Relative Cohomology for Tiling Spaces

Lorenzo Sadun
University of Texas at Austin

joint work with

Marcy Barge
Montana State University

What is cohomology in general? Cohomology is ...

1. A homeomorphism invariant that we can use to tell spaces apart.
2. Something computable by cut-and-paste rules (Mayer-Vietoris, etc.)
3. A functor that relates two or more spaces and the maps between them. (Naturality, relative cohomology, excision, etc.)
4. The setting for other topological structures, such as characteristic classes.

What is *tiling* cohomology? It is

1. A homeomorphism invariant that we can use to tell spaces apart. (so far, so good)
2. Not very computable by cut-and-paste rules. Most calculations rely on inverse limit structures.
3. Only studied one space at a time, not in a functorial setting.
4. We have a very limited understanding of what tiling cohomology tells us (gap labeling, deformations, spaces of measures, exact regularity). ?

One and a half out of four isn't good!

Review of relative (co)homology

Suppose X is a subspace of a topological space Y. The inclusion $X \hookrightarrow Y$ induces an inclusion of chains $i_* : C_k(X) \to C_k(Y)$ and we define $C_k(Y, X) = C_k(Y)/i_*(C_k(X))$. The snake lemma converts the short exact sequence of chain complexes:

$$0 \to C_*(X) \to C_*(Y) \to C_*(Y, X) \to 0$$

into a long exact sequence of homology groups:

$$\to H_k(X) \to H_k(Y) \to H_k(Y, X) \to H_{k-1}(X) \to \cdots$$

Dualize to get relative COhomology: $C^k(Y, X) =$ cochains on Y that vanish on X, snake lemma gives

$$\to H^k(Y, X) \to H^k(Y) \to H^k(X) \to H^{k+1}(Y, X) \to \cdots$$

The key tool for working with relative cohomology is excision:

If the closure of A lies in the interior of X, then $H^k(Y,X)$ is canonically isomorphic to $H^k(Y-A, X-A)$.

If $Y = X+$ extra pieces, then $H^k(Y,X)$ only sees a neighborhood of the extra pieces. (Cellular cohomology is based on this.)

If X is "nice", then $H^k(Y,X)$ is isomorphic to $\tilde{H}^k(Y/X)$.

But maps between tiling spaces are almost always surjections, and almost never injections. What to do?

If $f : X \to Y$ is a quotient map such that $f^* : C^k(Y) \to C^k(X)$ is 1–1, then define $C^k_Q(X,Y) = C^k(X)/f^*(C^k(Y))$
The snake lemma converts the short exact sequence of cochain complexes:

$$0 \to C^*(Y) \to C^*(X) \to C^*_Q(X,Y) \to 0$$

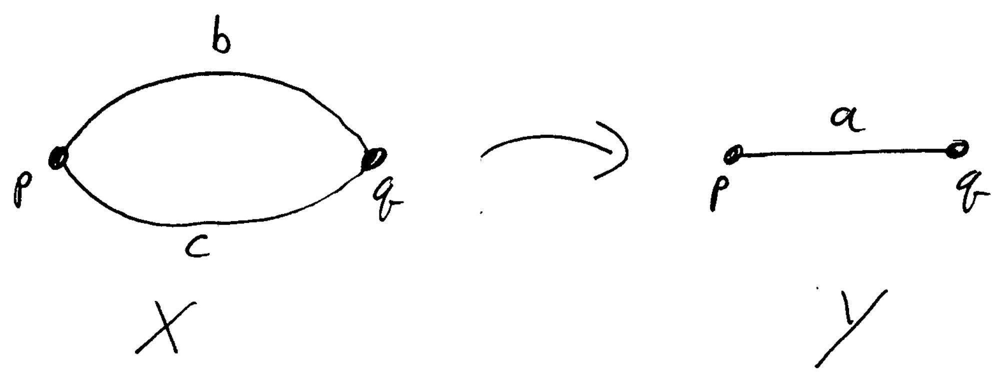
into a long exact sequence of cohomology groups:

$$H^k(Y) \to H^k(X) \to H^k_Q(X,Y) \to H^{k+1}(Y) \to \cdots$$

[Note: for tiling spaces you can think of cochains as either Čech cochains or pattern-equivariant cochains. For CW complexes you can use simplicial, singular, cellular, whatever. This works for all of them.]

