Asymptotic Behavior in Degenerate Parabolic Nonlinear equations and its application to Elliptic Eigenvalue Problems

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Introduction

Let us consider

$$\begin{cases} \triangle\varphi & = -\mu\varphi^p \qquad \text{in } \Omega, \\ \varphi & = 0 \qquad \text{on } \partial\Omega, \\ \varphi & > 0 \qquad \text{in } \Omega, \end{cases}$$

where Ω is smooth and bounded in $\mathbb{R}^n$ and $p > 0$.

- Let Ω be a strictly convex domain in $\mathbb{R}^n$.

  Are the level sets of the positive eigen-function $\varphi(x)$ convex?

  For Laplace operator,

  - $p = 1:$ $\log(\varphi)$, strictly concave.
  - $0 < p < 1:$ $\varphi^{\frac{1-p}{2}}$, strictly concave.

Known results for Laplacian

- $p = 1$,
  - Brascamp, Lieb (1976) , probability method
  - Korevaar (1983), analytical approach

- $0 < p < 1$,
  - Kawohl (1985), Korevaar's idea
  - Lee,Vazquez (2008), parabolic approach

- $1 < p < \frac{n+2}{n-2}$,
  - Lin (1994), for energy minimizer
  - Gladiali, Grossi (2004), for energy minimizing sequence
  - Lee, Vazquez (2008), $\exists\ \varphi$ having strictly convex level sets.

Fully Nonlinear Eigenvalue Problems

- We consider the following elliptic nonlinear eigenvalue problems

$$\begin{cases} F(D^2\varphi) & = -\mu\varphi^p \quad \text{in } \Omega, \\ \varphi & = 0 \quad \text{on } \partial\Omega, \\ \varphi & > 0 \quad \text{in } \Omega, \end{cases} \tag{NLEV}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$.

- Assumptions on Operators.

(F1) F is uniformly elliptic ;

$\exists \; 0 < \lambda \leqslant \Lambda < \infty$ (called ellipticity constants) s.t. for any symmetric matrices M and N,

$$\lambda \|N\| \leqslant F(M+N) - F(M) \leqslant \Lambda \|N\|, \quad \forall N \geqslant 0.$$

- If F is differentiable, $\lambda \mathbf{I} \leqslant (F_{ij}) \leqslant \Lambda \mathbf{I} \quad \left(F_{ij} = \frac{\partial F}{\partial m_{ij}} \right)$

- Pucci's extremal operators

  Let $0 < \lambda \leqslant \Lambda$. For a symmetric matrix M,

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \mathcal{M}^+(M) = \lambda \sum_{e_i<0} e_i + \Lambda \sum_{e_i>0} e_i$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \mathcal{M}^-(M) = \Lambda \sum_{e_i<0} e_i + \lambda \sum_{e_i>0} e_i,$$

  where $e_i = e_i(M)$ are the eigenvalues of M.

  Let $\mathcal{A}_{\lambda,\Lambda}$ be the set of all symmetric matrices whose eigenvalues lie in $[\lambda, \Lambda]$. Then,

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}(AM), \qquad \mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}(AM).$$

(F2) F is positively homogeneous of order one;

$$F(tM) = tF(M), \ \ \forall t \geqslant 0, \quad \forall M \in \mathcal{S}^n.$$

  - If F is differentiable, then F is a linear operator with constant coefficients.

Existence and Uniqueness of (NLEV)

Theorem ($p = 1$, Ishii and Yoshimura)

Let F *satisfy* $(F1), (F2)$. *Then,* $\exists$ *positive solution* $\varphi \in C^{1,\alpha}(\overline{\Omega})$ *of* (NLEV) *and the eigenvalue* $\mu > 0$ *is unique and simple.*

$\Rightarrow$ *uniqueness up to a constant multiple*

Theorem ($0 < p < 1$)

$\exists$ *a unique positive solution* $\varphi \in C^{0,1}(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$ *for a given* $\mu > 0$.

