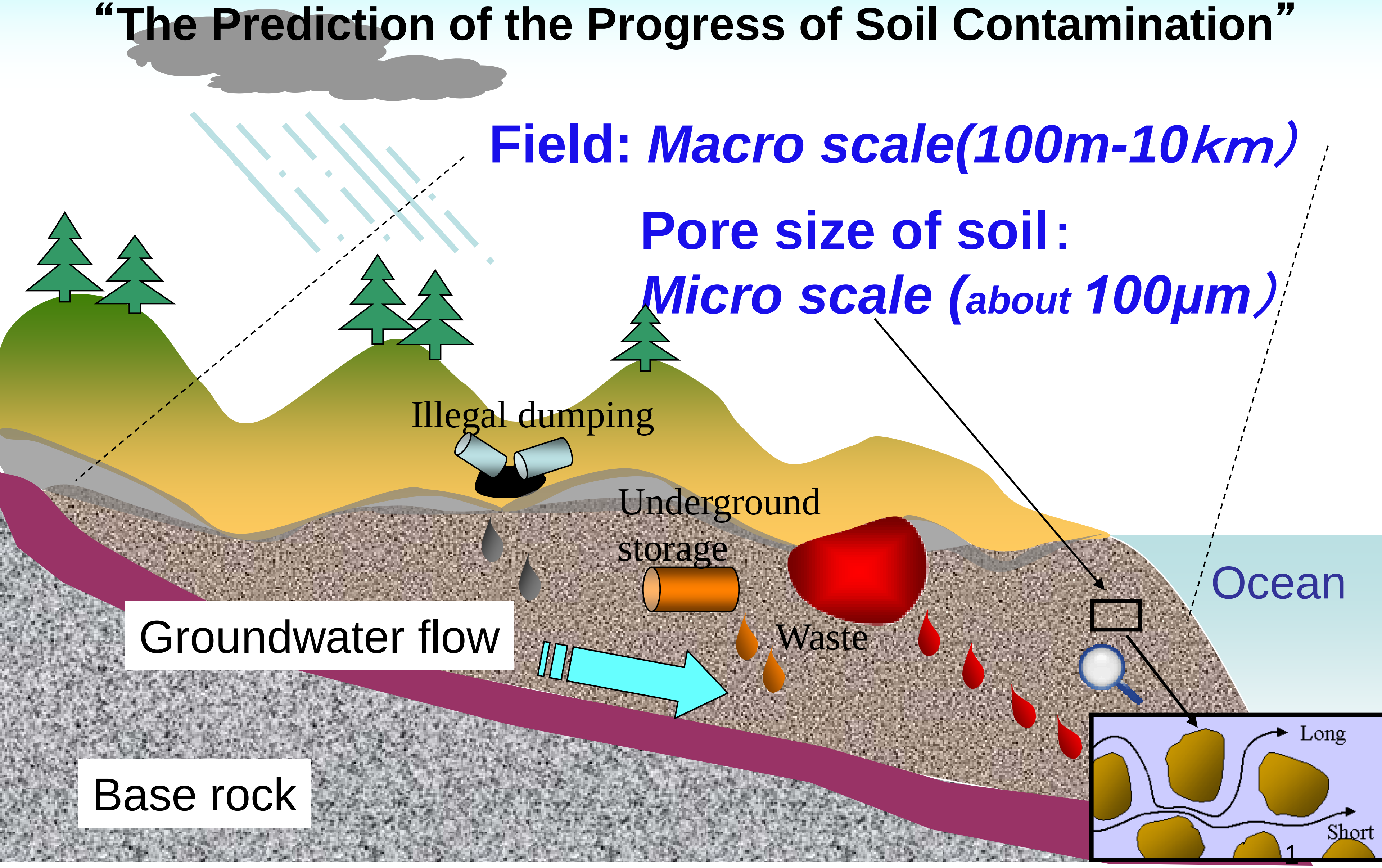
Issues Seen by Academia Engineering Researchers

