Complexity of entropy zero topological systems

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Let (X,T) be a dynamical system.

1. How do we measure randomness, complexity or chaoticity?

(1) **Topological entropy of** $(X, \mathcal{U}, T)$

$$h_t(T,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right)$$

where $\mathcal{U}$ is an open cover and
$N(\mathcal{U})$=minimum number of open sets of $\mathcal{U}$
necessary to cover X.

(2) **Entropy of** $(X, \mathcal{P}, \mu, T)$

$$h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n}\mathcal{H}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P})$$

$$= \lim_{n\to\infty} \frac{1}{n}\mathcal{H}(\sum_{A\in\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}} -\mu A log \mu A)$$

$$= \lim_{n\to\infty} \mathcal{H}(\mathcal{P}|\bigvee_{i=1}^{n-1} T^{-i}\mathcal{P})$$

where $\mathcal{P}$ is a partition of X.

When $h_\mu(T, \mathcal{P}) > 0$ or $h_t(T, \mathcal{U}) > 0$,then partitions or open covers are exponentially splitting.
(partitions or open covers have "independence").

(3) **Examples**

(i) $h_\mu(T) > 0$: chaotic systems

- geodesic flows
- billiards with enough dispersing boundaries
- a coin flipping

(ii)$h_\mu(T) = 0$: deterministic systems

- irrational rotations, interval exchange maps
- horocycle flows
- many examples of $\mathbb{Z}^2$ ($\mathbb{Z}^n$) -actions.

2. Properties of positive entropy.

(1) Bernoulli factor

(2) Shannon-McMillan-Breiman Theorem.

$$\mu(\mathcal{P}_0^{n-1}(x)) \sim 2^{-nh}$$

(3) Ornstein-Weiss return time property(a.e. convergence)

$$\frac{1}{n} \log \mathcal{R}_n(x) \to h$$

where $\mathcal{R}_n(x) = min\{k : x_0 x_1 ... x_{n-1} = x_k x_{k+1} ... x_{k+n-1}\}$.

(4) Joinings : If a system has the property of completely positive entropy, then it is disjoint from entropy zero systems.

3. Why Zero Entropy?

(1) Property of zero entropy systems are not much known.

(2) Growth rate of orbits

(i) exponential $\rightarrow$ positive entropy

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(ii) subexponential growth $\rightarrow$ zero entropy

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(iii) polynomial $\rightarrow$ zero entropy

(rotations, Toeplitz systems, billiards on polygons,...)

(3) General group action (X, σ) of zero entropy
has many interesting subdynamics

There exists a Z^2-action (X, σ) where $h(\sigma) = 0$ with the property that

(3a) (i) $h(\sigma^{(1,0)}) > 0$

(ii) the action $\sigma^{(1,0)}$ is mixing.

(iii) For $R_n = [0, n) \times [0, n)$,

$$\lim_{n\to\infty} \frac{1}{n} \mathcal{H}\Big(\bigvee_{(i,j)\in R_n} \sigma^{-(i,j)} \mathcal{P}\Big) > 0$$

(3b)(Katok and Thouvenot)

(i) $h(\sigma^{(p,q)}) = 0$ for $\forall (p,q) \in \mathbb{Z}^2$

(ii) For any given $0 < \alpha < 2$,

$$\overline{\lim} \frac{1}{n^\alpha} \mathcal{H}\Big(\bigvee_{(i,j) \in R_n} \sigma^{-(i,j)} \mathcal{P} \Big) > 0$$

(3c) (i) $0 < h(\sigma^{(p,q)}) < \infty$

(ii) Directional entropy is continuous.

(iii)

$$\lim_{n \to \infty} \frac{1}{n} \mathcal{H}\Big(\bigvee_{(i,j) \in R_n} \sigma^{-(i,j)} \mathcal{P} \Big) > 0$$

(3d) (i) $h(\sigma^{(p,q)}) = 0$ for $\forall (p,q) \in \mathbb{Z}^2$

(ii)

$$\lim_{n\to\infty} \frac{1}{n}\mathcal{H}(\bigvee_{(i,j)\in R_n} \sigma^{-(i,j)}\mathcal{P}) > 0$$

(3e) (i) $h(\sigma^{(1,0)}) > 0$

(ii) $h(\sigma^{(p,q)}) = 0$ for $\forall (p,q) \neq (n,0)$.