- Comparison Principle for $0 < p < 1$
- Perron's Method

Fully Nonlinear Parabolic Flows

- For $m > 0$,

$$\begin{cases} F(D^2u^m) & = u_t & \text{in } \Omega \times (0,\infty), \\ u(x,t) & = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) & = u_o(x) \geqslant 0 & \text{in } \Omega, \end{cases} \tag{PE}$$

  posed in a smooth bounded domain $\Omega \subset \mathbb{R}^n$.

- Example. $F = \triangle$,
  - $m = 1$: heat equation
  - $m > 1$: porous medium equation or slow diffusion equation
  - $0 < m < 1$: fast diffusion equation

- The positive eigenfunction $\varphi(x)$
  $=$ Limit of normalized function of $u(x,t)$ as $t \to +\infty$.
  - $m = \frac{1}{p}$

Main Result ($m = 1$)

Theorem

Suppose that F satisfies $(F1), (F2)$ and is concave and that Ω is convex.
If $\log u_o$ is concave, then $\log u(x,t)$ is concave in x for $t > 0$.

Theorem ($p = 1$)

Under the same condition, $\log \varphi(x)$ is concave,
where φ is the positive eigenfunction of (NLEV) with $p = 1$.

Remark

(i) *$\mathcal{M}^-_{\lambda,\Lambda}$ is a nontrivial example of the concave operators satisfying $(F1), (F2)$.*

(ii) *Concavity of F is required when we study geometric property of solutions.*

Main Result $(m > 1)$

Theorem

Suppose that F *satisfies* $(F1), (F2)$ *and is concave and that* Ω *is convex.*
Let u_o *satisfy* $-Cu_o \leqslant F(D^2 u_o^m) \leqslant 0$ *for* $C > 0$.
If $u_o^{\frac{m-1}{2}}$ *is concave, then* $u^{\frac{m-1}{2}}(x, t)$ *is concave in* x *for* $t > 0$.

Theorem $(0 < p = \frac{1}{m} < 1)$

Under the same condition, $\varphi^{\frac{1-p}{2}}(x) = \varphi^{\frac{m-1}{2m}}(x)$ *is concave,*
where φ *is the positive eigenfunction of* (NLEV) .

Remark

The condition on initial data u_o *might be removable.*

Parabolic Approach

1. Show convergence to the positive eigenfunction $\varphi(x)$ after suitable normalization of $u(x,t)$ as $t \to +\infty$
   - $$e^{\mu t} u(x,t) \qquad \text{for } m = 1$$
     $$t^{\frac{1}{m-1}} u(x,t) \qquad \text{for } m > 1$$
2. Some geometric quantity will be preserved under the flow.
   - $$f(u) = \log(u) \quad \text{for } m = 1$$
     $$f(u) = u^{\frac{m-1}{2}} \qquad \text{for } m > 1$$
   - the idea of K.-A. Lee and J.L. Vazquez.
3. The geometric property for Eigenvalue problem will be obtained in the limit as $t \to +\infty$.

Uniformly Parabolic Equation (m=1)

$$\begin{cases} F(D^2u) & = u_t & \text{in } \Omega \times (0,\infty), \\ u(x,t) & = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) & = u_o(x) \geqslant 0 & \text{in } \Omega. \end{cases} \qquad \text{(PE)}$$

- $\varphi(x)e^{-\mu t}$ is a similarity solution of (PE), where μ is the principal eigenvalue and φ is the solution of

$$\begin{cases} F(D^2\varphi) + \mu\varphi & = 0 & \text{in } \Omega \\ \varphi & = 0 & \text{on } \partial\Omega \\ \varphi & > 0 & \text{in } \Omega. \end{cases} \qquad \text{(NLEV)}$$

- Let $v(x,t) := e^{\mu t}u(x,t)$. Then v solves

$$F(D^2v) + \mu v = v_t.$$

Lemma (Uniform Convergence, $m = p = 1$)

Let F *satisfy* (F1), (F2). *Then,* $\exists \gamma^* > 0$ *s.t.*

$$v(x,t) = e^{\mu t}u(x,t) \to \gamma^*\varphi(x) \quad \textit{uniformly in } \overline{\Omega} \ \textit{ as } \ t \to +\infty.$$

Proof.