“The Prediction of the Progress of Soil Contamination”



Dr. Yuko Hatano, Department of Risk Engineering, University of Tsukuba

Physical backgrounds

Anomalous diffusion in heterogenuous media

e.g., soil

Fractional diffusion equation:

$$\partial_t^\alpha u - \Delta u = 0$$

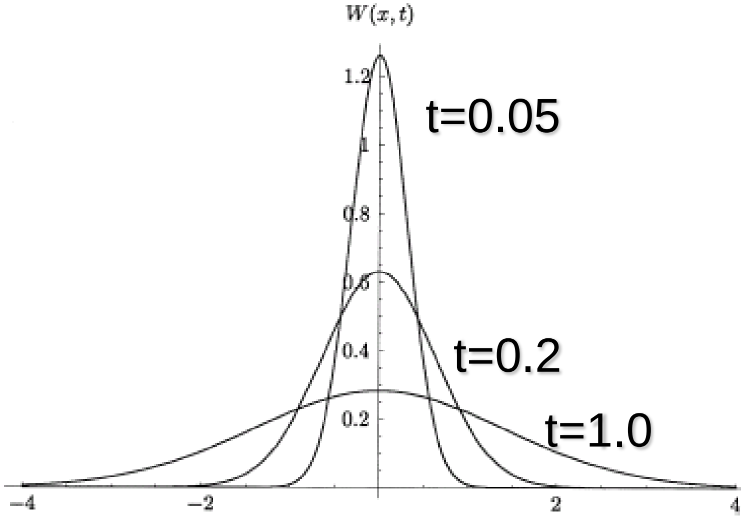
One good macroscopic model for slow diffusion

Long tail profile

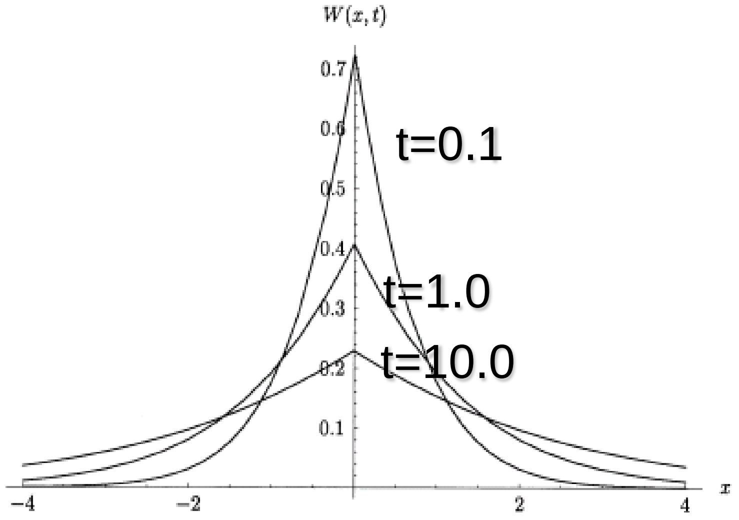
$$\|u(\cdot,t)\|_{L^2(\Omega)} \le \frac{C}{1+\lambda_1 t^{\alpha}}\|a\|_{L^2(\Omega)}, \quad t \ge 0.$$

$0 < \alpha < 1$: subdiffusion

α: small $\Longleftrightarrow$ Slow diffusion



Normal-diffusion（α=1）



Sub-diffusion（α=0.5）

Contents

- Backward problems
- Unique continuation: 1D case
- determination of fractional orders:
  $\partial_t^\alpha u + r\partial_t^\beta u = \Delta u$
- 1D inverse problem of determining order and one coefficient
- coefficient inverse problems
- determination of nonlinear terms

Motivation

For parabolic or hyperbolic equations, we have classical inverse problem:

- Backward problems
- Unique continuation
- coefficient inverse problems
- determination of nonlinear terms