(iii)

$$\lim \frac{1}{n}\mathcal{H}(\bigvee \sigma^{-(i,j)}\mathcal{P}) > 0$$

They have complexity and some independence

(4) Weakly(intermittent) chaotic systems(Manneville-Pomeau Maps)

$f(x) = x + x^t \pmod 1$, $t \geq 2$.

Note that $f'(0) = 1$.

- Sticky island examples.

•Properties

(i) A typical point(w.r.t. Lebesgue measure)

$$x = 1101\underbrace{00.\ .0.\ .\ .\ .0}101\underbrace{00.\ .0.\ .\ .\ .0}$$

(ii)There exists no finite invariant measure $\ll$ Lebesgue.

(iii)With respect to the infinite invariant measure it has positive entropy.

(iv) $\overline{\mathcal{O}(x)} = \{0,1\}^{\mathbb{Z}}$

(v) Complexity $\leftrightarrow$ Algorithmic Information Contents

(vi) The point "0" is called neutrally unstable fixed point.

4. Complexity of zero entropy system

(1) Low complexity case

(i) sequence entropy(Kushnirenko, Goodman(topological))
(ii) maximal pattern complexity(Kamae and Zamboni)
(iii) maximal pattern entropy(Hwang and Ye)

(2) Intermediate complexity case

Remark. Question by Milnor : Does there exist a finitely generated group of intermediate growth rate.

Answer by Grigorchuk, '80.(Recently by Grigorchuk and I.Pak)

Definition. Entropy dimension.

(i) topological entropy dimension

$$\overline{D}(T,\mathcal{U}) = inf\{\beta : \overline{lim}\frac{\log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})}{n^{\beta}} = 0\}$$

$$\overline{D}(T) = \sup\{\overline{D}(T,\mathcal{U}) : \mathcal{U}\ open\ covers\}$$

$$\underline{D}(T,\mathcal{U}) = inf\{\beta : \underline{lim}\frac{\log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})}{n^{\beta}} = 0\}$$

$$\underline{D}(T) = \sup\{\underline{D}(T,\mathcal{U}) : \mathcal{U}\ open\ covers\}$$

If $\overline{D} = \underline{D} = \alpha$, then we say $(X,\mathcal{U},T)$ has entropy dimension α.

- Topological entropy dimension is conjugacy invariant.

(ii)Metric entropy dimension has been defined analogously.

$$\overline{D}_\mu(T,\mathcal{P}) = \inf\{\beta : \overline{lim}\frac{\mathcal{H}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P})}{n^\beta} = 0\}$$

In the metric entropy dimension, $\overline{D}_\mu(T,\mathcal{P})$ and $\underline{D}_\mu(T,\mathcal{P})$ are not isomorphism invariant.

Modification of the definition.

Given $\epsilon > 0$, let

$$B(x,n,\epsilon) = \{y \in X : \overline{d}(\mathcal{P}_{[0,n)}(x), \mathcal{P}_{[0,n)}(y)) < \epsilon\}$$

Let $K(n,\epsilon)$ be the smallest number of balls necessary to cover a subset of X of measure at least $1-\epsilon$.

$$\overline{D}(\mathcal{P},\epsilon) = \inf\{\beta : \overline{\lim}\frac{1}{n^\beta}\log K(n,\epsilon) = 0\}$$

$$\overline{D}(\mathcal{P}) = \lim_{\epsilon\to 0}\overline{D}(\mathcal{P},\epsilon)$$

$$\overline{D} = \sup_{\mathcal{P}}\overline{D}(\mathcal{P})$$

We define $\underline{D}$ likewise.

