



# Words, Decomposition Rules and Invariants

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# Words

$w = X_1 X_2 \cdots X_r$  : word

$l(w) = r$  : length of  $w$

${}^t w = X_r \cdots X_2 X_1$  : reverse of  $w$

$l({}^t w) = l(w)$

# Matrix

$$\lambda = Y_1 Y_2 \cdots Y_{l(\lambda)}, \mu = Z_1 Z_2 \cdots Z_{l(\mu)}$$

$M(\lambda, \mu) = (m_{ij})$ :  $l(\lambda) \times l(\mu)$  matrix

$$m_{ij} = \begin{cases} Y_i & \text{if } Y_i = Z_j \\ \phi & \text{if } Y_i \neq Z_j \end{cases}$$

# Graph

$$\Gamma(\lambda, \mu)$$

Vertices

$$(i, j) : 1 \leq i \leq l(\lambda), 1 \leq j \leq l(\mu)$$

Arrows

$$(i, j) \rightarrow (k, \ell)$$

$$\underset{\text{def}}{\Leftrightarrow} i + 1 = k, j + 1 = \ell, m_{ij} \neq \phi, m_{k\ell} \neq \phi$$

# Component

$C(\lambda, \mu)$ :

the set of connected components of  $\Gamma(\lambda, \mu)$

$c \in C(\lambda, \mu)$

$c = [(i, j) \rightarrow (i+1, j+1) \rightarrow \cdots \rightarrow (i+p, j+p)]$

$\omega(c) = m_{ij} m_{i+1, j+1} \cdots m_{i+p, j+p}$

# Multiplicity

$$m_\nu(\lambda, \mu) = \#\{c \in C(\lambda, \mu) \mid \omega(c) = \nu\}$$

$$m_\nu(\lambda, \phi) = \delta_{\nu, \phi} \times l(\lambda)$$

$$m_\nu(\phi, \mu) = \delta_{\nu, \phi} \times l(\mu)$$

$$m_\nu(\phi, \phi) = \delta_{\nu, \phi}$$

the multiplicity of  $(\lambda, \mu)$  at  $\nu$

# Diagonal Calculus

	A	B	A	A	B
A	A	$\phi$	A	A	$\phi$
B	$\phi$	B	$\phi$	$\phi$	B
A	A	$\phi$	A	A	$\phi$
B	$\phi$	B	$\phi$	$\phi$	B

# Example

$$m_{ABA}(ABAB, ABAAB) = 1$$

$$m_{AB}(ABAB, ABAAB) = 3$$

$$m_A(ABAB, ABAAB) = 1$$

$$m_\phi(ABAB, ABAAB) = 10$$

$$m_\nu(ABAB, ABAAB) = 0 \quad \text{for other } \nu$$

# Tensor Product

Formula

$$V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus m_\nu(\lambda, \mu)}$$

$$V^{\oplus m} = \underbrace{V \oplus \cdots \oplus V}_m$$

# Dim Map

$$w = \text{word} \mapsto V_w \mapsto \dim(V_w) = l(w)$$

c.f. Grothendieck Groups, Witt Rings

$$\dim(V \oplus V') = \dim(V) + \dim(V')$$

$$\dim(V \otimes V') = \dim(V) \cdot \dim(V')$$

# Deformation

$$\phi \mapsto u_0 = u(\phi) = u(V_\phi) > 0 \quad (u_0 = 1 \Rightarrow \dim)$$

$$w = \text{word} \mapsto u(w) = u(V_w) > 0$$

$$u(w)^2 = \sum_{w'} m_{w'}(w, w) u(w')$$

Can uniquely solve with  $u(w) > 0$

$$V_w \otimes V_w = \bigoplus_{w'} V_{w'}^{\oplus m_{w'}(w, w)}$$

$$u(V_{w'} \oplus V_{w''}) = u(V_{w'}) + u(V_{w''})$$

$$u(V_{w'} \otimes V_{w''}) \neq u(V_{w'}) \cdot u(V_{w''})$$

# Universal Deformation

$f_w(t) \in \mathbf{R}[[t]]$ : formal power series in  $t$

$$\phi \mapsto f_\phi(t) = t$$

$$w = \text{word} \mapsto f_w(t)$$

$$f_w(t)^2 = \sum_{w'} m_{w'}(w, w) f_{w'}(t)$$

Can uniquely solve with  $f_w(0) > 0$

# Example

$$f_A(t) = 1, \quad f_{AA}(t) = 2, \quad f_{AAA}(t) = 3, \dots, \quad f_{\underbrace{AA\dots A}_n}(t) = n$$

$$f_{AB}(t) = 1 + 2t - 4t^2 + 16t^3 - 80t^4 + 488t^5 - 2688t^6 + \dots$$

$$f_{AAB}(t) = 2 + \frac{4}{3}t - \frac{16}{27}t^2 + \frac{128}{243}t^3 - \frac{1280}{2187}t^4 + \frac{14336}{19682}t^5 - \dots$$

$$f_{AABB}(t) = \frac{1+\sqrt{17}}{2} + \frac{8}{\sqrt{17}}t - \frac{64}{17\sqrt{17}}t^2 + \frac{1024}{289\sqrt{17}}t^3 - \dots$$

