Toward S-adic Rauzy fractals

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KIAS Workshop

Tribonacci's substitution [Rauzy '82]

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

$$1213121121312 1213 \cdots$$

$\sigma^{\infty}(1)$ is the fixed point of σ generated by 1
It is called the Tribonacci word

The Tribonacci substitution

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1.$$

The incidence matrix of σ is $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

It is primitive : there exists a power of M which contains only positive entries.

Its characteristic polynomial is $X^3 - X^2 - X - 1$. It admits one positive root $\beta > 1$ (the dominant eigenvalue) and two complex conjugates α, $\overline{\alpha}$, with $|\alpha| < 1$.

β is a Pisot number.

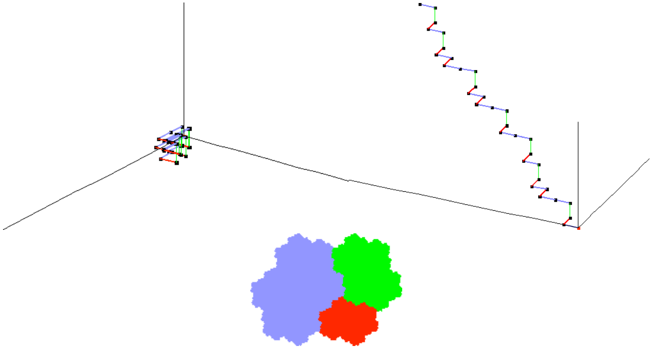
The Tribonacci fractal as a geometric representation of substitutive systems

Consider the Tribonacci substitution $1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 1$.
One represents $\sigma^{\infty}(1)$ as a broken line

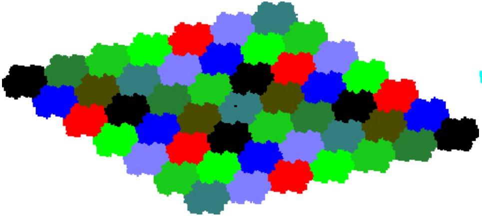
$$\mathbf{f} : \{1, 2, 3\}^* \to \mathbb{Z}^3,\ 1 \mapsto \vec{e}_1,\ 2 \mapsto \vec{e}_2,\ 3 \mapsto \vec{e}_3,$$

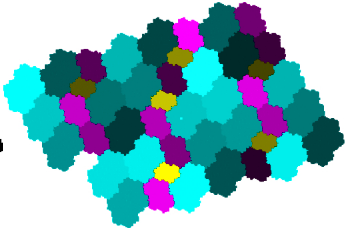
$$\mathbf{f}(w) = |w|_1 \vec{e}_1 + |w|_2 \vec{e}_2 + |w|_3 \vec{e}_3,$$

that we will be projected according to the eigenspaces of M.



Periodic and aperiodic tilings





Rauzy fractals : a geometric representation of substitutive systems

Let σ be a Pisot substitution : there exists a dominant eigenvalue α such that for every other eigenvalue λ,

$$\alpha > 1 > |\lambda| > 0.$$

Then σ is primitive.
σ is said to be a unit substitution if its incidence matrix has determinant ± 1.

Substitutive dynamical systems

Let σ be a primitive substitution over $\mathcal{A}$. Let u be generated by σ. Let S be the shift

$$S((u_n)_n) = (u_{n+1})_n$$

The symbolic dynamical system generated by σ is (X_σ, S) with

$$X_\sigma := \overline{\{S^n(u);\ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

Question Under which conditions is it possible to give a geometric representation of a substitutive dynamical system as a translation on an Abelian compact group ? (discrete spectrum)

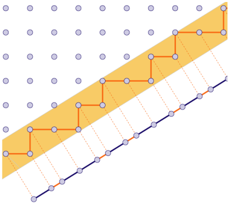
Remark Measure-theoretic discrete spectrum and topological discrete spectrum are equivalent for primitive substitutive dynamical systems [Host], see also [Cortez,Durand,Host,Maass]

