Exploring self-affine tilings with and without coincidences

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Self-affine tiling

Self-affine tiling is a tiling of $\mathbb{R}^d$, i.e., almost one-to-one covering by finite compact regular sets, having three properties:

- Finite local complexity,
- Repetitivity,
- Inflation-subdivision structure: $QT_j = \bigcup_i T_i + D_{ij}$

by an expanding matrix Q and finite sets D_{ij} of translations. We say that $(\#D_{ij})$ the **substitution matrix**.

To distinguish congruent tiles coming form different sources, we put **colors** of tiles by making pairs (T, c) where T is the tile and c is a color alphabet. This is necessary in the case of constant length substitution.

To simplify terminology, I usually do not write this. However, this coloring becomes important later on.

We consider the translation action by $\mathbb{R}^d$ and the orbit of $\mathcal{T}$. By taking its closure by natural topology, we arrive at the tiling dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$. We also use **duality** of Lagarias-Wang between tilings and point sets, to transfer the problem to **point set dynamics**.

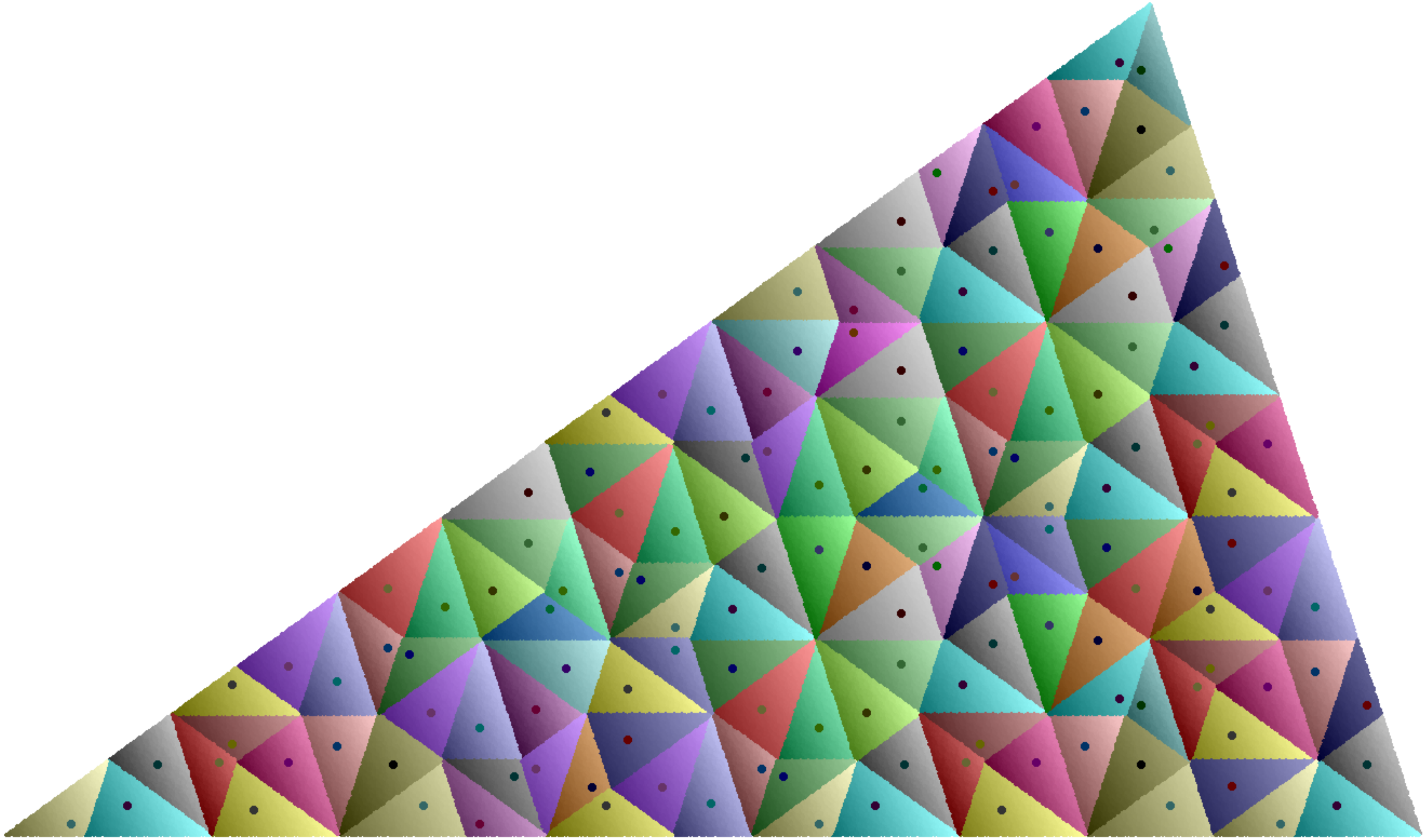


Figure 1: Penrose Tiling by Half-Kite and Half-Dart

- Finite Local Complexity (FLC) $\Leftrightarrow$ Compactness
- Repetitivity $\Leftrightarrow$ Minimality $+$ Unique Ergodicity.