We will study such kinds of inverse problems for fractional diffusion equations

Fractional diffusion equations

$x \in \Omega \subset \mathbb{R}^n$: bounded domain

$n-1 < \alpha < n$, $n \in \mathbb{N}$

$$\partial_t^{\alpha} u + \sum_{k=1}^{N} a_k(x)\partial_t^{\alpha_k} u = -Au$$

$$:= \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + c(x)u,\ t > 0$$

$$\partial_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} g(\tau) d\tau.$$

for $n - 1 < \alpha < n$, $n \in \mathbb{N}$

$$a_{ij} \in C^1(\overline{\Omega}), c \leq 0, \in C(\overline{\Omega})$$

Characteristics of $\partial_t^\alpha u + \ldots$

∂_t^α is with memory effect $\Longrightarrow$

1. not strong smoothing property
2. infinite propagation speed

IBVP for fractional diffusion equations

intermediate regularity between parabolic and hyperbolic equations

- Hölder maximal regularity:
  $\partial_t^\alpha u + Au = F(x, t)$
- Maximum principle
- Nonlinear dynamical system

Glance at comprehensive project

- Mathematics
  - Partial differential equation
  - Fractal structure of medium
  - Multiscale modelling, homogenization,....
- Engineering:

  risk management, environmental science,...
- Industry: many anomalous diffusions

Engine for project

Study Group for Solving Industrial Problems

actual working place by industry,

mathematicians,...

17 January - 23 January 2013

Graduate School of Mathematical Sciences

The University of Tokyo

Now mathematics!

1. Backward problem

Backward problem for $\alpha = 1$ severely ill-posed

Backward problem for $\alpha = 2$ well-posed

Backward problem for $\alpha \neq 1, \neq 2 \Longrightarrow$

moderately ill-posed!

Theorem 1 (i) Let $0 < \alpha < 1$.

$$\partial_t^\alpha u = -Au(x,t), \quad u|_{\partial\Omega} = 0,$$

$$u(x,T) = a_1(x), \quad x \in \Omega.$$

For $\forall T > 0$ and $\forall a_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution

$\exists^1 u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$

and $\|u(\cdot,0)\|_{L^2(\Omega)} \sim \|u(\cdot,T)\|_{H^2(\Omega)}$.

Remark:~ means both-sided estimate

(ii) Let $1 < \alpha < 2$.

$$\partial_t^\alpha u = -Au(x,t), \quad u|_{\partial\Omega} = 0$$

Then $\exists C > 0$ such that

$$\|u(\cdot,0)\|_{L^2(\Omega)} + \|\partial_t u(\cdot,0)\|_{L^2(\Omega)}$$
$$\le C(\|u(\cdot,T)\|_{H^2(\Omega)} + \|\partial_t u(\cdot,T)\|_{H^2(\Omega)})$$

with K. Ito (North Carolina State Univ.)
and K. Sakamoto

2. Unique continuation

With J. Liu and L. Yan (Southeast Univ.)

Unique continuation:

$\partial_t^\alpha u = -Au$ in $Q := \Omega \times (0, T)$

$u = \partial_\nu u = 0$ on $\Gamma \times (0, T)$

$\Longrightarrow \exists D \subset\subset Q$ where $u = 0$.

Main tool for $\alpha = 1$

Carleman estimate $\Leftarrow$ integration by part

Main differences from $\alpha = 1$

- Integration by part fails for ∂_t^α
- ∂_t^α has memory effect

Unique continuation

Theorem $\partial_t^{\frac{1}{2}} u - \partial_1^2 u = 0$ in Q, $u(x, 0) = 0$, $0 < x < \ell$
$u(0, t) = \partial_1 u(0, t) = 0$, $0 < t < T$ implies $u = 0$ in Q

Remark: Also conditional stability

Key: Carleman estimate (= L^2-weighted estimate)

- infinite propagation speed
- we need initial condition
- only 1D case.
- No Carleman estimate for general α

Unique continuation for general α

$$\partial_t^\alpha u = u_{xx}, \quad 0 < x < 1,\ t > 0,$$

$$u(0,t) = f(t), \quad t > 0$$

$$u(x,0) = 0, \quad 0 < x < 1.$$

Inverse problem Let $\omega \subset (0,\ell)$ be fixed. Determine $u(1,t)$, $0 < t < T$ by $u|_{\omega\times(0,T)}$.