- $\exists t_o > 0$ s.t. $0 < C_1\varphi(x) < u(x,t_o) < C_2\varphi(x)$ and hence for $t \geqslant t_o$,

$$C_1\varphi(x)e^{-\mu t} < u(x,t) < C_2\varphi(x)e^{-\mu t}$$

  by Comparison principle

  Then, $v(x,t)$ is bounded. From the Weak Harnack inequalities for v,

$$\sup_{s\geqslant 1} \|v(\cdot,\cdot+s)\|_{C^{\alpha}_{x,t}(\overline{\Omega}\times[0,+\infty))} < +\infty \quad \text{for } 0<\alpha<1.$$

- Let $\mathcal{A} := \{\text{all sequential limits of } \{v(\cdot,\cdot+s)\}_{s\geqslant 0}\}$ and

$$\gamma^* := \inf\{\gamma > 0 : \exists w \in \mathcal{A} \text{ s.t. } w(x,t) \leqslant \gamma\varphi(x) \text{ in } \Omega\times(0,\infty)\}.$$

$$\Rightarrow \quad 0 < C_1 < \gamma^* < C_2 < +\infty.$$

- We show $\mathcal{A} = \{\gamma^*\varphi\}$.

  (by Maximum principle, Regularity theory.)

  (i) $w \leqslant \gamma^*\varphi \quad \forall w \in \mathcal{A}$, (ii) $w = \gamma^*\varphi \quad \forall w \in \mathcal{A}$

□

Lemma (Log-concavity)

Suppose that F *satisfies* $(F1)$, $(F2)$ *and is concave and* Ω *is strictly convex. If* $\log(u_o)$ *is concave, then* $\log(u(x,t))$ *is concave in x for all* $t > 0$, *i.e.,*

$$D_x^2 \log(u(x,t)) \leqslant 0 \quad \textit{for all } t > 0.$$

Proof.

- Approximate the operator F by smooth, concave F^ε satisfying $(F1)$,

$$|F^\varepsilon(M) - F^\varepsilon_{ij}(M)M_{ij}| \leqslant C\varepsilon. \tag{F2'}$$

 (instead of $(F2)$,) where $F^\varepsilon_{ij}(M) := \frac{\partial F^\varepsilon}{\partial m_{ij}}(M)$.

- Assume $\log(u_o)$ is smooth and strictly concave.
- Let u^ε be the solution of

$$u_t = F^\varepsilon(D^2 u) \quad \text{in } \Omega \times (0, \infty),$$

 with u_o as initial data.

- We put $g^\varepsilon = \log(u^\varepsilon)$. Then g^ε solves

$$\partial_t g = e^{-g} F^\varepsilon \left(e^g (D^2 g + \nabla g \nabla g^t) \right).$$

Question $D^2 g^\varepsilon \leqslant 0$?

- In $\Omega \times (0, T]$, for small $\delta > 0$, define

$$Z(t) := \sup_{y \in \Omega, |e_\beta|=1} g^\varepsilon_{\beta\beta}(y,t) + \psi(t),$$

where $\psi(t) = -\delta \tan(K\sqrt{\delta} t)$ for $K > 0$ independent of $\varepsilon, \delta > 0$

- Suppose that $\exists\, t_o \geqslant 0$ s.t.

$$Z(t) := \sup_{y \in \Omega, |e_\beta|=1} g^\varepsilon_{\beta\beta}(y,t) + \psi(t) = 0 \quad \text{at } t = t_o$$

$$= g^\varepsilon_{\alpha\alpha}(x_o, t_o) + \psi(t_o),$$

and assume that t_o is the first time.

- We note that $Z(0) < 0$ and hence $t_o > 0$.

($\because$ $g^\varepsilon(\cdot, 0) = \log u_o$ is strictly concave.)

