Tilings associated with shift radix systems

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(joint work with V. Berthé, A. Siegel, P. Surer and J. Thuswaldner)

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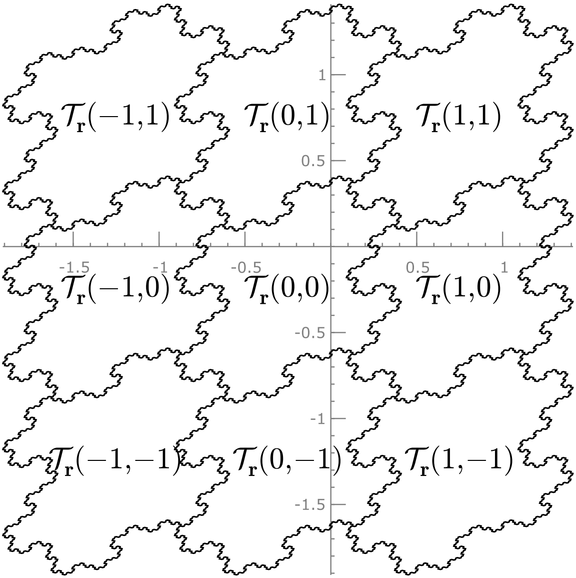
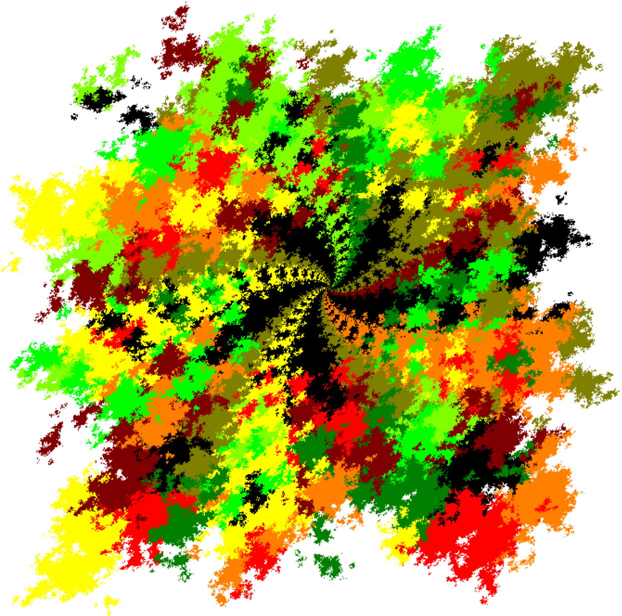
SRS tiles, $\mathbf{r} = (1/2, -1/2)$



SRS tiles, $\mathbf{r} = (9/10, -11/20)$



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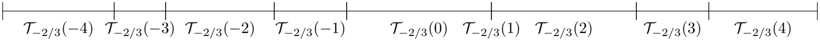

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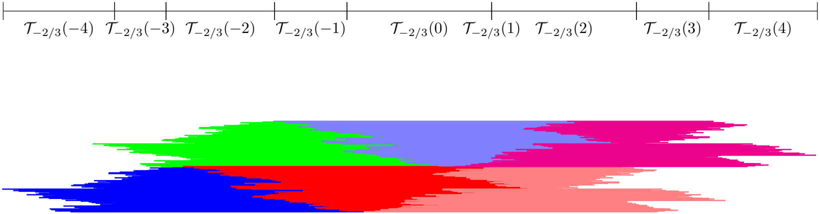
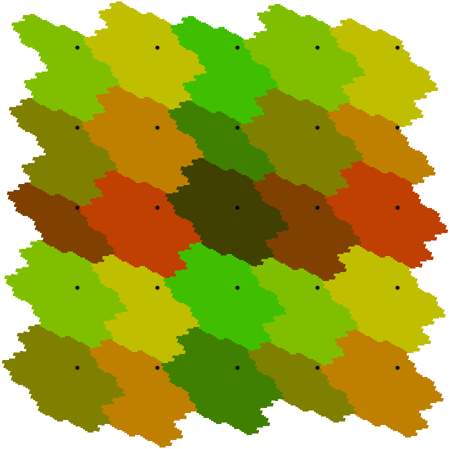


