Aperiodic Order and Quantum Mechanics

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The Mathematics of Aperiodic Order

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A Quantum Particle in a Discrete World

The state of the quantum system is described by a normalized element ψ of

$$\ell^2(\mathbb{Z}^d) = \left\{ \psi : \mathbb{Z}^d \to \mathbb{C} : \sum_{n \in \mathbb{Z}^d} |\psi(n)|^2 < \infty \right\}$$

The interpretation is as follows:

$$\mathrm{Prob}\,(\text{ particle is in } A \text{ }) = \sum_{n \in A} |\psi(n)|^2$$

A Quantum Particle in a Discrete World

The state changes with time according to the Schrödinger equation:

$$i\partial_t \psi = H\psi$$

Here, H is the Schrödinger operator $H = \Delta + V$, that is,

$$[H\psi](n) = \sum_{\|e\|_1 = 1} \psi(n+e) + V(n)\psi(n)$$

where the potential $V : \mathbb{Z}^d \to \mathbb{R}$ models the environment the quantum particle is exposed to.

A Quantum Particle in a Discrete World

The Schrödinger operator is self-adjoint:

$$\langle \phi, H\psi \rangle = \langle H\phi, \psi \rangle$$

The "allowed energies" are given by the spectrum of H:

$$\sigma(H) = \{E \in \mathbb{R} : (H - E \cdot I)^{-1} \text{ does not exist}\}$$

Moreover, for every $\psi \in \ell^2(\mathbb{Z}^d)$, there is a so-called spectral measure $d\mu_\psi$ so that

$$\langle \psi, g(H)\psi \rangle = \int_{\sigma(H)} g(E)\, d\mu_\psi(E)$$

A Quantum Particle in a Discrete World

Spectral measures are important because they are related to the long time behavior of the solutions to the Schrödinger equation.

Indeed, if $\psi(t)$ solves $i\partial_t\psi = H\psi$ and $d\mu$ is the spectral measure of $\psi(0)$, then

- the particle "travels freely" if $d\mu$ is absolutely continuous
- the particle "travels somewhat" if $d\mu$ is singular continuous
- the particle "does not travel" if $d\mu$ is pure point

Aperiodic Order in 1D

Suppose $V : \mathbb{Z} \to \mathbb{R}$ is an aperiodically ordered potential and consider the associated Schrödinger operator $H = \Delta + V$ in $\ell^2(\mathbb{Z})$.

There is a strong tendency for the following phenomena to occur:

- The spectrum of H is a Cantor set of zero Lebesgue measure.
- All spectral measures are purely singular continuous.

In this talk we will focus on the structure of the spectrum. Let us explain why it has a tendency to be a Cantor set of zero Lebesgue measure.

Lyapunov Exponents and Kotani's Theorem

The key to the zero measure result is that there is a specific subset of the spectrum that has zero Lebesgue measure by general principles and that turns out to be equal to the spectrum in aperiodically ordered situations.

To define this set, we need to recall the concept of a Lyapunov exponent. For $E \in \mathbb{R}$ and $n \geq 1$, consider the transfer matrix

$$M(n, E) = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} E - V(1) & -1 \\ 1 & 0 \end{pmatrix}$$

and the Lyapunov exponent

$$L(E) = \limsup_{n \to \infty} \frac{1}{n} \log \| M(n, E) \|.$$

Then, the set

$$\mathcal{Z} = \{E \in \mathbb{R} : L(E) = 0\}$$

is contained in the spectrum of H.

Lyapunov Exponents and Kotani's Theorem

Theorem (Kotani 1989)

If V takes finitely many values and is repetitive and aperiodic, then $\mathcal{Z}$ has zero Lebesgue measure.

To show that $\mathcal{Z}$ is actually equal to the spectrum of H, there are two approaches:

- analysis of induced trace maps
- uniformity of locally constant cocycles

Trace maps exist in substitutive (or at least S-adic) situations and have been studied since the early 1980's. The other approach is more general but provides less detailed information.

Lyapunov Exponents and Kotani's Theorem

The net result is that the spectrum is a zero-measure Cantor set in many situations, including the following:

- Fibonacci sequence
- all Sturmian sequences
- all (aperiodic) primitive substitution sequences
- all (aperiodic) linearly recurrent sequences
- all (aperiodic) sequences satisfying Boshernitzan's condition

Naturally, once zero Lebesgue measure has been established, the next step is a study of the fractal dimension of the spectrum. Let us recall the relevant notions.

Cantor Subsets of $\mathbb{R}$ and Fractal Dimensions

Let us consider a bounded Cantor subset of $\mathbb{R}$, that is, a set $C \subset \mathbb{R}$ that is compact, and which has empty interior and no isolated points.

Thus, C is "larger" than a point (a zero-dimensional object) and "smaller" than an interval (a one-dimensional object). To study the size of C more closely, various fractal dimensions are commonly considered.