If $\overline{D} = \underline{D}$, then we call $\alpha = \overline{D}$ the entropy dimension of T

Remark. $D(T \times S) = max\{D(T), D(S)\}$

Theorem. *(Ferenczi and Park) For each* $0 < \alpha < 1$, *there exists a dynamical system of entropy dimension* α.

(3) Examples

(i) **measurable case**

(ia) Ferenczi and Park : constructive example

(ib) Aaronson and Park : a skew product of a rotation with a

Bernoulli flow having metric entropy dimension $\leq 1/2$.

(ic) Ahn, Dou and Park : metric entropy dimension 0 with nontrivial topological entropy dimension

(ii) **topological case**

(iia) Park : constructive examples of any dimension $0 < \alpha < 1$

Idea.

$$B_{n+1} = \underbrace{B_n B_n ... B_n}_{b_{n+1}} \underbrace{00. \ . \ .0. \ . \ . \ .00}_{z_{n+1}} \underbrace{B_n B_n ... B_n}_{b_{n+1}}$$

(iib) Cassaigne : examples with minimality of any dimension $0 < \alpha < 1$

(iic) Dou, Huang and Park : minimal, weakly mixing ...

5. What are the properties?

(1) Dimension Set

Definition. We define the subset $\mathcal{D}(X,T) = \{\alpha :$ There exists an open cover $\mathcal{U}$ such that $D(X,\mathcal{U},T) = \alpha\} \subset [0,1]$ as the dimension set of (X,T).

Lemma. *If* $(X,T) \to (Y,S)$, *then* $\mathcal{D}(X,T) \geq \mathcal{D}(Y,S)$

Corollary. *The dimension set* $\mathcal{D}(X,T)$ *is homeomorphism invariant.*

- $\mathcal{D}(X,T)$ is not necessarily a closed subset.

- If $\mathcal{D}(X,T) \subset (0,1]$, then (X,T) is weakly mixing.

(2) Uniform dimension(Dou, Huang and Park)

Definition. We say (X,T) topologically K iff every nontrivial open cover has positive entropy.

Definition. We say (X,T) is of uniform dimension iff every factor has the same dimension. That is, $\mathcal{D}(X,T) = \alpha$ (Every non trivial open cover has entropy dimension α

• Given α, there exists a mimimal topological example of uniform dimension α.

(This property corresponds to K-mixing in the case of positive entropy.)

• (X,T) topologically K iff $\mathcal{D}(X,T) = 1$

(3) Joining

Definition. We say $(\widehat{X}, T \times S)$ is a joining of (X,T) and (Y,S) if

(a) $\widehat{X}$ is a closed invariant subset of X$\times$Y

(b) $\pi_X(\widehat{X}, T \times S) \cong (X, T)$ and $\pi_Y(\widehat{X}, T \times S) \cong (Y, S)$

Theorem(Dou, Hwang and P.) Let (X,T) be a TDS and (Y,S) be minimal. Suppose $\mathcal{D}(Y, S) < \mathcal{D}(X, T)$. Then (X, T) and (Y, S) are disjoint, that is ($X \times Y$, $T \times S$) is the only joining.

• If (X,T) is of uniformly positive entropy, then (X,T) is disjoint from the minimal zero entropy systems.

• If (X,T) has entropy dimension 0 and minimal, then it is disjoint from all positive entropy dimension systems.

• For any nontrivial minimal system (X,T), there exists a transitive system (Y,S) with $\overline{D}(S) = 0$ such that (X,T) and (Y,S) are not disjoint. (Minimality of the system of lower entropy is required.)

(4) Variational principle of entropy dimension? (Ahn, Dou and P.)

$D_{top}(T) = \sup_\mu D_\mu(T)$?

(It is clear from the definition that $D_\mu(T) \leq D_{top}(T)$.)

No!

For any given α, there exists an uniquely ergodic example $D_{top}(T) = \alpha$, but $D_\mu(T) = 0$.