- Boundary estimates; as $x \in \Omega \to \partial\Omega$

$$g^{\varepsilon}_{\alpha\alpha} = \frac{u^{\varepsilon}\, u^{\varepsilon}_{\alpha\alpha} - (u^{\varepsilon}_{\alpha})^2}{(u^{\varepsilon})^2} = \frac{u^{\varepsilon}_{\alpha\alpha}}{u^{\varepsilon}} - \frac{(u^{\varepsilon}_{\alpha})^2}{(u^{\varepsilon})^2} \;\to\; -\infty$$

1. $e_\alpha = e_\nu$, a normal vector to $\partial\Omega$,
    - $|\nabla u^{\varepsilon}| = -u^{\varepsilon}_{\nu} > 0$ on $\partial\Omega$ by Hopf's lemma
    - $u^{\varepsilon} = 0$ on $\partial\Omega$ and $|D^2 u^{\varepsilon}| < C$ in Ω.
2. $e_\alpha = e_\tau$, a tangential vector to $\partial\Omega$,
    - $u^{\varepsilon}_{\tau} = 0$ on $\partial\Omega$
    - Strict convexity of Ω $\Rightarrow$ $u^{\varepsilon}_{\tau\tau} = u^{\varepsilon}_{\nu}\kappa_\tau < -c_o < 0$ on $\partial\Omega$,
      where e_ν, outward normal to $\partial\Omega$, κ_τ = curvature of $\partial\Omega$ in e_τ

$\Rightarrow$ The maximum point x_o should be in Ω.

- $g^\varepsilon_{\alpha\alpha}$ satisfies

$$
\begin{aligned}
g_{\alpha\alpha,t} = {} & F^\varepsilon_{ij} \cdot \left(D_{ij} g_{\alpha\alpha} + D_i g_{\alpha\alpha} D_j g + D_i g D_j g_{\alpha\alpha} + 2 D_i g_\alpha D_j g_\alpha \right) \\
& + (g^2_\alpha - g_{\alpha\alpha}) \\
& \cdot \left\{ e^{-g} F^\varepsilon \left(e^g \left(D^2 g + \nabla g \nabla g^t \right) \right) - F^\varepsilon_{ij} \cdot (D_{ij} g + D_i g D_j g) \right\} \\
& + e^{-g} F^\varepsilon_{ij,kl} \cdot \left(e^g (D_{ij} g + D_i g D_j g) \right)_\alpha \cdot \left(e^g (D_{kl} g + D_k g D_l g) \right)_\alpha \\
\leqslant {} & F^\varepsilon_{ij} \cdot \left(D_{ij} g_{\alpha\alpha} + D_i g_{\alpha\alpha} D_j g + D_i g D_j g_{\alpha\alpha} + 2 D_i g_\alpha D_j g_\alpha \right) \\
& + |g^2_\alpha - g_{\alpha\alpha}| e^{-g} C \varepsilon,
\end{aligned}
$$

from (F2′), Concavity of F^ε,

where
$F^\varepsilon_{ij} := F^\varepsilon_{ij} \left(e^g \left(D^2 g + \nabla g \nabla g^t \right) \right)$, $F^\varepsilon_{ij,kl} := F^\varepsilon_{ij,kl} \left(e^g \left(D^2 g + \nabla g \nabla g^t \right) \right)$.

- At the maximum point x_o , we have
  - $\nabla_x g^\varepsilon_{\alpha\alpha} = 0, \quad D^2_x g^\varepsilon_{\alpha\alpha} \leqslant 0$
  - $g^\varepsilon_{\alpha\beta} = 0 \quad$ for $\beta \neq \alpha$

- At the maximum point (x_o, t_o),

$$\partial_t g^\varepsilon_{\alpha\alpha} \leqslant 2F^\varepsilon_{\alpha\alpha}(g^\varepsilon_{\alpha\alpha})^2 + K\varepsilon \leqslant 2\Lambda(g^\varepsilon_{\alpha\alpha})^2 + K\varepsilon$$

for $K := C(\Lambda, n)\left(1 + \max_{\Omega_{(-\eta)}\times(0,T)} \frac{|D^2 u_\varepsilon|}{u_\varepsilon^2}\right)$(from uniform $C^{2,\gamma}$-estimates.)