Example In the Fibonacci case (X_σ, S) is isomorphic to $(\mathbb{R}/\mathbb{Z}, R_{\frac{1+\sqrt{5}}{2}})$

A geometric representation of substitutive dynamical systems

Abelianisation Let d stand for the cardinality of $\mathcal{A}$

$$f\colon w \in \mathcal{A}^\star \mapsto (|w|_1, |w|_2, \cdots, |w|_d) \in \mathbb{N}^d$$



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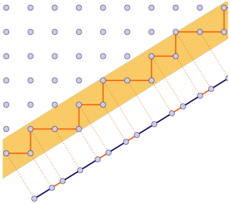
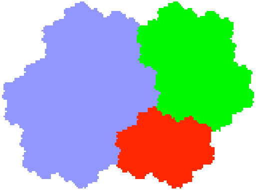
Let u be a periodic point of σ. Let π denote the projection onto the contracting eigenplane of σ along its expanding eigenline.

The Rauzy fractal of σ is defined as :

$$\mathcal{R}_\sigma := \overline{\{\pi \circ f(u_0 \cdots u_{n-1});\ n \in \mathbb{N}\}}.$$





How to reach nonalgebraic parameters ?

- We have considered so far iterations of a single substitution
- We now want to reach nonalgebraic parameters by considering convergent products of matrices
- We want to consider not only a substitution but a sequence of substitutions

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- Multidimensional continued fractions algorithms
- The *S*-adic conjecture : characterization/generation of symbolic flows of at most linear complexity

S-adic expansions

Theorem [Cassaigne] A symbolic flow X has at most linear complexity if and only if the first difference of the complexity $p_X(n+1) - p_X(n)$ is bounded, where $p_X(n)$ counts the number of factors of length n.

Theorem [Ferenczi] Let X be a minimal symbolic system on a finite alphabet $\mathcal{A}$ such that its complexity function $p_X(n)$ is at most linear ; then

- there exist a finite set of substitutions $\mathcal{S}$ over an alphabet $\mathcal{D} = \{0, ..., d-1\}$
- a substitution φ from $\mathcal{D}^\star$ to $\mathcal{A}^\star$
- and an infinite sequence of substitutions $(\sigma_n)_{n\geq 1}$ with values in $\mathcal{S}$

such that

- $|\sigma_1\sigma_2...\sigma_n(r)| \to +\infty$ when $n \to +\infty$, for any letter $r \in \mathcal{D}$
- and any word of the language of the system is a factor of

$$\varphi(\sigma_1\sigma_2...\sigma_n)(0)$$

for some n

S-adic expansions

Definition

A sequence u is said S-adic if there exist

- a finite set of substitutions $\mathcal{S}$ over an alphabet $\mathcal{D} = \{0, ..., d-1\}$
- a morphism φ from $\mathcal{D}^\star$ to $\mathcal{A}^\star$
- an infinite sequence of substitutions $(\sigma_n)_{n\geq 1}$ with values in $\mathcal{S}$

such that

$$u = \lim_{n\to+\infty} \varphi \circ \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0)$$

First remarks

- The fact that the lengths of the words tend to infinity, which generalizes the notion of everywhere growing substitutions, i.e., substitutions such that

$$\forall r,\ \exists n \in \mathbb{N},\ |\sigma_1 \sigma_2 ... \sigma_n(r)| \geq 2,$$

  is necessary to make Ferenczi's S-adic theorem nonempty
- To be S-adic is not a property of the sequence but a way to construct it