By unique ergodicity, there is a unique translation invariant measure μ. The isometry $U_z(f(x)) = f(x-z)$ is defined on $L^2(X_{\mathcal{T}}, \mu)$ for $z \in \mathbb{R}^d$. We say that $a \in \mathbb{R}^d$ is an eigenvalue with respect to an eigenfunction f, if

$$U_z(f) = \exp(2\pi\sqrt{-1}a \cdot z)f$$

for $z \in \mathbb{R}^d$ where $a \cdot z$ is the Euclidean inner product.

The dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is **pure discrete** if there is a complete orthonormal basis of $L^2(X_{\mathcal{T}}, \mu)$ by eigenfunctions.

Dynamical system is pure discrete if and only if the system is conjugate to the a translation action of a compact abelien group.

By many efforts of the participants in this room, we know that pure discreteness of dynamical system is equivalent to the fact that the dual point set is generated by a certain **cut and project scheme**. Thus pure discreteness of tiling dynamics is equivalent to **pure point diffractivity** of point sets, gaining strong motivation from the study of aperiodic order.

Substitution dynamical system

Let σ be a substitution over $\{0, 1, \ldots, m-1\}^*$. Associating entries of Perron Frobenius eigenvalue of the incidence matrix M_σ, we obtain a suspension self-similar tiling.

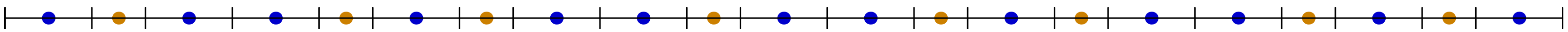


Figure 2: Fibonacci Tiling

FLC is clear from the construction. Repetitivity is equivalent to **primitivity** of substitution matrix. Repetitivity allows us to extend it to a self-similar tiling of $\mathbb{R}$. Together with translation action by $\mathbb{R}$, we got a strictly ergodic dynamics $(X_{\mathcal{T}}, \mathbb{R})$.

For 1-dim substitution tiling dynamics, it is not weakly mixing if and only if the Perron Frobenius root of M_σ is a **Pisot number**. In general, under **Pisot family condition**:

Any conjugate γ *of an eigenvalue of* Q *with* $|\gamma| \geq 1$ *is again an eigenvalue of* Q*,*

the dual point set associated to tilings becomes a Meyer set and vice versa under a mild condition. See Lee-Solomyak [9]. Under Pisot family condition, the dynamical system has d-linearly independent eigenvectors.

Solomyak's criterion of pure discreteness

$(X_{\mathcal{T}}, \mathbb{R}^d)$ is **pure discrete** iff

$$\text{density } \mathcal{T} \cap (\mathcal{T} - Q^n v_i) \to 1,$$

for d-linearly independent return vectors $v_1, \ldots, v_d$ in $\mathbb{R}^d$. This is the almost periodicity in self-affine setting.

Overlap algorithm by Solomyak [11]

Take a return vector v from $\{v_1, v_2, \ldots, v_d\}$ and denote by T, U the tiles in $\mathcal{T}$. Denote by $T \simeq U$ if T and U are translationally equivalent. We say that (T, v, U) is an **overlap** if

$$\left(\overset{\circ}{T} - v\right) \bigcap \overset{\circ}{U} \neq \emptyset.$$

We say that the overlap is a **coincidence** if it is of the form $(T, 0, U)$ and $T \simeq U$. Using inflation-subdivision, (QT, Qv, QU) is subdivided into overlaps and we draw edges from (T, v, U) to new overlaps. If Q satisfies Pisot family condition then the number of overlaps generated by this iteration is finite up to translational equivalence of overlaps.

Eventually we get a finite graph. From each overlaps there is a path to a coincidence, then the system is pure discrete and the converse is also true.

Overlap coincidence gives an algorithm to determine pure discreteness. However this is difficult to execute, because it is hard to tell whether $T - v$ and U have an inner point in common. One can do this for 1-dim substitutive tiling, but it is already difficult in 2-dim polygonal tiling to implement. Moreover, self-affine tiles often show fractal shapes !

Potential Overlap Algorithm

Recently with Jeong-Yup Lee, we invented an easy practical algorithm [1].

First of all, we use duality of tiling and point sets in the sense of Lagarias-Wang. We choose a reference point $c(T)$ for each T such that all points of T is within a ball of radius R. We say that (T, v, U) is a potential overlap, if

$$|c(T) - v - c(U)| \leq 2R$$

Clearly, an overlap must be a potential overlap. Then we apply the same overlap algorithm for potential overlaps.

For computation by translational equivalence, we only have to keep the record of (i, v, j) where i, j are the colors of T, U.

Because the dual point set becomes a Meyer set, the number of overlaps is finite. Moreover, **a bound for the number of iterations is given by the number of overlaps** (see [1]).

If $(T - v) \cap U = \emptyset$, then from such potential overlap there are no infinite paths, because the distance between them becomes larger by the application of Q and eventually all descendants becomes non potential overlaps.