$\Rightarrow$ at $t = t_o$,

$$\begin{aligned} 0 \leqslant Z'(t_o) &= \partial_t g^\varepsilon_{\alpha\alpha}(x_o, t_o) + \psi_t(t_o) \\ &\leqslant \psi_t + 2\Lambda\psi^2 + K\varepsilon \leqslant \psi_t + K(\psi^2 + \varepsilon), \end{aligned}$$

but $\psi(t) = -\delta \tan(K\sqrt{\delta}t)$,

$$\psi_t + K(\psi^2 + \varepsilon) < \frac{K(-\delta^{3/2} + \delta^2)}{\cos(K\sqrt{\delta}t)} < 0$$

for $0 < \varepsilon \ll \delta$ and for $K\sqrt{\delta}t < \frac{\pi}{4}$, which is a contradiction.

Therefore,

$$Z(t) = \sup_{y\in\Omega, |e_\beta|=1} g^{\varepsilon}_{\beta\beta}(y,t) + \psi(t) < 0,$$

i.e., for any e_β with $|e_\beta| = 1$,

$$\partial_{\beta\beta} \log(u^{\varepsilon}) < -\psi(t) = \delta \tan(K\sqrt{\delta}t) \leqslant \delta$$

for $0 < t < \min\left(\frac{\pi}{4K\sqrt{\delta}}, T\right)$ and for $0 < \varepsilon \ll \delta$.

- Letting $\varepsilon \to 0$ and $\delta \to 0$, we conclude that

$$\partial_{\beta\beta} \log(u) \leqslant 0 \quad \text{in} \quad \Omega \times (0,T].$$

□

Corollary (Log-concavity, $p = 1$)

Suppose that F *satisfies* $(F1)$, $(F2)$ *and is concave. If* Ω *is convex, the eigenfunction* $\varphi(x)$ *is log-concave, i. e.,* $D^2 \log(\varphi(x)) \leqslant 0$ *in* Ω.

Proof.

- Take $\operatorname{dist}(x, \partial\Omega)$ as initial data $u_o(x)$.
  - If Ω is convex, then $\operatorname{dist}(x, \partial\Omega)$ is concave and also log-concave.
- Let u solve (PE) with $u_o(x) = \operatorname{dist}(x, \partial\Omega)$.

$\Rightarrow$ $u(x,t)$ is log-concave for any $t > 0$, i.e.,

$$\frac{1}{2}\left(\log u(x,t) + \log u(y,t)\right) - \log u\left(\frac{x+y}{2}, t\right) \leqslant 0.$$

- Uniform Convergence: $\|e^{\mu t}u(x,t) - \gamma^* \varphi(x)\|_{C^0_x(\overline{\Omega})} \to 0$ as $t \to \infty$.

$$\Rightarrow \quad \frac{1}{2}\left(\log \varphi(x) + \log \varphi(y)\right) - \log \varphi\left(\frac{x+y}{2}\right) \leqslant 0.$$

Degenerate parabolic Equation ($m > 1$)

$$\begin{cases} F(D^2 u^m) = u_t & \text{in } \Omega \times (0, \infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x,0) = u_o(x) > 0 & \text{in } \Omega, \end{cases} \qquad \text{(DPE)}$$

- (DPE) is degenerate at $u = 0$
  $\because$ diffusion coefficient $= m u^{m-1}$
- Gas flow in a porous medium,
  Underground water infiltration
- $u =$ density, $v := u^{m-1} =$ pressure,
- Scaling property
  - u is a solution of (DPE)
    $\Rightarrow$ So is $\tilde{u}(x,t) := Au(Bx, Ct)$ when $C = A^{m-1}B^2$.