Theorem (J.Liu-L.Yan-Yamamoto)

Let $\frac{1}{2} < \alpha < 1$ and $u(1,0) = 0$.

$\exists C > 0$ such that

$$\|u(1,\cdot)\|_{L^2(0,T)} \le C(\|h\|_{L^2(\omega\times(0,T))} + \|f\|_{L^2(0,T)}).$$

- Case $\alpha = 1$: severely ill-posed, logarithmic conditional stability
- Case $\frac{1}{2} < \alpha < 1$: unconditional Lipschitz stability $\|u(1,\cdot)\|_{L^2(0,T)} \sim \|u\|_{L^2(\omega\times(0,T))}$ if $f = 0$.

$\Longleftarrow$ regularity in non-homogeneous boundary value problem

(by K. Fujishiro)

Interpretation

Fractional diffusions equation:

no strong smoothing property

The inverse problem is more well-posed than the classical diffusion equation.

3. Determination of fractional orders

Let $0 < \beta < \alpha < 1$.

$u_{\alpha,\beta}$:

$\partial_t^\alpha u + r_0 \partial_t^\beta u = -Au$ in $\Omega \times (0, T)$

$u|_{\partial\Omega} = 0$

$u(x, 0) = a(x)$, $x \in \Omega$

Inverse Problem: Let $x_0 \in \Omega$, $T > 0$ be fixed.

Determine (α, β) by $u_{\alpha,\beta}(x_0, t)$, $0 < t < T$.

Theorem (uniqueness) Let: $a \not\equiv 0, \geq 0$, smooth (e.g., $a \in H_0^3(\Omega)$ in 3D case). Then $u_{\alpha_1,\beta_1}(x_0, t) = u_{\alpha_2,\beta_2}(x_0, t)$, $0 < t < T$ implies $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

Remark ?? Uniqueness for

$$\partial_t^{\alpha_0} + \sum_{k=1}^{N} r_k \partial_t^{\alpha_k}$$

Other determination of orders

$u_{\alpha,\gamma}$:

$\partial_t^\alpha u = -A^\gamma u$ in Q

$u|_{\partial\Omega} = 0$, $u(\cdot, 0) = a$

Theorem (uniqueness) Let: $a \not\equiv 0, \geq 0$, smooth (e.g., $a \in H_0^3(\Omega)$ in 3D case). If $u_{\alpha_1,\gamma_1}(x_0, t) = u_{\alpha_2,\gamma_2}(x_0, t)$, $0 < t < T$, then $\alpha_1 = \alpha_2$ and $\gamma_1 = \gamma_2$.

4. Determination of one derivative order and coefficient

J.Cheng (Fudan Univ.) -
J. Nakagawa (Nippon Steel)
- Yamamoto - T.Yamazaki

$$\partial_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right), \; 0 < x < \ell, \; 0 < t < T,$$

$u(x,0) = \delta:$ delta function, $u_x(0,t) = u_x(\ell,t) = 0$

Inverse problem: Determine $\alpha \in (0,1)$ and $p(x)$ by $u(0,t)$, $0 < t < T$.

Answer: Uniqueness holds.

Key to proof: Gel'fanf-Levitan theory + eigenfunction expansion

G. Li, D. Zhang, X. Jia (Shandong Univ. Tech.) and M. Yamamoto:

Numerical reconstruction of p and α

5. Inverse source problem

with K. Sakamoto, Z. Li

$$\partial_t^\alpha u(x,t) = -Au(x,t) + \mu(t)f(x),\ (x,t) \in Q$$

A: symmetric uniformly ellitpic with
x-dependent coefficients, e.g., $A = -\Delta$

Let: $u(\cdot,0)$, $\partial_A u|_{\partial\Omega\times(0,T)}$: conormal: given.