(5) Independence

Definition. Let $S = \{s_1 < s_2 < ...\}$ be an increasing sequence of integers. We define

$$\overline{D}(S) = inf\{\beta \geq 0 : \limsup_{n\to\infty} \frac{n}{(s_n)^\beta} = 0\}$$

the upper dimension of the sequence S. Similarly, we define

$$\underline{D}(S) = inf\{\beta \geq 0 : \liminf_{n\to\infty} \frac{n}{(s_n)^\beta} = 0\}$$

the lower dimension of the sequence S.

When $\overline{D}(S) = \underline{D}(S) = \alpha$, we say the sequence S has dimension α and denote it by $D(S)$.

Example. If $S = \{n^2\}$, then $D(S) = \frac{1}{2}$.
If S has positive density, then $D(S) = 1$.

Definition. We say an in increasing sequence of integers $S = \{s_1 < s_2 < ...\}$ is an entropy generating sequence of $\mathcal{U}$ if

$$\liminf_{n\to\infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n} T^{-s_i}\mathcal{U}) > 0.$$

Remark. If S is an entropy generating sequence for $(X, T, \mathcal{U})$ and $(X, T, \mathcal{U})$ and $(Y, S, \mathcal{V})$ are conjugate, then it is also an entropy generating sequence for $(Y, S, \mathcal{V})$.

Definition. We say an open cover $\mathcal{U} = \{U_1, ...U_k\}$ is independent along W if for any $s \in \{1, ..., k\}^W$, we have $\bigcap_{w \in W} T^{-w} U_{s(w)} \neq \emptyset$.

Lemma. Let (X, T) be a TDS and $\mathcal{U}$ be a finite open cover. For any $\tau \in (0, 1]$ and $0 < \eta < \tau$ and finite c, there exists $N \in \mathbb{N}$(depending on τ, η, c) such that if a finite subset B of $\mathbb{Z}_+$ with $|B| \geq N$ and $\mathcal{N}(\bigvee_{i \in B} T^{-i}\mathcal{U}) \geq e^{c|B|^\tau}$, then there exists $W \subset B$ such that $|W| \geq |B|^\eta$ and $\mathcal{U}$ is independent along W

(A generalization of the result by Kerr and Li)

Theorem. Let $\mathcal{U}$ be a finite generating (diam($\bigcap_i T^{-i}\mathcal{U}$)=0) open cover such that $\overline{D}(T,\mathcal{U}) > 0$. There exists an entropy generating sequence $F \subset \mathbb{Z}$ such that $\overline{D}(F) = \overline{D}(T,\mathcal{U})$.

Sketch of the proof.

Assume $\overline{D}(T,\mathcal{U}) > 0$ and let $\{\tau_j\} \subset (0, \overline{D}(T,\mathcal{U}))$ be an increasing sequence such that $\lim_{j\to\infty} \tau_j = \overline{D}(T,\mathcal{U})$. We choose $a > 0$ so that

$$\limsup_{n\to+\infty} \frac{1}{n^{\tau_j}} \log \mathcal{N}(\bigvee_{i=1}^{n} T^{-i}\mathcal{U}) > a \text{ for } j \in \mathbb{N}.$$

Let $\tau_{j-1} < \eta_j < \tau_j$ for $j \in N$. By the above Lemma , there exists $N_j \in N$ such that for every finite set B with $|B| \geq N_j$ and $\mathcal{N}(\bigvee_{i\in B} T^{-i}\mathcal{U}) \geq e^{\frac{a}{2}|B|^{\tau_j}}$, we can find $W \subseteq B$ with $|W| \geq |B|^{\eta_j}$ and $\{A_1, A_2, \cdots, A_k\}$ is independent along W. Take $1 = n_1 < n_2 < \cdots$ such that $(n_{j+1} - n_j)^{\eta_j} \geq jn_j + N_j$ and

$$\mathcal{N}(\bigvee_{i=n_j+1}^{n_{j+1}} T^{-i}\mathcal{U}) \geq e^{\frac{a}{2}(n_{j+1}-n_j)^{\tau_j}}$$

for each $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, there exist $W_j \subseteq \{n_j + 1, n_j + 2, \cdots, n_{j+1}\}$ with $|W_j| \geq (n_{j+1} - n_j)^{\eta_j}$ and $\{A_1, A_2, \cdots, A_k\}$ is independent along W_j. For any nonempty set $B \subseteq W_j$ and $s = (s(z))_{z \in B} \in \{1, 2, \cdots, k\}^B$, we can find $x_s \in \bigcap_{z \in B} T^{-z} A_{s(z)}$.