- Barenblatt (sub-) solutions

$$V(x,t) = t^{-\alpha}\left(c - k\frac{|x|^2}{t^{\beta}}\right)_+^{\frac{1}{(m-1)}},$$

where $\alpha = \frac{n\Lambda}{2\lambda+n(m-1)\Lambda}$, $\beta = \frac{2\lambda}{2\lambda+n(m-1)\Lambda}$, $k = \frac{1}{2(2\lambda+n(m-1)\Lambda)}$
and any $c > 0$.

: source-type solution of (DPE)

- waiting time , free boundary problem

- We assume

$$0 < c_o \operatorname{dist}(x, \partial\Omega) \leqslant u_o(x) \leqslant C_o \operatorname{dist}(x, \partial\Omega) \quad \text{in } \Omega$$

for some $0 < c_o < C_o < +\infty$.

Asymptotics

Lemma (Aronson-Benilan inequality)

Let F *satisfy* $(F1)$ *and be concave and let* $v = u^{m-1}$. *Then,*

$$F(D^2 u^m) = u_t \geqslant -C(m)\frac{u}{t} \qquad \textit{and} \qquad v_t \geqslant -C(m)\frac{v}{t}.$$

Lemma (Uniform Convergence, $m = \frac{1}{p} > 1$)

Let F satisfy $(F1), (F2)$. *Then,*

$$t^{\frac{1}{m-1}} u(x,t) \;\to\; \varphi^{\frac{1}{m}}(x) \quad \textit{uniformly in } \overline{\Omega} \;\; \textit{as} \;\; t \to +\infty,$$

where φ *is the positive eigenfunction of* (NLEV) *with eigenvalue* $\frac{1}{m-1}$.

- We note that for any $\tau > 0$,

$$U(x,t) := \frac{\varphi^{\frac{1}{m}}(x)}{(\tau + t)^{\frac{1}{m-1}}} \quad \text{is a similarity solution of (DPE) with } m > 1.$$

Square root concavity of the pressure

- Approximate the equation: for $0 < \eta < 1$,

$$\begin{cases} F(D^2 u_\eta^m) & = \partial_t u_\eta & \text{in } \Omega \times (0,\infty), \\ u_\eta(x,t) & = \eta & \text{on } \partial\Omega \times (0,\infty), \\ u_{\eta,o}(x) & > \eta & \text{in } \Omega. \end{cases} \qquad \text{(DPE')}$$

- For each $\eta > 0$, the equation for $g_\eta := u_\eta^m$:

$$m g_\eta^{1-\frac{1}{m}} F(D^2 g_\eta) = \partial_t g_\eta$$

becomes a uniformly parabolic equation. $(\because \quad u_\eta(x,t) \geqslant \eta > 0\ )$

- Let $g := u^m$ and $g_\eta := u_\eta^m$.

Let $w := \sqrt{v} = u^{\frac{m-1}{2}} = g^{\frac{m-1}{2m}}$ and $w_\eta := \sqrt{v_\eta} = u_\eta^{\frac{m-1}{2}} = g_\eta^{\frac{m-1}{2m}}$.

Lemma (Uniform Lipschitz estimates)

$$|\nabla_x u_\eta^m| = |\nabla_x g_\eta| < C \qquad \textit{uniformly in} \quad \Omega \times (0,T].$$

$\Rightarrow$ It suffices to show concavity of $w_\eta = u_\eta^{\frac{m-1}{2}}$ for each $\eta > 0$.

Lemma (Boundary estimates)

Let F satisfy $(F1)$, $(F2)$ and be concave and let Ω be strictly convex. Assume $-Cu_o \leqslant F(D^2 u_o^m) \leqslant 0$ for $C > 0$. Then, for small $\eta > 0$, and for any e_α,

$$w_{\eta,\alpha\alpha}(x,t) = \frac{m-1}{2m g_\eta^{2-\frac{m-1}{2m}}} \left(g_\eta g_{\eta,\alpha\alpha} - \frac{m+1}{2m} g_{\eta,\alpha}^2 \right) \leqslant -\frac{c_o}{\eta^{\frac{m+1}{2}}}$$

on $(x,t) \in \partial\Omega \times (0,T]$, where $c_o > 0$ is independent of $\eta > 0$.