A very simple example:

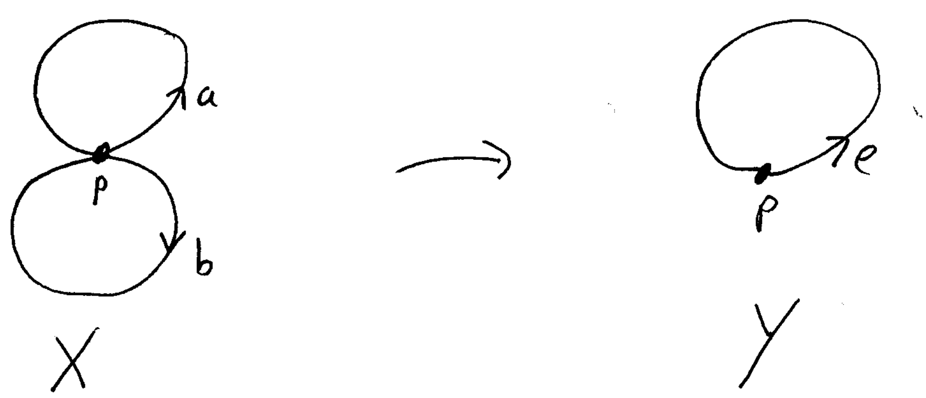
Let X be a circle and Y an interval:



Using cellular cochains, $C^0(X) = \mathbb{Z}^2$, with generators p' and q', $C^1(X) = \mathbb{Z}^2$ with generators b' and c', $C^0(Y) = \mathbb{Z}^2$ with generator p' and q', and $C^1(Y) = \mathbb{Z}$ with generator a'. Since $f^*(a') = b' + c'$, we have $C^0_Q(X, Y) = 0$, $C^1_Q(X, Y) = \mathbb{Z}$ with generator $b' = -c'$.

A slightly more complicated example:

Let X be a figure 8 and Y a circle, with f identifying the two loops of X:



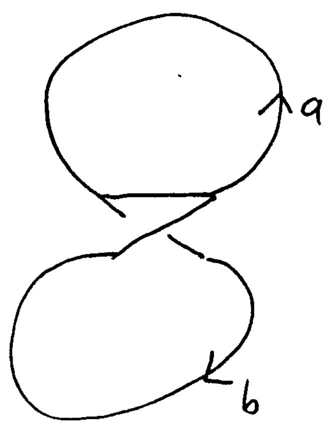
$C^0(X) = C^0(Y) = \mathbb{Z}$, with generator p' so $C^0(X < Y) = 0$. $C^1(X) = \mathbb{Z}^2$ with generators a' and b', $C^1(Y) = \mathbb{Z}$ with generator e'. Since $f^*(e') = b' + a'$, $H^1_Q(X, Y) = C^1_Q(X, Y) = \mathbb{Z}$ with generator $a' = -b'$.

A tiling example (finally)

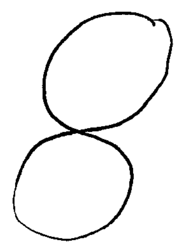
Period doubling substitution: $a \longrightarrow ba$, $b \longrightarrow aa$.

The period-doubling space Ω_{PD} is the inverse limit under substitution of

$\Gamma_{PD} =$

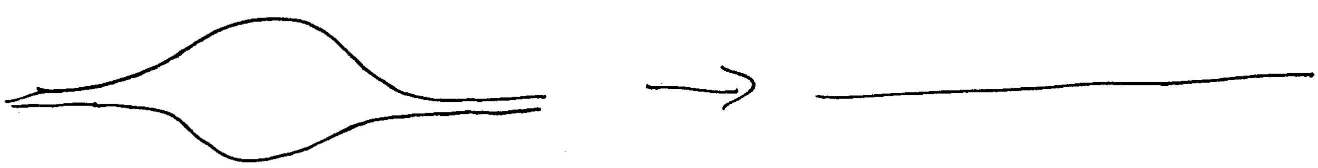


$\sim$



The dyadic solenoid S_2 is the inverse limit of S^1 under doubling. $H^1_Q(\Omega_{PD}, S_2) = \lim H^1_Q(\Gamma_{PD}, S^1) = \lim(\mathbb{Z}, -1) = \mathbb{Z}$, since $\sigma^*(a') = a' + 2b' = -a'$. Since $H^1(S_2) = \mathbb{Z}[1/2]$, $H^1(\Omega_{PD}) = \mathbb{Z}[1/2] \oplus \mathbb{Z}$.