The Hausdorff dimension of C is defined as follows. Consider

$$h^\alpha(C) = \lim_{\varepsilon \downarrow 0} \inf_{\varepsilon-\text{covers}} \sum_{n=1}^{\infty} |I_n|^\alpha$$

and let

$$\dim_H(C) = \inf\{\alpha > 0 : h^\alpha(C) = 0\}.$$

Cantor Subsets of $\mathbb{R}$ and Fractal Dimensions

The upper and lower box counting dimensions of C are defined as follows. Consider

$$N_C(\varepsilon) = \#\{j \in \mathbb{Z} : [j\varepsilon, (j+1)\varepsilon) \cap C \neq \emptyset\}$$

and let

$$\dim_B^+(C) = \limsup_{\varepsilon \downarrow 0} \frac{\log N_C(\varepsilon)}{\log \frac{1}{\varepsilon}}$$

and

$$\dim_B^-(C) = \liminf_{\varepsilon \downarrow 0} \frac{\log N_C(\varepsilon)}{\log \frac{1}{\varepsilon}}$$

These dimensions obey the inequalities

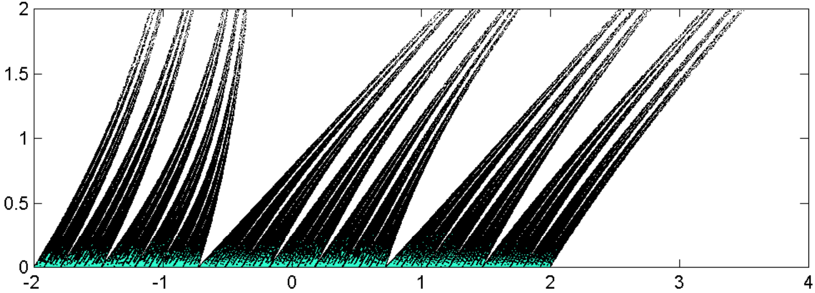
$$\dim_H(C) \leq \dim_B^-(C) \leq \dim_B^+(C).$$

The Fibonacci Hamiltonian

The Fibonacci Hamiltonian in $\ell^2(\mathbb{Z})$ is given by

$$[H_{\lambda,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + \lambda\chi_{[1-\alpha,1)}(n\alpha + \omega \mod 1)\psi(n),$$

where $\lambda > 0$, $\alpha = \frac{\sqrt{5}-1}{2}$, and $\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. The spectrum $\sigma(H_{\lambda,\omega})$ is independent of ω and will be denoted by Σ_λ.



The set $\{(E,\lambda) : E \in \Sigma_\lambda,\ 0 \le \lambda \le 2\}$.

The Fibonacci Hamiltonian

Theorem (D-Embree-Gorodetski-Tcheremchantsev 2008)

For $\lambda > 16$, we have

$$\dim_H(\Sigma_\lambda) = \dim_B^-(\Sigma_\lambda) = \dim_B^+(\Sigma_\lambda)$$

and

$$\lim_{\lambda\to\infty} \dim(\Sigma_\lambda) \cdot \log \lambda = \log(1 + \sqrt{2})$$

The Square Fibonacci Hamiltonian

The square Fibonacci Hamiltonian in $\ell^2(\mathbb{Z}^2)$ is the Schrödinger operator

$$[H_\lambda^{(2)}\psi](n) = \sum_{\|e\|_1=1} \psi(n+e) + V(n)\psi(n)$$

where $V(n) = V(n_1, n_2)$ is given by

$$\lambda \left(\chi_{[1-\alpha,1)}(n_1\alpha \mod 1) + \chi_{[1-\alpha,1)}(n_2\alpha \mod 1)\right)$$

with $\lambda > 0$ and $\alpha = \frac{\sqrt{5}-1}{2}$ as above.

The theory of tensor products of Hilbert spaces and operators implies that $\sigma(H_\lambda^{(2)}) = \Sigma_\lambda + \Sigma_\lambda$.

Numerical Results of Even-Dar Mandel and Lifshitz

The structure of $\sigma(H_\lambda^{(2)}) = \Sigma_\lambda + \Sigma_\lambda$ undergoes a number of transitions as λ runs from zero to infinity. One can identify the following regimes:

- $0 < \lambda < \lambda_1$: The spectrum has no gaps.
- $\lambda_1 < \lambda < \lambda_2$: The spectrum has finitely many gaps.
- $\lambda_2 < \lambda < \lambda_3$: The spectrum has infinitely many gaps, but does contain intervals.
- $\lambda_3 < \lambda < \lambda_4$: The spectrum contains no intervals, but has positive measure.
- $\lambda_4 < \lambda < \infty$: The spectrum is a zero-measure Cantor set.