Let $X_B = \{x_s : s \in \{1,2,\cdots,k\}^B\}$. It is clear that for any $t \in \{1,2,\cdots,k\}^B$, we have

$$|\bigcap_{z\in B} T^{-z}A^c_{t(z)} \cap X_B| \le (k-1)^{|B|}.$$

Combining this fact with $|X_B| = k^{|B|}$, we get

$$\mathcal{N}(\bigvee_{z\in B} T^{-z}\mathcal{U}) \ge \frac{k^{|B|}}{(k-1)^{|B|}} \text{ for any } B \subseteq W_j.$$

Put $F = \bigcup_{i=1}^{\infty} W_j$ and write $F = \{t_1 < t_2 < \cdots\}$.

Theorem. $D(X,T) = \sup_F\{\overline{D}(F)$: F is an entropy generating sequence for $\mathcal{U}$ $\}$. If it has a finite generating open cover, then $D(X,T) = \overline{D}(F)$ for some F.

Let $(X, \mathcal{F}, \mu, T)$ be a measurable dynamical system.

Definition. Given a partition $\mathcal{P}$, we define

$$D_\mu(T, \mathcal{P}) = \begin{cases} \sup\{\overline{D}(F)\} & \text{F is an entropy generating sequence} \\ 0 & \text{if there is no entropy generating sequence} \end{cases}$$

We define $D_\mu(X, T) = \sup_{\mathcal{P}} D_\mu(T, \mathcal{P})$

Properties

- $D_\mu(X, T)$ is an isomorphism invariant.
- If $\mathcal{P}$ is a generating partition, then $D_\mu(X, T) = D_\mu(T, \mathcal{P})$.
- $D_\mu(X, T) = D_\mu(X, T^k)$ for any k.

Definition. Dimension set for T is

$\mathcal{D}(X,T) = \{\alpha$: There exists a partition $\mathcal{P}$ such that $D(T,\mathcal{P}) = \alpha\} \subset [0,1]$

We say (X,T) has a uniform dimension, $\mathcal{D}(X,T) = \{\alpha\}$, iff every nontrivial partition has entropy dimension α.

Theorem If $\mathcal{D}(X,T) \geq \mathcal{D}(Y,S)$, then (X,T) and (Y,S) are disjoint.

Corollary An α-uniform dimension ergodic system is disjoint from β-uniform dimension ergodic system.

Questions

Aaronson and Park's example : entry dimension is $\leq 1/2$.

$$-\frac{1}{\sqrt{n}} \log \mu(\mathcal{P}_0^{n-1}(x))$$

converges in distribution.

We do not have Shannon-McMillan-Breiman Theorem.

1. What kind of "regularity" do we have in the size of atoms?

2. Suppose $(X, \mathcal{F}, \mu, T)$ has entropy dimension $0 < \alpha < 1$. Does the Equipartition Property imply the Return Time Property?

$\lim_{n\to\infty} \frac{\log R_n(x)}{n^\alpha}$ exists ?, $R_n = \min\{m \geq 1 : x_1^n = x_{m+1}^{m+n}\}$

3.Do we have smooth models for entropy dimension $0 < \alpha < 1$?

4. Does there exist an example satisfying $D_{top}(T) = \sup_\mu D_\mu(T)$?

5. What can we say about the dimension set, $\mathcal{D}$?

6. α-entropy.

$$\lim_{n\to+\infty}\frac{1}{n^{\alpha}}\log\mathcal{N}(\bigvee_{i=1}^{n}T^{-i}\mathcal{U})$$

exists?

If $(X,\mathcal{F},\mu,T)$ has positive α-entropy, then does it have a factor of smaller α-entropy?

7. Definition of α-Bernoulli? α- K?

Thank You!!