SRS tiles, $\mathbf{r} = (1/\beta, \beta - 1)$, $\beta^3 = \beta^2 + \beta + 1$



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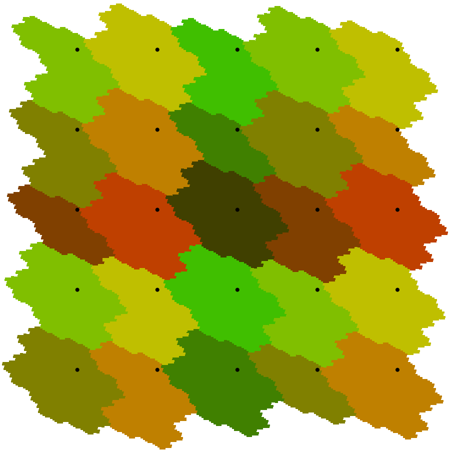

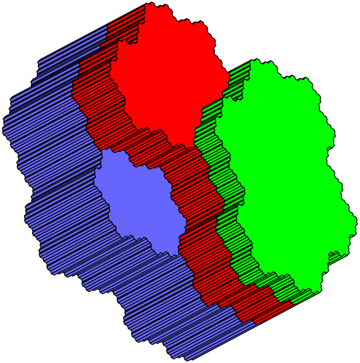

SRS tiles

The shift radix system (SRS) associated with $\mathbf{r} \in \mathbb{R}^d$ is the dynamical system $(\mathbb{Z}^d, \tau_{\mathbf{r}})$ defined by

$$\tau_{\mathbf{r}} : \ \mathbb{Z}^d \to \mathbb{Z}^d, \ \mathbf{x} = (x_1, x_2, \ldots, x_d) \mapsto (x_2, \ldots, x_d, -\lfloor \mathbf{r}\mathbf{x} \rfloor)$$

(cf. Akiyama–Borbély–Brunotte–Pethő–Thuswaldner 2005).

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$$\tau_{\mathbf{r}}(\mathbf{x}) = M_{\mathbf{r}}\mathbf{x} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \{\mathbf{r}\mathbf{x}\} \end{pmatrix}, \ M_{(r_0,\ldots,r_{d-1})} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -r_0 & -r_1 & \cdots & -r_{d-2} & -r_{d-1} \end{pmatrix}$$

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Assume $\varrho(M_\mathbf{r}) < 1$. For $\mathbf{x} \in \mathbb{Z}^d$, the set

$$\mathcal{T}_\mathbf{r}(\mathbf{x}) = \lim_{n\to\infty} M_\mathbf{r}^n \tau_\mathbf{r}^{-n}(\mathbf{x})$$

(limit with respect to the Hausdorff metric) is called SRS tile.

Basic properties of SRS tiles

$\mathcal{T}_{\mathbf{r}}(\mathbf{x})$ is compact, $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^d\}$ is locally finite,

$$\bigcup_{\mathbf{x} \in \mathbb{Z}^d} \mathcal{T}_{\mathbf{r}}(\mathbf{x}) = \mathbb{R}^d.$$

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$\mathcal{T}_{\mathbf{r}}(\mathbf{x})$ is not necessarily the closure of its interior. (Example: $\mathcal{T}_{\mathbf{r}}(\mathbf{x}) = \{\mathbf{0}\}$ for $\mathbf{r} = \left(\frac{9}{10}, -\frac{11}{20}\right)$, $\mathbf{x} = (1, -1)$, since $\tau_{\mathbf{r}}^{-5}(\mathbf{x}) = \{\mathbf{x}\}$.)

Conjecture

The boundary of $\mathcal{T}_{\mathbf{r}}(\mathbf{x})$ has zero Lebesgue measure.

Periodic sequences and finiteness property

Each sequence $(\tau_{\mathbf{r}}^n(\mathbf{x}))_{n\geq 0}$ is eventually periodic.