However, if $T - v$ and U are touching at their boundaries, there should be an infinite path from this potential overlap, our main problem is to say that such contribution is small. In a potential overlap graph $\mathcal{G}$, take a subset of vertices which leads to coincidences and take an induced graph $\mathcal{G}_{coin}$ and we also take $\mathcal{G}_{res}$ the induced graph to the reminder vertices. We have proved

Theorem 1 (A. & J.Y.Lee [1])**.** $(X_{\mathcal{T}}, \mathbb{R}^d)$ is pure discrete if and only if $\rho(G_{res}) < \rho(G_{coin})$.

To show this, we have to simultaneously prove a conjecture folklore:

Theorem 2 (A. & J.Y.Lee [1]). The boundary of tiles of self-affine tiling $\mathcal{T}$ has Hausdorff dimension less than d. However we use a slightly modified Hausdorff dimension by **quasi norm**.

We also implemented a Mathematica program.

`http://mathweb.sc.niigata-u.ac.jp/~akiyama/Research1.html`

It works for all self-affine tilings, including non-unit scaling and non lattice-based tilings.

Example: Pisot Substitution with small trace

Fix positive m and B.

There are only finitely many primitive substitutions over m letters and PF roots $\leq B$.

However the number of substitutions quickly becomes large. By our program, we computed:

Cubic Pisot unit substitutions having a fix point, and $\mathrm{Tr}(M_\sigma) \leq 2$ is pure discrete.

There are 8,403 such substitutions up to symmetry, all of them are pure discrete.

For irreducible Pisot substitution, balanced pair algorithm is the quickest known way. However, our program seems equally quick and more efficient in cases, for e.g., a non-unit Pisot case,

$$0 \to 0122, \quad 1 \to 22, \quad 2 \to 0.$$

However my machine could not settle:

$$0 \to 0211, \quad 1 \to 22, \quad 2 \to 10$$

possibly by memory shortage. (Later, I finished this by machine in RIMS, Kyoto, with larger memory. Anyway we did not find a counter example.)

Example: Non pure substitutions

An example by Bernd Sing. Tiling dynamical system of substitution on 4 letters:

$$0 \to 0\overline{1},\ 1 \to 0,\ \overline{0} \to \overline{0}1,\ \overline{1} \to \overline{0}$$

is not purely discrete (c.f. [10]). This substitution is a skew product of Fibonacci substitution by a finite cocycle over $\mathbb{Z}/2\mathbb{Z}$.

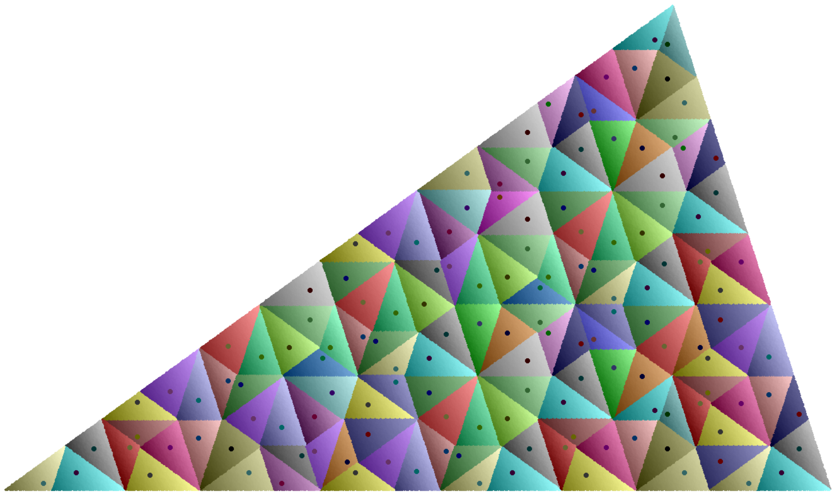
We computed many other examples of the same kind by different finite groups. One impressive example is

$$0 \to 0\overline{1},\ 1 \to 0\overline{2},\ 2 \to 0,\ \overline{0} \to \overline{0}1,\ \overline{1} \to \overline{0}2,\ \overline{2} \to \overline{0}$$

which is pure discrete. If the cocycle gives a trivial coboundary in the sense of Host [6], then the system is not pure discrete. However it gives non trivial coboundary, both pure and non-pure occur.

Example: Penrose Tiling

We confirmed directly that



is pure discrete. Computation takes half a day by our program by a machine equipped with large memory. We also examined several tilings by same triangles with different orientations.

Example: Einstein problem

Recently J. Socolar and J. Taylor discovered an **aperiodic monotile**, which solves the famous 'Einstein' problem.

This is just a hexagonal tile with matching condition. This can tiles the plane with its rotated and reflected pieces, but only in an aperiodic way.

We used an equivalent half-hexagonal tiling by 168 different protiles. **One week computation** saids it is pure discrete. Currently many people are analyzing this construction.