Inverse source problems: $\Gamma \subset \partial\Omega$:

- Type I: Determine $f(x)$, $x \in \Omega$, from $u|_{\Gamma\times(0,T)}$, for given $\mu(t)$.
- Type II: Determine $\mu(t)$, $0 \le t \le T$, from $u|_{\Gamma\times(0,T)}$ for given $f(x)$.

Uniqueness for Type I

Let: $f \in H_0^3(\Omega)$, $\mu \in C^1[0,T]$, $\mu \not\equiv 0$,

$\Gamma \subset \partial\Omega$: arbitrary sub-boundary

Then $u(x,t) = 0$, $x \in \Gamma$, $0 < t < T$

implies $f = 0$ in Q.

Sketch of Proof

1. We can prove:

   $$\partial_t^\alpha v = -Av \text{ in } Q,$$

   $$v|_{\Gamma\times(0,T)} = \partial_A v|_{\partial\Omega\times(0,T)} = 0$$

   $$\Longrightarrow v = 0 \text{ in } Q$$

2. Duhamel's principle

$\partial_t^\alpha u = -Au + \mu(t)f$, $u|_{\partial\Omega} = 0$, $u(\cdot, 0) = 0$

$\partial_t^\alpha v = -Av$, $v|_{\partial\Omega} = 0$, $v(\cdot, 0) = f$

$$u(x,t) = \int_0^t \theta(t-s)v(x,s)ds \quad \text{in } Q$$

Here $\theta(t) = D_t^{\alpha-1}\mu(t) := \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \frac{\mu(s)}{(t-s)^{1-\alpha}}ds$

Stability for Type II

Let f be smooth and $f(x_0) \neq 0$. Then $\|\partial_t^\alpha u(x_0, \cdot)\|_{C[0,T]} \sim \|\mu\|_{C[0,T]}$.

Without $f(x_0) \neq 0$?

Uniqueness Let Γ be arbitrary subboundary and $f \not\equiv 0$.

Then $u|_{\Gamma\times(0,T)} = 0 \Rightarrow \mu = 0$ in $(0, T)$

Counter-example for data $u(x_0, t)$ with $f(x_0) = 0$:

$\Omega = (0,1)$, $f(x) = \cos \pi x$, $x_0 = \frac{1}{2}$

And $u(x_0, t) = 0$, $0 < t < T$ for any $\mu(t)$.

One point observation does not guarantee the uniqueness!

6. Inverse coefficient problem by Carleman estimate

by Y. Zhang (Fudan Univ.) and M. Yamamoto

$u(p)(x,t)$:

$\partial_t^{1/2} u = \partial_x^2 u(x,t) + p(x)u(x,t)$ in Q

$u(\cdot,0)$, $u(0,t)$: given

Inverse Problem: fixed $t_0 \in (0,T)$, determine $p(x)$ by $\partial_x u(0,\cdot)$, $u(\cdot,t_0)$.

Conditional stability in inverse coefficient problem

Assume a priori boundedness for p, q and $u(p), u(q)$ and $u(p)(x, t_0), u(q)(x, t_0) \neq 0$, $0 \leq x \leq \ell$.

For small $\delta > 0$, $\exists C = C(\delta) > 0$ and $\exists \kappa = \kappa(\delta) \in (0, 1)$ such that

$$\|p - q\|_{H^2(0,\ell-\delta)} \leq C(\|(u(p) - u(q))(\cdot, t_0)\|_{L^2(0,\ell)} + \|\partial_x(u(p) - u(q))(0, \cdot)\|_{L^2(0,T)})^{\kappa}$$

Key to the proof

- Carleman estimate
- Bukhgeim-Klibanov method

Carleman estimate can be proved for
$\boldsymbol{\alpha = 1}$/[natural number].
$\Longrightarrow$ very limited!