Every sequence is S-adic

Let $u = u_0 u_1 u_2 \cdots \in \mathcal{A}^{\mathbb{N}}$. We define for all $n \in \mathbb{N}$ substitutions σ_n over the alphabet $\mathcal{A} \cup \{\ell\}$

$$\sigma_a(b) = b, \forall b \in \mathcal{A},\ \sigma_a(\ell) = \ell a$$

One has

$$|\sigma_{u_0} \circ \sigma_{u_1} \circ \cdots \circ \sigma_{u_n}(\ell)| \to \infty$$

but for all $a \in \mathcal{A}$ and for all n

$$|\sigma_{u_0} \circ \sigma_{u_1} \circ \cdots \circ \sigma_{u_n}(a)| = 1$$

We project by erasing ℓ :

$$\varphi(a) = a, \forall a \in \mathcal{A},\ \varphi(\ell) = \varepsilon.$$

One has

$$u = \lim_{n \to +\infty} \varphi \circ \sigma_{u_0} \circ \sigma_{u_1} \circ \cdots \circ \sigma_{u_n}(\ell)$$

Arithmetic dynamics

Arithmetic dynamics [Sidorov-Vershik] arithmetic codings of dynamical systems that preserve their arithmetic structure

Numeration dynamics [Keane]

Arithmetic dynamics

Arithmetic dynamics [Sidorov-Vershik] arithmetic codings of dynamical systems that preserve their arithmetic structure

Numeration dynamics [Keane]

- Numeration system. Example : Beta-expansions $\sum_{i\geq 1} b_i\beta^{-i}$, $T_\beta : x \mapsto \{\beta x\}$
- Artithmetic codings of automorphisms of the torus [Schmidt]

Arithmetic dynamics

Arithmetic dynamics [Sidorov-Vershik] arithmetic codings of dynamical systems that preserve their arithmetic structure

Numeration dynamics [Keane]
Example Let $R_\alpha : \mathbb{T} \to \mathbb{T},\ x \mapsto x + \alpha$ mod 1. One gets by coding trajectories according to a finite partition an isomorphism between

$$(R_\alpha, \mathbb{T}) \sim (X_\alpha, T)$$

where T is the shift and $X_\alpha \subset \{0,1\}^{\mathbb{N}}$
We also can define a further isomorphism of an arithmetic nature via an odometer

$$(R_\alpha, \mathbb{T}) \sim (K_\alpha, \mathrm{Od})$$

$$\begin{tikzcd} \mathbb{R}/\mathbb{Z} \arrow[r, "R_\alpha"] \arrow[d, "\mathrm{Ostr.}"'] & \mathbb{R}/\mathbb{Z} \arrow[d, "\mathrm{Ostr.}"] \\ K_\alpha \arrow[r, "\mathrm{Od}"'] & K_\alpha \end{tikzcd}$$

Ostrowski expansion of real numbers

The base is given by the sequence $(\theta_n)_{n\geq 0}$, where $\theta_n = (q_n\alpha - p_n)$.

Every real number $-\alpha \leq \beta < 1-\alpha$ can be expanded uniquely in the form

$$\beta = \sum_{k=1}^{+\infty} c_k \theta_{k-1},$$

where

$$\begin{cases} 0 \leq c_1 \leq a_1 - 1 \\ 0 \leq c_k \leq a_k \text{ for } k \geq 2 \\ c_k = 0 \text{ if } c_{k+1} = a_{k+1} \\ c_k \neq a_k \text{ for infinitely many odd integers.} \end{cases}$$

Primitivity and proper S-adic systems

- An s-adic expansion is said primitive if there exists ℓ such that for all $a, b \in \mathcal{A}$ and for all n, then b occurs in $\sigma_{i_n} \circ \cdots \circ \sigma_{i_{n+\ell}}(a)$
- An S-adic expansion is said to have bounded partial quotients if every substitution comes back with bounded gaps in the S-adic expansion

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- An S-adic expansion is said to have bounded partial quotients if every substitution comes back with bounded gaps in the S-adic expansion
- A substitution over $\mathcal{A}$ is said (b, e)-proper if there exist two letters $b, e \in \mathcal{A}$ such that for all $a \in A$ $\sigma(a)$ begins with b and ends with e.
- An S-adic system is said to be proper if there exist (b, e) such that every substitution is (b, e)-proper.
- A subshift which is generated by a proper and primitive S-adic sequence/system is called a proper primitive S-adic subshift