$(\mathbb{Z}^d, \tau_{\mathbf{r}})$ satisfies the finiteness property if

$$\forall\, \mathbf{x} \in \mathbb{Z}^d \;\exists\, n \in \mathbb{N} \text{ such that } \tau_{\mathbf{r}}^n(\mathbf{x}) = \mathbf{0}.$$

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$\mathbf{0} \in \mathcal{T}_{\mathbf{r}}(\mathbf{x})$ iff $(\tau_{\mathbf{r}}^n(\mathbf{x}))_{n\geq 0}$ is purely periodic.

$(\mathbb{Z}^d, \tau_{\mathbf{r}})$ has the finiteness property iff $\mathbf{0}$ is an exclusive point of $\mathcal{T}_{\mathbf{r}}(\mathbf{0})$, i.e., $\mathbf{0} \notin \bigcup_{\mathbf{x}\neq\mathbf{0}} \mathcal{T}_{\mathbf{r}}(\mathbf{x})$.

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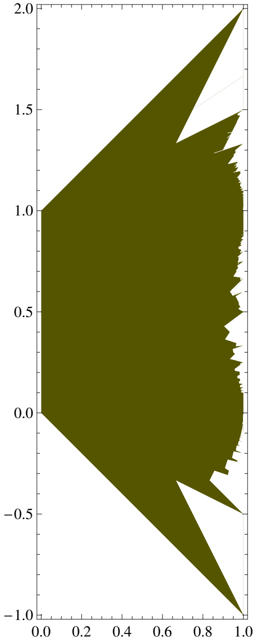
Characterization of SRS with finiteness property is complicated, see figure on the right for $\mathbf{r} \in \mathbb{R}^2$. (Akiyama–Brunotte–Pethő–Thuswaldner 2006, Surer 2007)



Weak tilings

Lemma

Let $\mathbf{t} \in \mathcal{T}_{\mathbf{r}}(\mathbf{x})$*, then* $\mathbf{t} = \lim_{n\to\infty} M_{\mathbf{r}}^n \mathbf{z}_n$ *with* $\tau_{\mathbf{r}}^n(\mathbf{z}_n) = \mathbf{x}$*.*
$\mathbf{t}$ *is an exclusive point of* $\mathcal{T}_{\mathbf{r}}(\mathbf{x})$*, i.e.,* $\mathbf{t} \notin \bigcup_{\mathbf{y}\neq\mathbf{0}} \mathcal{T}_{\mathbf{r}}(\mathbf{y})$*, iff*

$$\exists\, n: \ \tau_{\mathbf{r}}^n(\mathbf{z}_n + \mathbf{y}) = \mathbf{x} \quad \forall\, \mathbf{y} \in \mathbb{Z}^d \text{ with } \|\mathbf{y}\| \leq 2R,$$

where $R = \sum_{n=0}^{\infty} \left\| M_{\mathbf{r}}^n (0, \ldots, 0, 1)^t \right\| < \infty$*.*

Theorem

Assume that one of the following conditions hold.

- $\mathbf{r} \in \mathbb{Q}^d$,
- $(X - \beta)(X^d + r_{d-1}X^{d-1} + \cdots + r_0) \in \mathbb{Z}[X]$ *for some* $\beta > 1$,
- $r_0, \ldots, r_{d-1}$ *are algebraically independent over* $\mathbb{Q}$.

Then $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^d\}$ *forms a weak tiling of* $\mathbb{R}^d$ *(i.e., any two distinct tiles have disjoint interiors) iff there exists an exclusive point, in particular if* $(\mathbb{Z}^d, \tau_{\mathbf{r}})$ *satisfies the finiteness property.*

Weak m-tilings

$\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^d\}$ forms a weak m-tiling of $\mathbb{R}^d$ if every point of $\mathbb{R}^d$ is contained in at least m different tiles $\mathcal{T}_{\mathbf{r}}(\mathbf{x})$ and no point is in the interior of $m+1$ different tiles $\mathcal{T}_{\mathbf{r}}(\mathbf{x})$. ($m = 1$: weak tiling)

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Then $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^d\}$ *forms a weak m-tiling of* $\mathbb{R}^d$ *for some* $m \geq 1$.