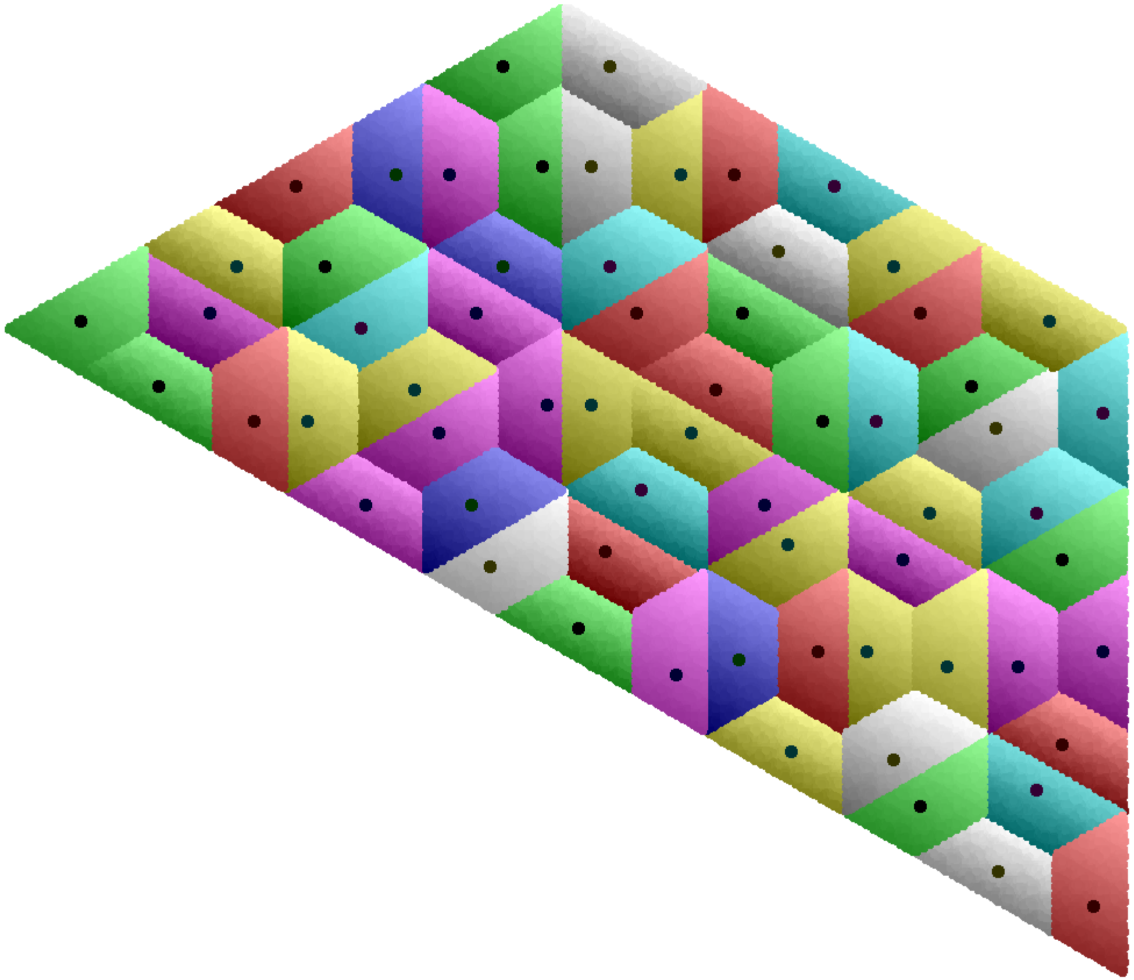


Figure 3: Half-Hex Tiling by Monotile

Example: endomorphisms of free group (Dekking [4, 5], Kenyon [7])

We consider a self similar tiling generated a boundary substitution:

$$\begin{aligned}\theta(a) &= b \\ \theta(b) &= c \\ \theta(c) &= a^{-1}b^{-1}\end{aligned}$$

acting on the boundary word $aba^{-1}b^{-1}, aca^{-1}c^{-1}, bcb^{-1}c^{-1}$, representing three fundamental parallelogram. The associated tile equation is

$$\begin{aligned} \alpha A_1 &= A_2 \\ \alpha A_2 &= (A_2 - 1 - \alpha) \cup (A_3 - 1) \\ \alpha A_3 &= A_1 - 1 \end{aligned}$$

with $\alpha \approx 0.341164 + i1.16154$ which is a root of the polynomial $x^3 + x + 1$.