S-adicity and complexity [Cassaigne]

There exists an S-adic sequence with an S-adic expansion having bounded partial quotients, and with each substitution being primitive, whose complexity is quadratic

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Let

$$f: \ a \mapsto aab, \ b \mapsto b, \ g: \ a \mapsto b, \ b \mapsto a.$$

The substitution f has quadratic complexity and the substitution $f \circ g \circ f$ is primitive

The substitutions $f \circ g$ and $g \circ f$ are primitive and appear with bounded gaps

Let us consider the sequence u defined as the limit when n tends to infinity of

$$f \circ g \circ f^2 \circ g \circ f^3 \circ g \circ f^4 \circ \cdots \circ f^n \circ g(b)$$

One has

$$u = \lim_{n \to +\infty} (f \circ g \circ f) \circ (f \circ g \circ f) \circ f \circ (f \circ g \circ f) \cdots \circ (f \circ g \circ f) \circ f^n \circ (f \circ g \circ f) \cdots$$

The complexity of u is quadratic

Linear recurrence : a measure of aperiodic order

Let u be a given recurrent sequence and let W be a factor of the sequence u.

- A return word over W is a word V such that VW is a factor of the sequence u, W is a prefix of VW and W has exactly two occurrences in VW
- A sequence is linearly recurrent if there exists a constant $C > 0$ such that for every factor W, the length of every return word V of W satisfies $|V| \leq C|W|$
- Such a sequence always has at most linear complexity [Durand-Host-Skau]
- But this condition is strictly stronger than having at most linear complexity
- A Sturmian sequence is linearly recurrent if and only if the partial quotients in the continued fraction expansion of its angle/slope are bounded

Tilings and long-range aperiodic order

Discrete planes with irrational normal vector are

- repetitive (uniform recurrence)
- aperiodic

Assume we have a "substitutive" arithmetic discrete plane

Multidimensional substitutive tilings $\rightsquigarrow$ Local/matching rules
[S. Mozes, C. Goodman-Strauss]

Can we recognize/characterize a given "substitutive" arithmetic discrete plane by local inspection ?

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Can we recognize/characterize a given "substitutive" arithmetic discrete plane by local inspection ?

Yes in the Tribonacci case $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$
[Bressaud-Sablik-Pytheas Fogg'09]

From discrete planes to tilings via... number theory

Fact : Arithmetic discrete planes are repetitive.

Repetitivity function : Let N be the smallest integer N such that every ball of radius N in the tiling contains all configurations of radius n. We set $R(n) := N$.

Linear repetitivity : there exists C such that $R(n) \leq Cn$ for all n.

Open problem : Characterize the discrete planes which have linear repetitivity.

Discrete lines : one has linear repetitivity iff and the slope of the line has bounded partial quotients in its continued fraction expansion.

LR and S-adicity

Theorem [F. Durand]

- A proper primitive S-adic subshift is a LR subshift
- LR implies primitive S-adic
- LR is equivalent with primitive and proper S-adic

A primitive S-adic subshift is not necessarily an LR subshift

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Proof

$$\sigma : a \mapsto acb,\ b \mapsto bab,\ c \mapsto cbc$$

$$\tau : a \mapsto abc,\ b \mapsto acb,\ c \mapsto aac$$

We consider the S-adic expansion

$$v := \lim_{n \to +\infty} \sigma \circ \tau \circ \sigma^2 \circ \tau \circ \cdots \circ \sigma^n \tau(a)$$

The sequence v is primitive S-adic, it is not LR, it has linear complexity

[F. Durand, Corrigendum and Addendum to "LR Subshsifts have a finite number of non-periodic factors"]

Examples of S-adic expansions

- Arnoux-Rauzy sequences

  $p(n) = 2n + 1$ + one special factor of each length

$$\sigma_1 : \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{array} \qquad \sigma_2 : \begin{array}{l} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{array} \qquad \sigma_3 : \begin{array}{l} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{array}$$