7. Inverse coefficient problem by integral transform

with L. Miller (Univ. Paris Ouest)

Let $0 < \alpha < 2$.

$u(p)$: $\partial_t^\alpha u = \Delta u + p(x)u$, $u|_{\partial\Omega} = 0$,

$u(\cdot, 0) = a$, $\partial_t u(\cdot, 0) = 0$ (if $1 < \alpha < 2$)

$w(p)$: $\partial_t^2 w = \Delta w + p(x)w$, $w|_{\partial\Omega} = 0$,

$w(\cdot, 0) = a$, $\partial_t w(\cdot, 0) = 0$

Theorem (Bazhlekova 2001)

$u(p)(x, t) = \int_0^\infty K_\alpha(t, s)w(p)(x, s)ds$ in Q

Here $K_\alpha(t,s) = \frac{1}{t^\gamma}\Phi_\gamma\left(\frac{s}{t^\gamma}\right)$, $\gamma = \frac{\alpha}{2}$,

$\Phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n+1-\gamma)}$: Wright function

Lemma 1 (Miller-Yamamoto)

$\int_0^\infty K_\alpha(t,s)f(x,s)ds = 0$ for $x \in \Omega$, $t > 0$ and $||f(\cdot,t)||_{L^2(\Omega)} \le \exists M e^{t\omega}$ with $\omega > 0 \Longrightarrow f = 0$ in Q.

Lemma 2 (Sakamoto-Yamamoto)

$u : (0,T] \longrightarrow L^2(\Omega)$ is analytic in $\left\{z \in \mathbb{C},\ z \neq 0,\ |\arg z| < \frac{\pi}{2}\right\}$.

Theorem (Miller and Yamamoto)

Let $\exists \omega \subset \Omega$ with $\partial\omega \supset \partial\Omega$ and $p = q$ in ω.
Assume that $\inf_{\Omega\setminus\overline{\omega}} |a| > 0$.
Then $u(p) = u(q)$ in $\omega \times (0, T)$
implies $p = q$ in Ω.

Sketch of Proof

$u(p) \Longleftrightarrow w(p)$, $u(q) \Longleftrightarrow w(q)$

$u(p) = u(q)$ in $\omega \times (0, T) \Longrightarrow$

$u(p) = u(q)$ in $\omega \times (0, \infty)$ by Lemma 2

$\Longrightarrow$

$\int_0^\infty K_\alpha(t,s)(w(p) - w(q))(x,s)ds = 0$,

$x \in \omega$, $t > 0$

$\Longrightarrow w(p) = w(q)$ in $\omega \times (0, T)$

$\Longrightarrow p = q$ in Ω

by Imanuvilov and Yamamoto: 2001

8. Determination of nonlinearity

With Y. Luchko (Beuth Tech. Hochschule Berlin) -
W. Rundell, L.Zuo (Texas A&M Univ.)

Let $0 < \alpha < 1$.

$u(f)$: $\partial_t^\alpha u = \Delta u + f(u(x,t))$ in Q

$\partial_\nu u(x,t) = g(x,t)$, $x \in \partial\Omega$, $0 < t < T$,

$u(x,0) = a(x)$, $x \in \Omega$.

We set

$\mathcal{F} = \{f \in C^1(\mathbb{R}) : f'(r) < 0, r \in (m, M)\}$,

$m = \inf_{x\in\Omega} a(x)$, $M = \sup_{x\in\Omega} a(x)$.

Uniqueness Let $f_1, f_2 \in \mathcal{F}$. If $u(f_1)(x_0, t) = u(f_2)(x_0, t)$, $0 \le t \le T$ with some $x_0 \in \Omega$ and $T > 0$, then there exist constants m_1, M_1 with $m \le m_1 < M_1 \le M$ such that $f_1(r) = f_2(r)$ for $m_1 \le r \le M_1$

Remark

- For analytic f_1, f_2, we have global uniqueness.
- numerical method

Thank you very much!