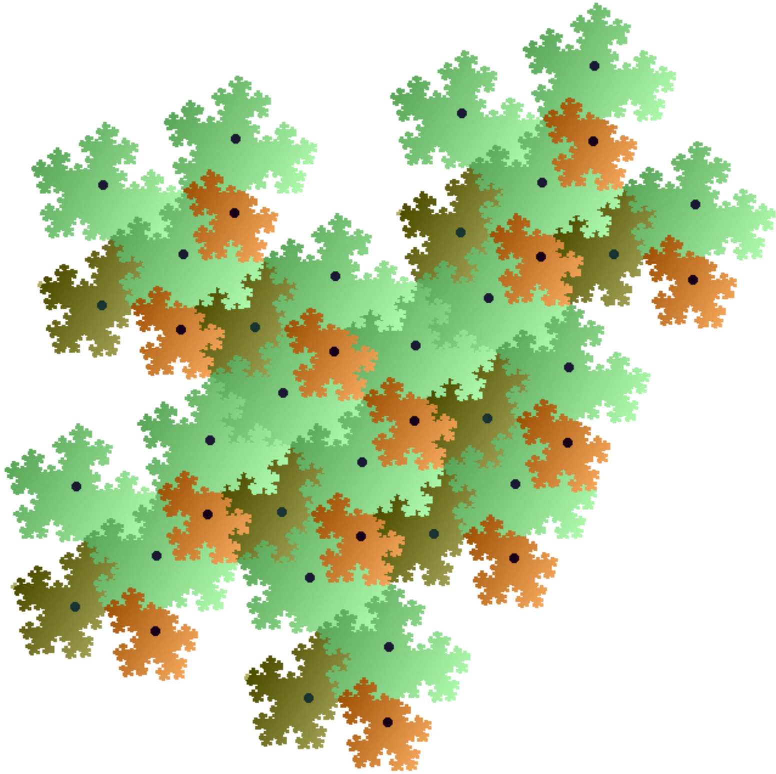


Figure 4: Tiling by boundary endomorphism

The tiling dynamical system has pure discrete spectrum.

Kenyon-Vershik's sofic cover

Kenyon-Vershik [8] introduced a geometric realization of hyperbolic toral automorphism. This construction gives a sofic system that has the toral automorphism as a factor. T. Sadahiro gave me an example of such construction in 4-dim case which is one to one whose characteristic polynomial is:

$$x^4 - x^3 - x^2 + x + 1$$

with 27 states and digits in $\{0, 1\}$.

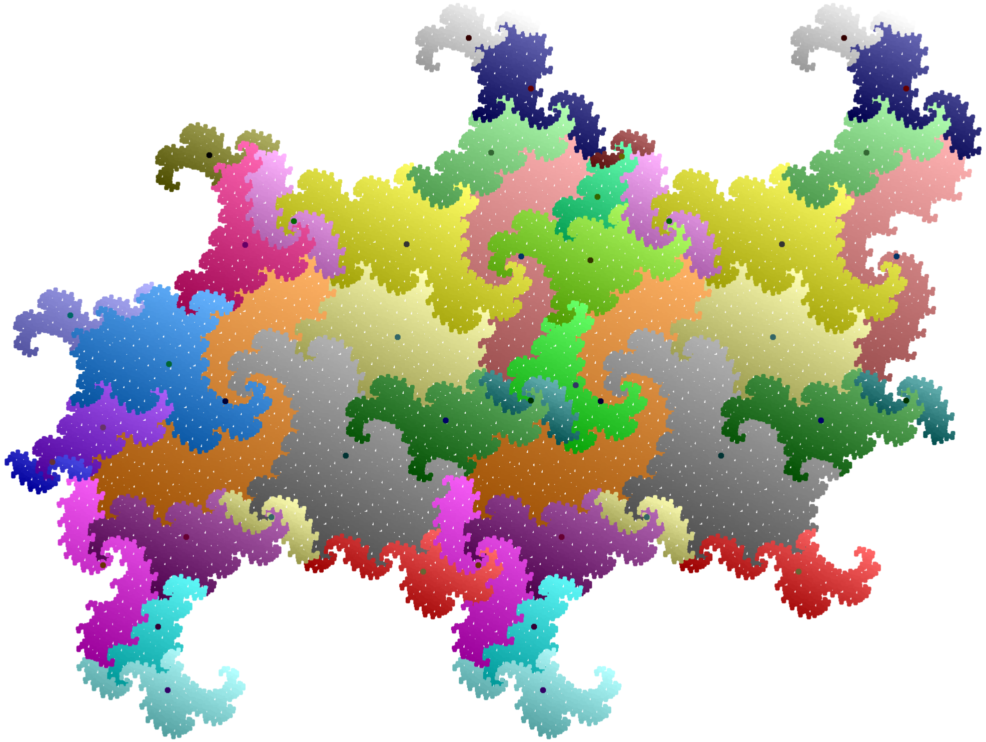


Figure 5: Sofic cover by Kenyon-Vershik

Our program saids this system is pure discrete.