- Multidimensional continued fractions
  - Jacobi-Perron algorithm
  - Brun algorithm (=modified JP)

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  Periodic Arnoux-Rauzy substitutions are Pisot [Arnoux-Ito]
  There exist AR sequences with unbounded partial quotients wich are not uniformly balanced [J. Cassaigne, S. Ferenczi, L. Zamboni]

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- Multidimensional continued fractions
  - Jacobi-Perron algorithm

    [Sh. Ito, M. Ohtsuki, Paralelogram tilings and Jacobi-Perron algorithm, Tokyo J. Math. 1994]

  - Brun algorithm (=modified JP)

S-adic expansions

One considers

$$u = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$$

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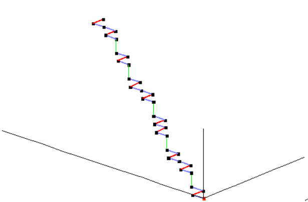
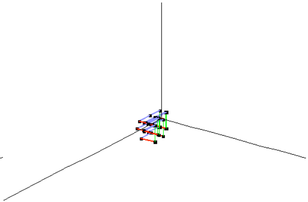
Geometrically





Let p_k be the perfix of u of length k. Do the $f(p_k)$ remain at a bounded distance of a line ?

S-adic expansions

One considers

$$u = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$$

Algebraically

Theorem of Perron–Frobenius type [Furstenberg]

One considers an infinite product of matrices

$$E_1 \cdots E_k \cdots$$

with entries in $\mathbb{N}$. One assumes that there exists a matrix B with strictly positive entries s.t. there exist $i_1 < j_1 < \cdots < i_k < j_k$ s.t.

$$B = E_{i_1} \cdots E_{j_1}, \cdots, B = E_{i_k} \cdots E_{j_k}, \cdots.$$

Then, the intersection of the cones

$$\cap_k \, E_1 \cdots E_k(\mathbb{R}_+^n)$$

is unidimensional.

Convergence speed ? Type of convergence ? Weak ? strong ?

S-adic expansions

One considers

$$u = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$$

Combinatorially

- Frequencies with bounded remainders and balance

$$\exists C,\ \forall i \in \mathcal{A},\ \exists f(i) \text{ t.q. } \forall N\ |\mathrm{Card}\{k \le N,\ u_k = i\} - Nf(i)| \le C$$

S-adic expansions

One considers

$$u = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$$

Arithmetically

• Weak and strong convergence of multidimensional continued fraction algorithms

Theorem

There exists $\delta > 0$ s.t. for almost every (α, β), there exists $n_0 = n_0(\alpha, \beta)$ s.t. for all $n \geq n_0$

$$|\alpha - p_n/q_n| < \frac{1}{q_n^{1+\delta}}$$

$$|\beta - r_n/q_n| < \frac{1}{q_n^{1+\delta}},$$

where p_n, q_n, r_n are given by Brun/Jacobi-Perron.

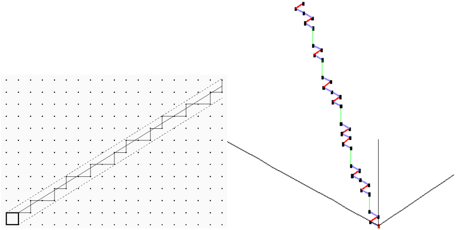
Brun [Ito-Fujita-Keane-Ohtsuki '93+'96] ; Jacobi-Perron [Broise-Guivarc'h '99]

Multidimensional continued fractions

If we start with two parameters (α, β), one looks for two rational sequences (p_n/q_n) et (r_n/q_n) with the same denominator that satisfy

$$\lim p_n/q_n = \alpha, \lim r_n/q_n = \beta.$$

Geometrically



Dynamically

translation on the torus : $R_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2,\ (x, y) \mapsto x + (\alpha, \beta)$