# Example

$$w = AB$$

$$V_{AB} \otimes V_{AB} = V_{AB} \oplus V_\phi \oplus V_\phi$$

$$f_w(t)^2 = f_w(t) + 2t$$

$$(a_0 + a_1 t + a_2 t^2 + \cdots)^2 = (a_0 + a_1 t + a_2 t^2 + \cdots) + 2t$$

$$a_0^2 = a_0 \Rightarrow a_0 = 1 > 0$$

$$2a_0 a_1 = a_1 + 2 \Rightarrow 2a_1 = a_1 + 2 \Rightarrow a_1 = 2$$

$$2a_0 a_2 + a_1^2 = a_2 \Rightarrow 2a_2 + 4 = a_2 \Rightarrow a_2 = -4$$

$$f_w(t) = f_{AB}(t) = 1 + 2t - 4t^2 + \cdots$$

	A	B
A	A	$\phi$
B	$\phi$	B

# Word Invariants

$w$ : word

$$w \mapsto V_w \mapsto V_w \otimes V_w = \bigoplus_{w'} V_{w'}^{\oplus m_{w'}(w,w)}$$

$$f_w(t)^2 = \sum_{w'} m_{w'}(w, w) f_{w'}(t)$$

$$f_w(0) > 0$$

$$w \mapsto \begin{cases} f_w(t) = \sum_{i \geq 0} a_i t^i \in \mathbf{R}[[t]] \\ K_w = \mathbf{Q}(a_0, a_1, a_2, \dots) / \mathbf{Q} \end{cases}$$

# Rationality Theorem

$w = X_1 X_2 \cdots X_n, \quad X_i \in \{A, B\}, \quad A \in \{X_i, X_{i+1}\}$

Golden Mean Shift Type

$A \in \{X_1, X_n\}$

$$\Rightarrow \begin{cases} f_w(t) \in \mathbf{Q}[[t]] \\ K_w = \mathbf{Q} \\ f_w(0) = a_0 = \# \{i \mid X_i = A\} \end{cases}$$

# Example

$$K_{BAAB} = \mathbf{Q}(\sqrt{17})$$

$$K_{BAAAB} = \mathbf{Q}(\sqrt{33})$$

$$K_{BAAAAB} = \mathbf{Q}(\sqrt{57})$$

$$K_{BAAAAAB} = \mathbf{Q}(\sqrt{89})$$

$$K_{B\underbrace{AA\cdots A}_k B} = \mathbf{Q}(\sqrt{4k^2 - 4k + 9})$$

# Example

$$K_{BAAB} = \mathbf{Q}(\sqrt{17})$$

$$K_{BAABAAB} = \mathbf{Q}(\sqrt{45 + 4\sqrt{17}})$$

$$K_{BAABAABAAB} =$$

$$\mathbf{Q}(\sqrt{45 + 4\sqrt{17}}, \sqrt{89 + 4\sqrt{17} + 4\sqrt{45 + 4\sqrt{17}}})$$

.....

$$\mathbf{Q} \subset K_w \subset \mathbf{R}, \quad [K_w : \mathbf{Q}] = 2^r$$

# Bi-infinite Words

$\mathbf{T} = \cdots X_{-2} X_{-1} X_0 X_1 X_2 \cdots$  Bi-infinite Word

$S(\mathbf{T}) = \{X_i X_{i+1} \cdots X_j \mid i \leq j\}$  Finite Subwords

$\lambda, \mu \in S(\mathbf{T}) \quad \lambda \prec \mu \Leftrightarrow \lambda = \text{subword of } \mu$

$W(\mathbf{T}) = S(\mathbf{T}) \cup \{ \phi \} \quad \phi \prec \forall \lambda \in S(\mathbf{T})$