Example: Arnoux-Furukado-Harriss-Ito tiling

Arnoux-Furukado-Harriss-Ito [2] recently gave an explicit Markov partition of the toral automorphism for the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

which has two dim expanding and two dim contractive planes. They defined 2-dim substitution of 6 polygons. Let $\alpha = -0.518913 - 0.66661\sqrt{-1}$ a root of $x^4 - x^3 + 1$. The multi colour Delone set is given by 6×6 matrix:

$$\begin{pmatrix} \{\} & \{z/\alpha\} & \{z/\alpha\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\} & \{z/\alpha\} & \{z/\alpha\} & \{\} \\ \{\} & \{\} & \{\} & \{\} & \{\} & \{z/\alpha\} \\ \{z/\alpha\} & \{\} & \{\} & \{\} & \{\} & \{\} \\ \{\} & \{z/\alpha + 1 - \alpha\} & \{\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\} & \{(z-1)/\alpha + \alpha\} & \{\} & \{\} \end{pmatrix}$$

and the associated tiling for contractive plane is:

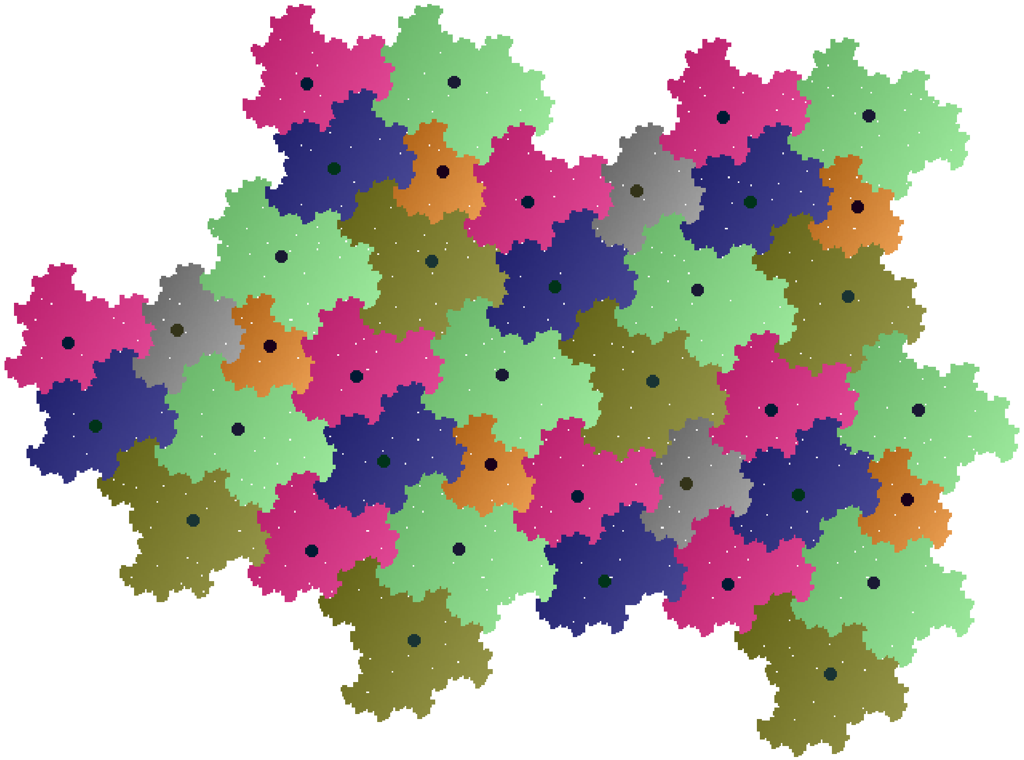


Figure 6: AFHI Tiling

Our program saids it is purely discrete.

Non pure example: Fractal Chair.

Bandt discovered a non-periodic tiling in [3] whose setting comes from crystallographic tiles. This is a 3-reptile defined by:

$$-I\omega\sqrt{3}A = A \cup (A+1) \cup (\omega A + \omega)$$

where $\omega = (1+\sqrt{-3})/2$ is the 6-th root of unity.