Continued fractions

- Euclid's algorithm Starting with two numbers, one subtracts the smallest to the largest
- Unimodularity

$$\det \begin{bmatrix} p_{n+1} & q_{n+1} \\ p_n & q_n \end{bmatrix} = \pm 1$$

Rem $SL(2, \mathbb{N})$ is a finitely generated free monoid. It is generated by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$SL(3, \mathbb{N})$ is not finitely generated. Consider the family of matrices

$$\begin{pmatrix} 1 & 0 & n \\ 1 & n-1 & 0 \\ 1 & 1 & n-1 \end{pmatrix}$$

These matrices are undecomposable for $n \geq 3$ [Rivat]

Multidimensional Euclid's algorithms : a zoo of algorithms

- Jacobi-Perron : we subtract the first one to the two other ones with $0 \leq x_1, x_2 \leq x_3$

$$(x_1, x_2, x_3) \mapsto (x_2 - [\frac{x_2}{x_1}]x_1, x_3 - [\frac{x_3}{x_1}]x_1, x_1)$$

- Brun : we subtract the second largest and we reorder with $x_1 \leq x_2 \leq x_3$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - x_2)$$

- Poincaré : we subtract the previous one and we reorder with $x_1 \leq x_2 \leq x_3$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2 - x_1, x_3 - x_2)$$

- Selmer : we subtract the smallest to the largest and we reorder with $x_1 \leq x_2 \leq x_2$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - x_1)$$

- Fully subtractive : we subtract the smallest one to all the largest ones and we reorder with $x_1 \leq x_2 \leq x_3$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2 - x_1, x_3 - x_1)$$

Poincaré algorithm [Nogueira'95]

$$(x_1, x_2, x_3) \mapsto (x_1, x_2 - x_1, x_3 - x_2), \; x_1 \leq x_2 \leq x_3$$

$$1/\varphi^2 + 1/\varphi = 1$$

$$\begin{array}{lll}
1/\varphi^2 & 1/\varphi & 100 \\
1/\varphi^3 & 1/\varphi^2 & 100 - 1/\varphi \\
1/\varphi^4 & 1/\varphi^3 & 100 - 1/\varphi - 1/\varphi^2 \\
\cdots & \cdots & \cdots \\
1/\varphi^{k+1} & 1/\varphi^k & 100 - \sum_{i<k} 1/\varphi^i
\end{array}$$

Jacobi-Perron vs. Ostrowski

Jacobi-Perron

$$(\alpha, \beta) \mapsto (\{\beta/\alpha\}, \{1/\alpha\})$$

Ostrowski

$$(\alpha, \beta) \mapsto (\{1/\alpha\}, \{\beta/\alpha\})$$

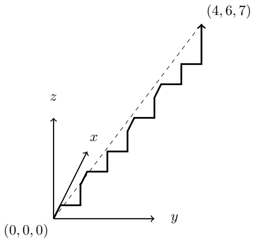
Discrete lines and continued fractions

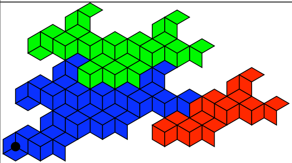
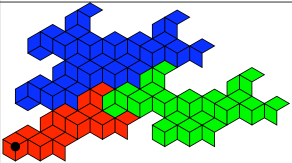
- We apply to $\vec{u}$ a finite sequence of steps under the action of a generalized three-dimensional Euclid's algorithm T together with a choice of Euclid's substitutions $\sigma^{(n)}$ associated with the produced matrices
- One has
$$\vec{u} = M^{(1)} \cdots M^{(N)} \vec{u}^{(N)},$$
where the vector $\vec{u}^{(N)} \in X_0$ has only two coordinates equal to 0, and one coordinate equal to 1
- Let $i_N \in \{1, 2, 3\}$ stand for the index of the nonzero coordinate of $\vec{u}^{(N)}$
- We consider the discrete segment coded by
$$\sigma^{(1)} \cdots \sigma^{(n)}(i_N)$$

[In collaboration with S. Labbé]