${}^t \mathbf{T} = \cdots X_2 X_1 X_0 X_{-1} X_{-2} \cdots$

# Algebraic Structures

Kellendonk Product

$$\lambda = Y_1 \cdots Y_{l(\lambda)}, \mu = Z_1 \cdots Z_{l(\mu)} \in S(\mathbf{T})$$

$$1 \leq i, j \leq l(\lambda), \quad 1 \leq k, \ell \leq l(\mu)$$

$$(i, \lambda, j) \bullet (k, \mu, \ell) = (m, \nu, n) \quad \text{if matching in } S(\mathbf{T})$$

$$Y_1 \dots \dots \dots Y_j \dots Y_{l(\lambda)}$$

$$Z_1 \dots Z_k \dots \dots \dots Z_{l(\mu)}$$

$$W_1 \dots W_m \dots \dots \dots W_n \dots W_{l(\nu)} = \nu \in S(\mathbf{T})$$

$m$  = position of  $Y_i$ ,  $n$  = position of  $Z_\ell$

$$(i, \lambda, j) \bullet (k, \mu, \ell) = \mathbf{z} \quad \text{otherwise}$$

# Example

$\mathbf{T} = \cdots ABAABABAABABAAB \cdots$ : Fibonacci Tiling

$\lambda = ABAA, \mu = AABA, \nu = ABAABA$

$$(1, \lambda, 3) \bullet (1, \mu, 3) = (1, \nu, 5) : \left. \begin{array}{c} ABAA \\ AABA \end{array} \right\} \Rightarrow ABAABA \in S(\mathbf{T})$$

$$(1, \lambda, 4) \bullet (1, \mu, 3) = \mathbf{z} : \left. \begin{array}{c} ABAA \\ AABA \end{array} \right\} \Rightarrow ABAAAABA \notin S(\mathbf{T})$$

$$(1, \lambda, 2) \bullet (1, \mu, 3) = \mathbf{z} : \left. \begin{array}{c} ABAA \\ AABA \end{array} \right\} \Rightarrow \times \times \times \quad (\text{no match})$$

# Monoids

$$\mathbf{M} = \{(i, \lambda, j) \mid \lambda \in S(\mathbf{T}), 1 \leq i, j \leq l(\lambda)\} \cup \{\mathbf{z}, \mathbf{e}\}$$

$$(i, \lambda, j) \bullet (k, \mu, \ell) = \begin{cases} (m, \nu, n) \\ \mathbf{z} \end{cases}$$

$$\mathbf{m} \bullet \mathbf{z} = \mathbf{z} \bullet \mathbf{m} = \mathbf{z}, \quad \mathbf{m} \bullet \mathbf{e} = \mathbf{e} \bullet \mathbf{m} = \mathbf{m} \quad (\mathbf{m} \in \mathbf{M})$$

$\mathbf{M} = \mathbf{M}(\mathbf{T})$ : monoid with zero

# Bialgebras

$\mathbf{A} = \mathbf{A}(\mathbf{T}) = \mathbf{C}[\mathbf{M}] = \bigoplus_{\mathbf{m} \in \mathbf{M}} \mathbf{C}\mathbf{m}$  : monoid algebra  
bialgebra

$$\mathbf{A} \rightarrow \mathbf{A} \otimes \mathbf{A} \quad \mathbf{m} \mapsto \mathbf{m} \otimes \mathbf{m}$$

$$\mathbf{A} \rightarrow \mathbf{C} \quad \mathbf{m} \mapsto 1$$

# Standard Modules

$V$  : standard  $\mathbf{A}$  - module

$$\Leftrightarrow \dim V < \infty, \quad \underset{\text{def}}{\mathbf{z}(V)} = 0$$

$$\#\{x \in \mathbf{A} \mid x : \text{group like}, xV \neq 0\} < \infty$$

$$\lambda \in S(\mathbf{T})$$

$$V_\lambda = \mathbf{A}.(1, \lambda, 1) \left/ \right. \mathbf{A}.(1, \lambda, 1) \cap \left( \sum_{\substack{\lambda \prec \mu, \lambda \neq \mu \\ p, q}} \mathbf{C}(p, \mu, q) \right)$$

$V_\phi = \mathbf{C}$  : trivial  $\mathbf{A}$  - module

# Tensor Products

Formula

$T : \text{Bi-infinite Word}$

$\lambda, \mu \in W(T)$

$$\Rightarrow V_\lambda \otimes V_\mu = \bigoplus_{\nu \in W(T)} (V_\nu)^{\oplus m_\nu(\lambda, \mu)}$$

# Characterizations

Fact  $T, T'$ : Bi-infinite Words

$$T \sim_{l.i.} T' \text{ or } T \sim_{l.i.} {}^t T'$$

locally indistinguishable

$$\Leftrightarrow A(T) \cong A(T') \text{ or } A(T) \cong A({}^t T')$$

as bialgebras

# Groups and Lie Algebras

$\mathbf{T} \rightarrow \mathbf{T}^*$  Modification

$$X \mapsto X' X'' X'''$$

$L \subset A(\mathbf{T}^*)$  Lie Algebra

$$L = L_+ \oplus L_0 \oplus L_- \quad \text{Triangular Decomposition}$$

$$U(L) = U(L_+)U(L_0)U(L_-) \quad \text{Additive Gauss Decomp.}$$

$G \subset A(\mathbf{T}^*)^\times$  Group

$$G = G_+ G_{\mp} G_0 G_\pm \quad \text{Gauss Decomposition}$$

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