The period doubling result shouldn't surprise, since $f : \Omega_{PD} \to S_2$ is 1–1 except on two asymptotic orbits that are identified. Morally same as



or

To make this rigorous need more tools.

Theorem: (Excision) Suppose that $f : X \to Y$ *is a quotient map that induces an injection on cochains. Suppose that* $Z \subset X$ *is an open set such that* $f|_{\bar{Z}}$ *is a homeomorphism onto its image. Then* $H^*_Q(X, Y)$ *is isomorphic to* $H^*_Q(X - Z, Y - f(\bar{Z}))$.

Proof: Relate quotient cohomology to the ordinary relative cohomology of a mapping cylinder and use ordinary excision.

Application: use this on approximants and take limits to get theorems on tiling spaces.

Theorem: If $f : X \to Y$ is a factor map of tiling spaces that is 1–1 except on the k-fold suspension of an n−k dimensional tiling subspace X' of X, then $H^*_Q(X,Y) = H^{*-k}_Q(X', f(X'))$.

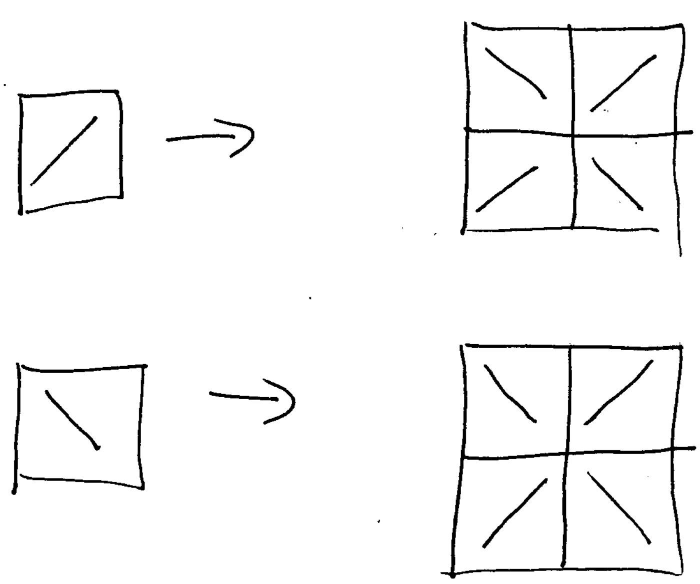
Example: For the period-doubling mapping to the solenoid, X' is 2 points that map to a single point, so $H^1(X,Y) = H^0_Q(X', f(X')) = \mathbb{Z}$.

Example: The half-hex tiling maps to $S_2 \times S_2$, with the map 1-1 except on three orbits that are identified. This is 2-fold suspension of $X' = 3$ points mapping to one point, so $H^2_Q(\Omega_{HH}, S_2 \times S_2) = \mathbb{Z}^2$ and $H^0_Q = H^1{}_Q = 0$. $H^2(\Omega_{HH}) = \mathbb{Z}[1/4] \oplus \mathbb{Z}^2$, with the $\mathbb{Z}[1/4]$ coming from the dyadic structure and the $\mathbb{Z}^2$ coming from the asymptotic orbits. Meanwhile, $H^1(\Omega_{HH})$ comes entirely from the dyadic structure.