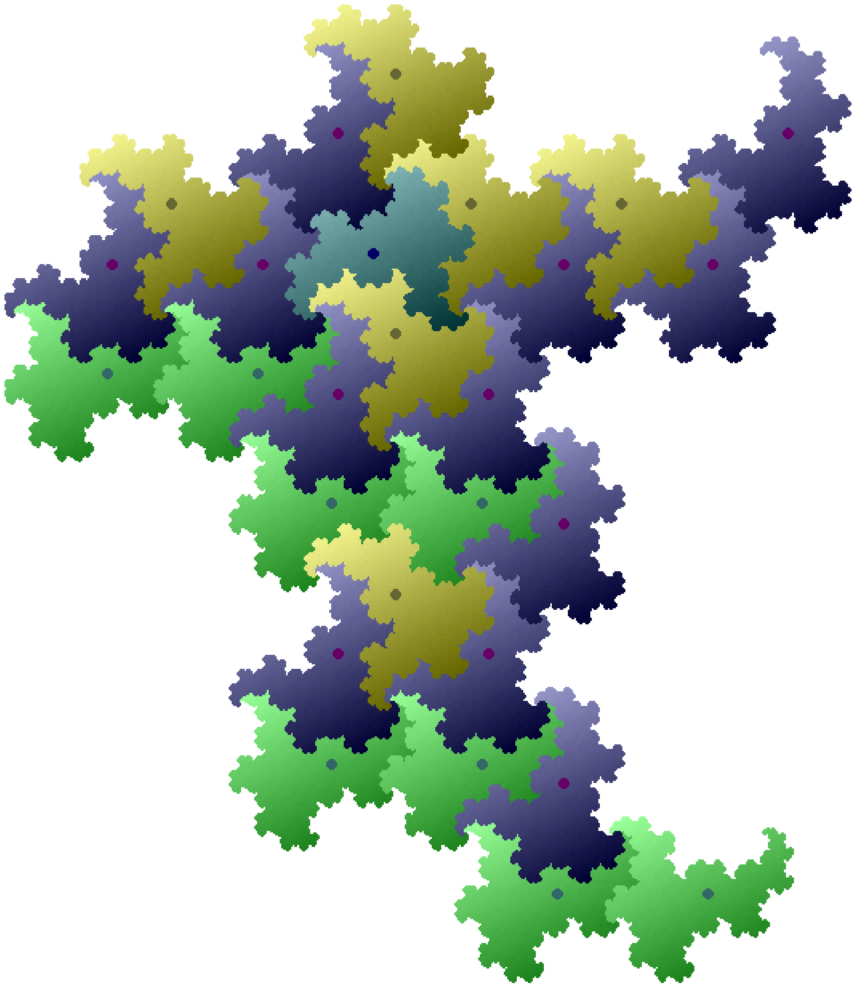


Figure 7: Fractal chair tiling

Fractal chair tiling is **not purely discrete**! An overlap creates new overlaps without any coincidence. One can draw an overlap graph.

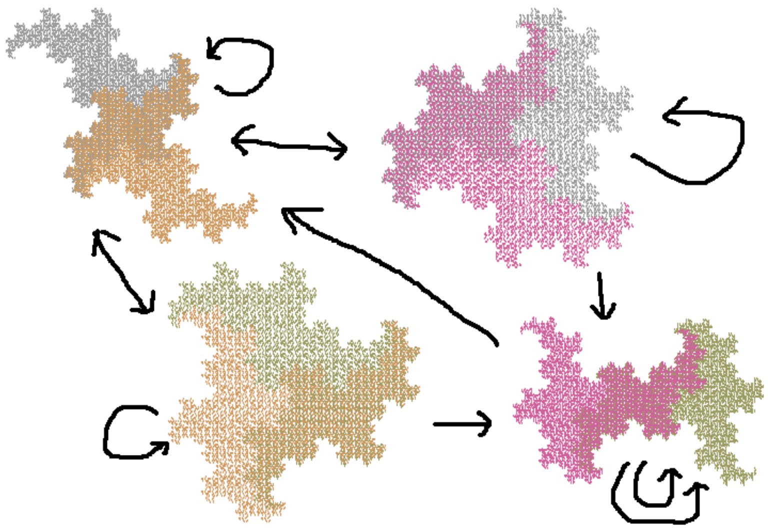


Figure 8: Overlap graph of fractal chair

However, from this overlap graph, one can construct another tiling which explains well this non pureness.