Example

Take $u = [4, 6, 7]$ and take the fully subtractive algorithm

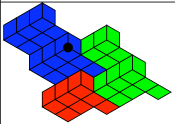
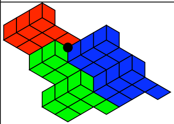


Vecteur Algorithm	$(8, 13, 17)$ `poincare`	
p	Under: 	Over: 
$E_1^*(\sigma)(p)$		

`word: 12312323312312323312312323123233123233`

Complexity $= [1, 3, 5, 7, 9, 11, 13, 15, 17, 18, 19, 20, 21, 21, 21, 21, 21, 21, 21, 20]$

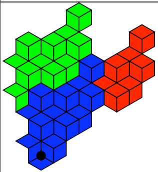
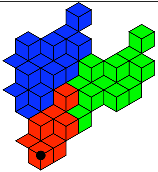
Distance of the word $= 1.8626$

Vecteur Algorithm	$(7, 11, 13)$ `jacobi perron`	
p	Under: 	Over: 
$E_1^*(\sigma)(p)$		

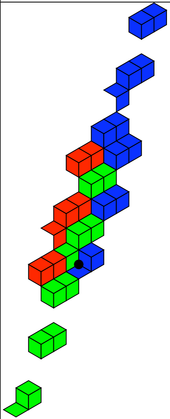
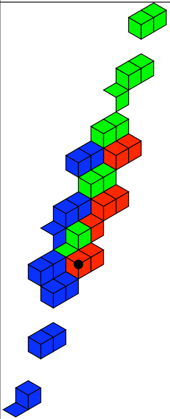
`word: 1233123231232312331232312323123`

Complexity $= [1, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 14, 14, 14, 14, 14, 14, 14, 14, 13]$

Distance of the word $= 1.1521$

Vecteur Algorithm	$(7, 11, 13)$ `brun`	
p	Under: 	Over: 
$E_1^*(\sigma)(p)$		

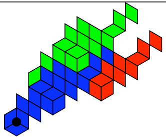
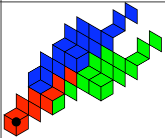
`word: 1231232312312323312323123123233`
Complexity $= [1, 3, 5, 7, 9, 11, 13, 14, 15, 16, 16, 16, 16, 16, 16, 16, 16, 15, 14, 13]$
Distance of the word $= 1.7997$

Vecteur Algorithm	$(7, 11, 13)$ `selmer`	
p	Under: 	Over: 
$E_1^*(\sigma)(p)$		

`word: 132313223132231323132231321322 3`

Complexity $= [1, 3, 6, 8, 10, 12, 14, 15, 16, 17, 18, 19, 19, 19, 18, 17, 16, 15, 14, 13]$

Distance of the word $= 1.3847$

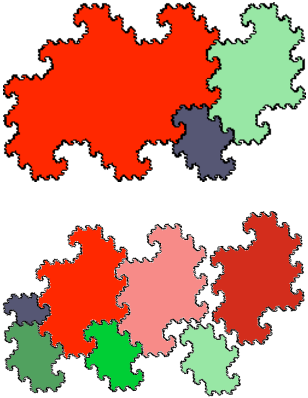
Vecteur	$(7, 11, 13)$	
Algorithm	`fully subtractive`	
p	Under: 	Over: 
$E_1^*(\sigma)(p)$		

`word: 1231232312312323312323123123233`

Complexity $= [1, 3, 5, 7, 9, 11, 13, 14, 15, 16, 16, 16, 16, 16, 16, 16, 16, 15, 14, 13]$

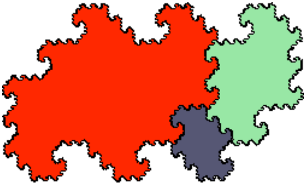
Distance of the word $= 1.7997$

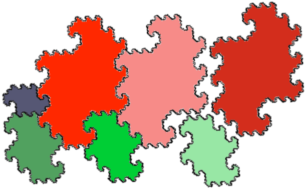
GIFS structure



$$\sigma(1) = 112, \sigma(2) = 113, \sigma(3) = 1$$

GIFS structure





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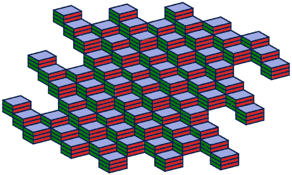

As a consequence of the GIFS structure...