Arithmetic Sums of Cantor Sets

Lemma

Suppose $C_1, C_2 \subset \mathbb{R}$ are Cantor sets with

$$\dim_B^+(C_1) + \dim_B^+(C_2) < 1.$$

Then, $C_1 + C_2$ is a Cantor set of zero Lebesgue measure.

Proof.

Choose $d_j' > \dim_B^+(C_j)$ with $d_1' + d_2' < 1$. For $\varepsilon > 0$ small enough, C_j can be covered by $\varepsilon^{-d_j'}$ intervals of length ε. Thus, $C_1 + C_2$ is contained in $\varepsilon^{-d_1'} \cdot \varepsilon^{-d_2'}$ intervals of length 2ε. Its Lebesgue measure is therefore bounded from above by $2\varepsilon^{1-d_1'-d_2'}$ and hence it must be zero. In particular, the set $C_1 + C_2$ has empty interior. On the other hand, it is clear that $C_1 + C_2$ is compact and has no isolated points. $\square$

Square Fibonacci Hamiltonian at Large Coupling

Corollary

For λ sufficiently large, $\sigma(H_\lambda^{(2)}) = \Sigma_\lambda + \Sigma_\lambda$ is a Cantor set of zero Lebesgue measure.

Proof. By [D-Embree-Gorodetski-Tcheremchantsev 2008], there is $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$,

$$\dim_B^+(\Sigma_\lambda) < \frac{1}{2},$$

and therefore

$$\dim_B^+(\Sigma_\lambda) + \dim_B^+(\Sigma_\lambda) < 1.$$

By the previous lemma, $\Sigma_\lambda + \Sigma_\lambda$ is a Cantor set of zero Lebesgue measure. $\square$

The Newhouse Gap Lemma

Given a Cantor set $C \subset \mathbb{R}$, its thickness $\tau = \tau(C)$ is defined as follows. Consider a bounded gap G of C and one of its boundary points $b \in \partial G$. Form the interval B from b through C all the way to the next gap of C that is longer than G. Let

$$\tau(C) = \inf_b \frac{|B|}{|G|}.$$

Gap Lemma (Newhouse 1979)

Let $C_1, C_2 \subset \mathbb{R}$ be Cantor sets with thickness τ_1, τ_2, respectively. If $\tau_1 \cdot \tau_2 > 1$, then we have one of the following:

(i) *C_1 is contained in a gap of C_2,*

(ii) *C_2 is contained in a gap of C_1,*

(iii) *$C_1 \cap C_2 \neq \emptyset$.*

A Consequence of the Newhouse Gap Lemma

Corollary

Suppose $C \subset \mathbb{R}$ is a Cantor set with thickness $\tau(C) > 1$. Then, $C + C$ is an interval.

Proof. Denote $\min C = c_1$ and $\max C = c_2$. We claim that

$$C + C = [c_1 + c_1, c_2 + c_2].$$

The inclusion "$\subseteq$" is obvious, so let us prove the inclusion "$\supseteq$."

Take an arbitrary point $x \in [c_1 + c_1, c_2 + c_2]$. Then, $x \in C + C$ if and only if $0 \in C + C - x = C - (x - C)$. Therefore,

$$x \in C + C \iff C \cap (x - C) \neq \emptyset.$$

A Consequence of the Newhouse Gap Lemma

Since $\tau(C) \cdot \tau(x - C) = \tau(C) \cdot \tau(C) > 1$, the Gap Lemma implies that *a priori* there are only four possibilities:

1. the intervals $[c_1, c_2]$ and $[x - c_2, x - c_1]$ are disjoint;
2. the set C is contained in a finite gap of the set $(x - C)$;
3. the set $(x - C)$ is contained in a finite gap of the set C;
4. $C \cap (x - C) \neq \emptyset$.

But the case (1) contradicts the assumption $x \in [c_1 + c_1, c_2 + c_2]$, and the cases (2) and (3) are clearly impossible. Therefore, we must have $C \cap (x - C) \neq \emptyset$ and hence $x \in C + C$. $\square$

Square Fibonacci Hamiltonian at Small Coupling

Theorem (D.-Gorodetski)

For $\lambda > 0$ sufficiently small, we have $\lambda^{-1} \lesssim \tau(\Sigma_\lambda) \lesssim \lambda^{-1}$

Square Fibonacci Hamiltonian at Small Coupling

Theorem (D.-Gorodetski)

For $\lambda > 0$ sufficiently small, we have $\lambda^{-1} \lesssim \tau(\Sigma_\lambda) \lesssim \lambda^{-1}$ and $\lambda \lesssim 1 - \dim \Sigma_\lambda \lesssim \lambda$.

Corollary (D.-Gorodetski)

For $\lambda > 0$ sufficiently small, $\sigma(H_\lambda^{(2)}) = \Sigma_\lambda + \Sigma_\lambda$ is an interval.