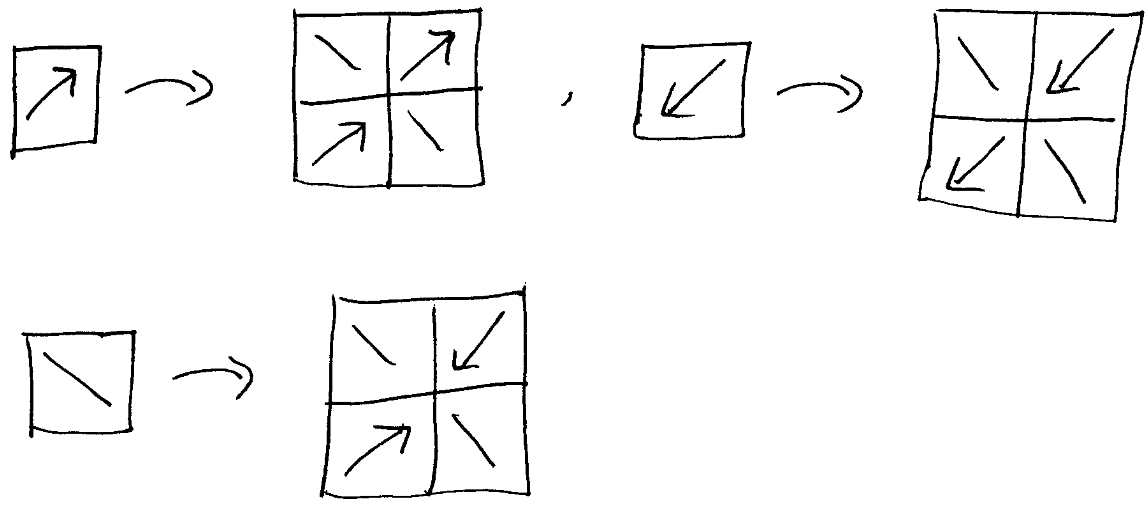
The chair tiling

The chair tiling maps almost 1–1 to $S_2 \times S_2$, maps 2:1 on small set, and 5:1 over a single orbit. Since 2:1 set isn't a suspension, can't apply tricks directly. Instead, define an intermediate model Ω_{hc} such that $\Omega_{chair} \longrightarrow \Omega_{hc}$ and $\Omega_{hc} \longrightarrow S_2 \times S_2$ are each 1-1 except over a suspension.

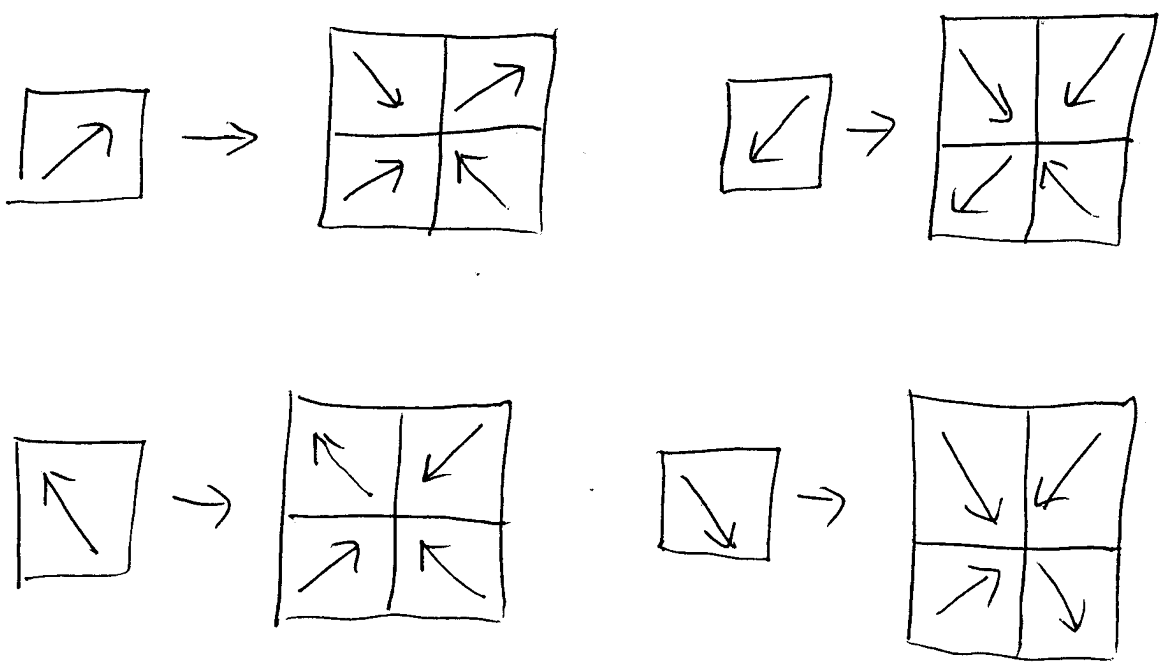
$S_2 \times S_2$ is inverse limit of substitution:



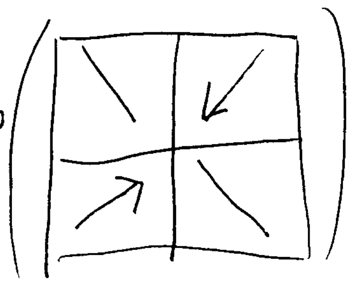
Ω_{hc} is inverse limit of substitution



Ω_{chair} is inverse limit of substitution



The map $\Omega_{chair} \to \Omega_{hc}$ is 1-1 except ~~on~~ over

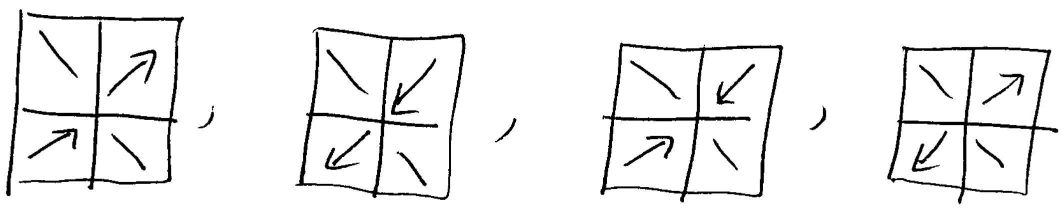
σ^{∞} 

and limits of translates in NW/SE direction.

where it is 2:1. This is the suspension of 2 copies of S_2 being identified, so $H^2_Q(\Omega_{chair}, \Omega_{hc}) = \mathbb{Z}[1/2]$ and $H^1_Q(\Omega_{chair}, \Omega_{hc}) = \mathbb{Z}$.

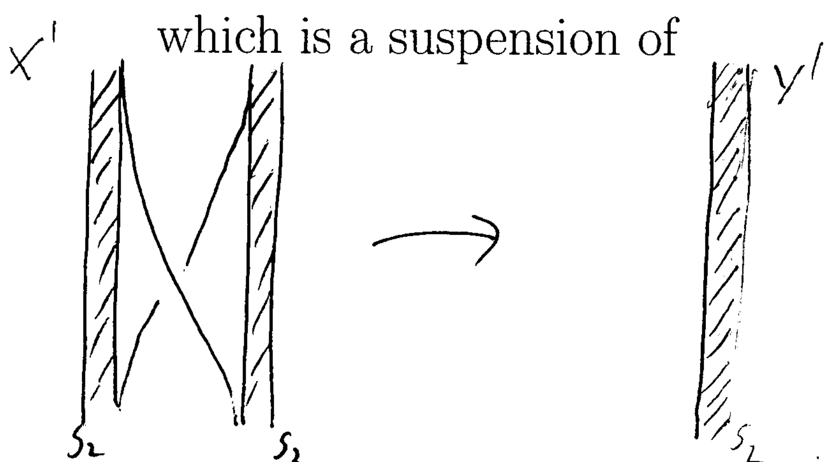
$$H^k_Q(2 \text{ solenoids}, 1 \text{ solenoid}) = H^k(\text{one solenoid}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}[\frac{1}{2}] & k=1 \end{cases}$$

The map $\Omega_{hc} \to S_2 \times S_2$ is 1-1 except on



& translates in NE-SW direction

which is a suspension of



We get $H^2_Q(\Omega_{hc}, S_2 \times S_2) = \mathbb{Z}[1/2] \otimes \mathbb{Z}$, $H^1_Q = H^0_Q = 0$.

$$H^k_Q(X', Y') = \begin{cases} \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z} & k=1 \\ 0 & k=0 \end{cases}$$

Combining the two quotient computations is not a direct sum. The $\mathbb{Z}$ in $H^2_Q(\Omega_{hc}, S_2 \times S_2)$ almost cancels the $\mathbb{Z}$ in $H^1_Q(\Omega_{chair}, \Omega_{hc})$. The upshot is $H^2_Q(\Omega_{chair}, S_2 \times S_2) = \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}_3$, $H^1_Q = H^0_Q = 0$.

Torsion in quotient cohomology, not in absolute cohomology:

$H^2(S_2 \times S_2) = \mathbb{Z}[1/4]$, $H^2(\Omega_{chair}) = \frac{1}{3}\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2$. $H^1(\Omega_{chair}) = H^1(S_2 \times S_2) = \mathbb{Z}[1/2]^2$.

Future application: Tiling spaces with finite matching rules, modeling substitution systems. Chaim Goodman-Strauss' construction gives a space Ω_{FT} that maps almost 1-1 to Ω_ϕ, where the multiple:1 set is a collection of suspensions of 0 and 1-dimensional tiling spaces.