Theorem Let σ be a Pisot irreducible unimodular substitution
The tiles $\mathcal{R}_\sigma(i)$ are solutions of the GIFS

$$\mathcal{R}_\sigma(i) = \bigcup_{\substack{j \in A, \\ i \xrightarrow{(p,i,s)} j}} h_\sigma(\mathcal{R}_\sigma(j)) + \pi_c \circ f(p)$$

where

- h_σ is the restriction of M_σ on its contracting hyperplane
- π_c is the projection on the contracting hyperplane along the expanding line (Pisot hypothesis)

Some properties of the GIFS

The tiles $\mathcal{R}_\sigma(i)$ are solutions of the GIFS

$$\mathcal{R}_\sigma(i) = \bigcup_{\substack{j \in A, \\ i \xrightarrow{(p,i,s)} j}} h_\sigma(\mathcal{R}_\sigma(j)) + \pi_c \circ f(p)$$

- The union $\cup_{i \in \mathcal{A}} \mathcal{R}_\sigma(i)$ is a disjoint union up to sets of zero measure
- The tiles $\mathcal{R}_\sigma(i)$ have non-emtpy interior
- The boundaries of $\mathcal{R}_\sigma$ and $\mathcal{R}_\sigma(i)$ have zero measure
- The tiles $\mathcal{R}_\sigma(i)$ are the closure of their interior
- The subtiles that occur in each decomposition of $\mathcal{R}_\sigma(i)$ are disjoint in measure
- If σ satisfies the strong coincidence conditions, then the tiles $\mathcal{R}_\sigma(i)$ are disjoint in measure

Zooming in : Intersections of subtiles

- The tiles $\mathcal{R}_\sigma(i)$ are solutions of the GIFS

$$\mathcal{R}_\sigma(i) = \bigcup_{\sigma(j)=pis} h_\sigma(\mathcal{R}_\sigma(j)) + \pi_c \circ f(p)$$

- They have non-empty interior (Pisot)
- The mapping h_σ contracts the Lebesgue measure by $1/\beta$

$$\forall\, i \in \mathcal{A},\ \ \mu_{d-1}(\mathcal{R}_\sigma(i)) \leq \sum_{j \in A} 1/\beta m_{ij}\, \mu_{d-1}(\mathcal{R}_\sigma(j))$$

  with $M_\sigma = [m_{ij}]$
- We thus have

$$M_\sigma\, [\mu(\mathcal{R}_\sigma(j))] \geq \beta\, [\mu(\mathcal{R}_\sigma(j))]$$

- Since β is the Perron–Frobenius eigenvalue of M_σ, one gets the reverse inequality

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How to propagate this information ?

Strong coincidence and synchronization

- The strong coincidence condition gives

$$\forall j_1, j_2 \in \mathcal{A},\ \exists k,\ \exists i,\ \sigma^k(j_1) = pis,\ \sigma^k(j_2) = pis'$$

$$\text{with } f(p) = f(p')$$

- The tiles $\mathcal{R}_\sigma(i)$ are solutions of the k-order GIFS

$$\mathcal{R}_\sigma(i) = \bigcup_{\sigma^k(j)=pis} h_\sigma^k(\mathcal{R}_\sigma(j)) + \pi_c \circ f(p)$$

- For every (j_1, j_2) there exists a common letter i and $f(p)$ s.t.

$$h_\sigma^k(\mathcal{R}_\sigma(j_1)) + \pi_c \circ f(p)$$

  and

$$h_\sigma^k(\mathcal{R}_\sigma(j_2)) + \pi_c \circ f(p)$$

  both occur in the k-th order GIFS equation

- We use the fact that the subtiles are disjoint in measure in the k-th order iteration