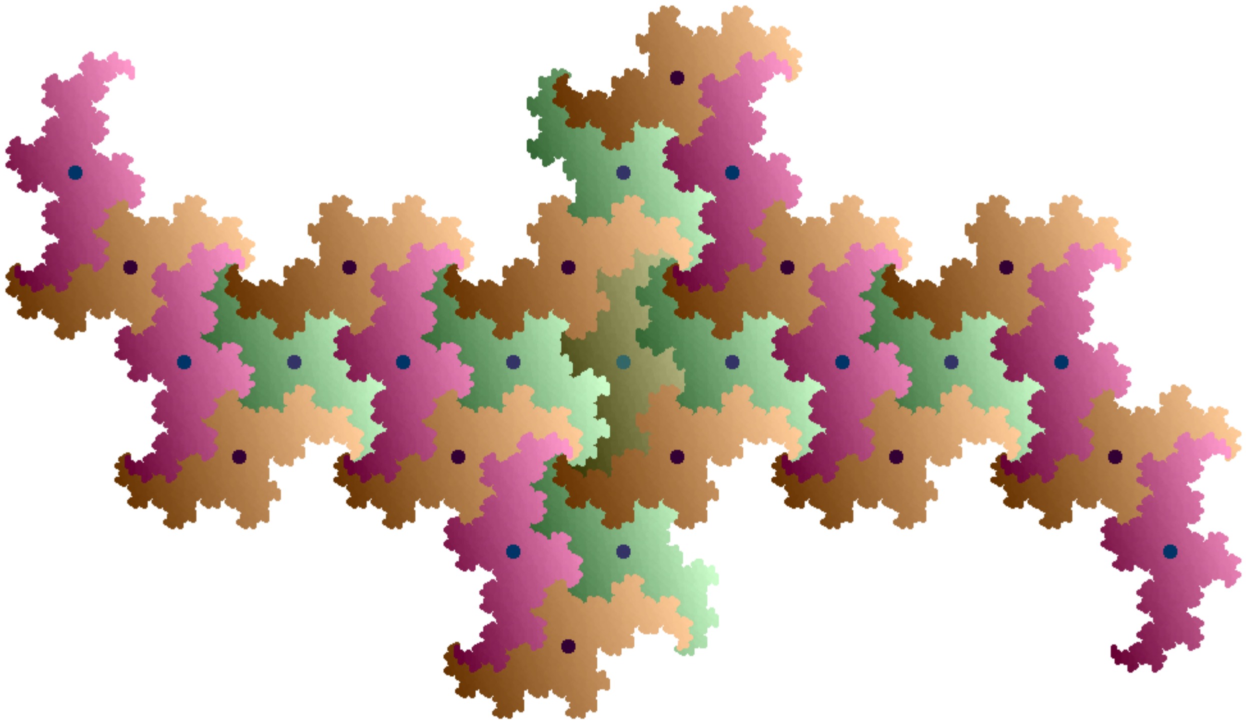


Figure 9: Tiling from overlaps

The 2nd tiling associates different colors to translationally equivalent tiles. Forgetting colors of tiles, it is periodic.

Non pure example: 4IET

Arnoux-Furukado-Ito studied self-inducing 4 interval exchange and studied associated substitutions:

$$1 \to 1241224,\ 2 \to 1224,\ 3 \to 1243334,\ 4 \to 124334$$

with characteristic polynomial $(x^2 - 3x + 1)(x^2 - 6x + 1)$, a Pisot family case, and

$$1 \to 124,\ 2 \to 1224,\ 3 \to 124334,\ 4 \to 12434$$

with totally real $x^4 - 7x^3 + 13x^2 - 7x + 1$; non Pisot family case. Both are non pure.

We studied a conjugate substitution of the former one

$$a \to adbdbd,\ b \to adbdbdbd, c \to adcccd,\ d \to adccd$$

This is of course not pure. The overlap graph gives another substitution:

$$a \to gegedbfca,\ b \to cdbff,\ c \to cdbfca,\ d \to gegedbf,$$
$$e \to gegedbfca,\ f \to cdbff,\ g \to ge$$

which is again non pure. The 2nd overlap graph gives:

$$a \to aff,\ b \to bed,\ c \to cde,\ d \to affcded,$$
$$e \to bedbede,\ f \to cdeafff$$

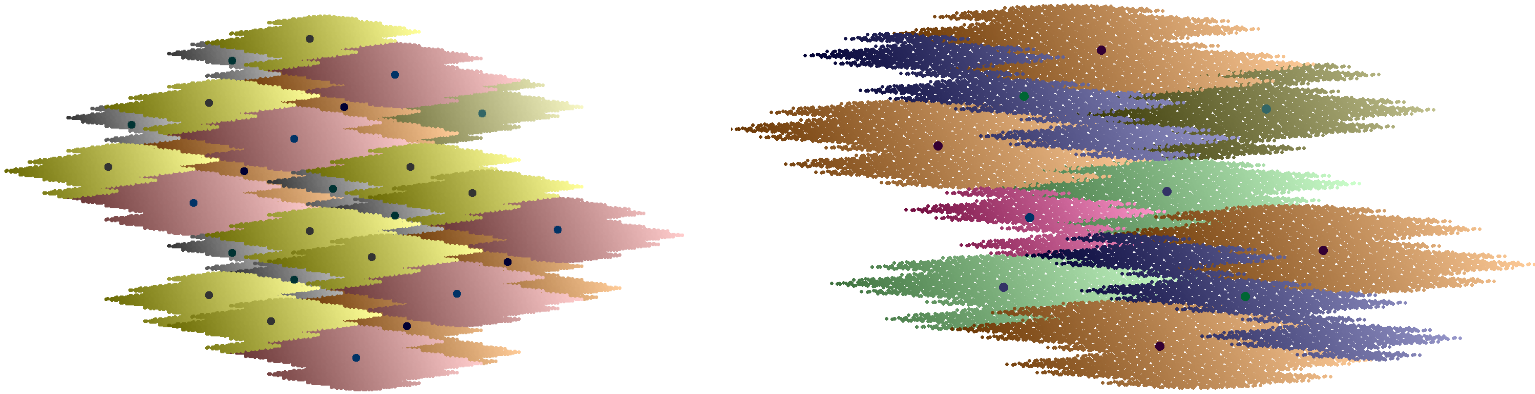
and the 3rd one is based on the Pisot substitution

$$1 \to 122,\ 2 \to 1221222$$

which is pure!

Realization of 4IET substitution.

From 4IET substitutions discussed above, Arnoux-Furukado-Ito made tilings on the contractive plane in two hyperbolic cases:



(a) Biquadratic (b) Totally real

Figure 10: Tilings by 4IET

Our program saids both are purely discrete. The geometric realizations and the original substitution system behaves quite differently.

Pisot Conjecture : Working hypothesis

1. Q satisfies Pisot family condition (Necessary).

2. Non pure part if exists, can be explained by a tiling whose congruent tiles have different colors, like Fractal Chair. Thus Φ_σ is irreducible, the tiling dynamics should be pure discrete.

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