Remark

(i) *The boundary estimate holds if $|D^2 u_\eta^m| = |D^2 g_\eta|$ is uniformly bounded in $\overline{\Omega} \times (0,T]$ w.r.t. $\eta > 0$.*

(ii) *To get (i), we assume $-Cu_o \leqslant F(D^2 u_o^m) \leqslant 0$ for $C > 0$.*

($\Rightarrow \partial_t u/u$ is bounded.)

(iii) *In general, we need to prove a weighted $C^{2,\gamma}$- estimate of u_η up to the boundary.*

Lemma (Square root concavity of the pressure)

Suppose that F *satisfies* $(F1), (F2)$ *and is concave and* Ω *is strictly convex. Assume "Boundary Estimate" holds.*
If $\sqrt{u_o^{m-1}}$ *is concave, then* $\sqrt{v} = \sqrt{u^{m-1}}(x, t)$ *is concave in* x *for* $t > 0$.

Proof.

- Fix $\eta > 0$. Then $g_\eta = u_\eta^m$ solves a uniformly parabolic equation .

- Approximate the equation: for $0 < \varepsilon, \eta < 1$,

$$F^\varepsilon(D^2(u_\eta^\varepsilon)^m) \quad = \partial_t u_\eta^\varepsilon \qquad \text{in } \Omega \times (0, \infty),$$

  with smooth, concave F^ε satisfying $(F1)$

$$|F^\varepsilon(M) - F_{ij}^\varepsilon(M) M_{ij}| \leqslant C\varepsilon. \qquad \text{(F2')}$$

- $w^\varepsilon := (u_\eta^\varepsilon)^{\frac{m-1}{2}} = (g_\eta^\varepsilon)^{\frac{m-1}{2m}}$ satisfies

$$\partial_t w = \frac{m-1}{2} w^{\frac{m-3}{m-1}} F^\varepsilon\left(\frac{2m}{m-1} w^{\frac{3-m}{m-1}}\left(w^2 D^2 w + \frac{m+1}{m-1} w \nabla w \nabla w^t\right)\right).$$

Question $D^2 w^\varepsilon \leqslant 0$?

- In $\Omega \times (0, T)$, for small $\delta > 0$, assume

$$\sup_{y \in \Omega, |e_\beta|=1} w^\varepsilon_{\beta\beta}(y, t) + \psi(t) = 0 \quad \text{at } t = t_o$$
$$= w^\varepsilon_{\underline{\alpha\alpha}}(0, t_o) + \psi(t_o),$$

where $\psi(t) = -\varepsilon - e^{-1/\delta} e^{Kt} \tan(K\sqrt{\delta} t)$, and $K > 0$ is independent of $\varepsilon, \delta, \eta > 0$.

- $t_o > 0$ from the initial condition.
- The maximum point $x = 0$ is interior from "Boundary estimates".
- We use the function

$$Z(x, t) = \sum_{\alpha, \beta} w^\varepsilon_{\alpha\beta} \xi^\alpha \xi^\beta, \qquad \text{"tilted } w^\varepsilon_{\underline{\alpha\alpha}}\text{" around } (0, t_o),$$

where $\xi^\beta(x) = \delta_{\overline{\alpha}\beta} + c_{\overline{\alpha}} x^\beta + \frac{1}{2} c_{\overline{\alpha}} c_\gamma x^\gamma x^\beta$, and $\vec{\xi}^t := (\xi^1, \cdots, \xi^n)$ and look at the evolution of

$$Y(x, t) := Z(x, t) + \psi(t) |\vec{\xi}(x)|^2.$$

The remaining argument is similar to the case of $m = 1$.

Corollary ($0 < p = \frac{1}{m} < 1$)

Let F satisfy $(F1), (F2)$ and be concave and let Ω be convex. Then, $\varphi^{\frac{m-1}{2m}} = \varphi^{\frac{1-p}{2}}$ is concave.

Thank you.