Conjecture

For any $\mathbf{r} \in \mathbb{R}^d$ *(with* $\varrho(M_{\mathbf{r}}) < 1$*),* $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^d\}$ *forms a weak m-tiling of* $\mathbb{R}^d$ *for some* $m \geq 1$.

Rational bases

Akiyama–Frougny–Sakarovitch 2008 considered expansions of positive integers in rational bases p/q, with coprime integers $p > q \geq 1$, of the form

$$N = \frac{1}{q}\sum_{n=0}^{\infty} b_n \Big(\frac{p}{q}\Big)^n \qquad (b_n \in \mathcal{N} = \{0, \ldots, p-1\}).$$

These number systems correspond to SRS with $\mathbf{r} = (-q/p)$.

Theorem

The tiling $\{\mathcal{T}_{-2/3}(N) \mid N \in \mathbb{Z}\}$ consists of intervals with infinitely many different lengths.

Expanding algebraic numbers and CNS

Let β be an expanding algebraic number (i.e., all its Galois conjugates lie outside the unit circle) with minimal polynomial $a_d X^d + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$, $a_0 \geq 2$, and $\mathcal{N} = \{0, \ldots, a_0 - 1\}$.
If, for each $x \in \mathbb{Z}[\beta]$,

$$x = c_0 + c_1\beta + \ldots + c_\ell \beta^\ell \qquad (c_i \in \mathcal{N}),$$

then we call $(\beta, \mathcal{N})$ a canonical number system (CNS).

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Let $\Lambda_\beta = \mathbb{Z}[\beta] \cap \beta^{-1}\mathbb{Z}[\beta^{-1}]$. Λ_β is a $\mathbb{Z}$-module generated by $W_0 = a_d$ and $W_k = \beta W_{k-1} + a_{d-k}$, $1 \leq k < d$.
If β is an algebraic integer, i.e., $|a_d| = 1$, then $\Lambda_\beta = \mathbb{Z}[\beta]$.

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For every $x \in \mathbb{Z}[\beta]$, there exist unique $c_0 \in \mathcal{N}$, $y \in \mathbb{Z}[\beta]$ such that

$$x = c_0 + \beta y.$$

If $x \in \Lambda_\beta$, then $y \in \Lambda_\beta$. Let $x = \sum_{k=0}^{d-1} x_k W_k$, $\mathbf{x} = (x_0, \ldots, x_{d-1})$, $y = \sum_{k=0}^{d-1} y_k W_k$, $\mathbf{y} = (y_0, \ldots, y_{d-1})$, then

$$\mathbf{y} = \tau_{\mathbf{r}}(\mathbf{x}) \quad \text{with} \quad \mathbf{r} = \left(\tfrac{a_d}{a_0}, \tfrac{a_{d-1}}{a_0}, \ldots, \tfrac{a_1}{a_0}\right).$$

$(\beta, \mathcal{N})$ is a CNS iff $(\mathbb{Z}^d, \tau_{\mathbf{r}})$ has the finiteness property.

Tiles associated with expanding algebraic numbers

For $x \in \Lambda_\beta = \mathbb{Z}[\beta] \cap \beta^{-1}\mathbb{Z}[\beta^{-1}]$, $x = \sum_{k=0}^{d-1} p_k \beta^k$, define the tile

$$\mathcal{G}_\beta(x) = (p_0, \ldots, p_{d-1})^t$$
$$+ \left\{ \sum_{i=1}^{\infty} B^{-i}(c_i, 0, \ldots, 0)^t \;\middle|\; c_i \in \mathcal{N}, \beta^n x + \sum_{i=1}^{n} c_i \beta^{n-i} \in \Lambda_\beta \ \forall n \geq 0 \right\},$$

where B is a companion matrix of $X^d + \frac{a_{d-1}}{a_d} X^{d-1} + \cdots + \frac{a_0}{a_d}$.

If $|a_d| = 1$, then $\Lambda_\beta = \mathbb{Z}[\beta]$, thus

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The tiles are only self affine if $|a_d| = 1$.

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The tiles are only self affine if $|a_d| = 1$.

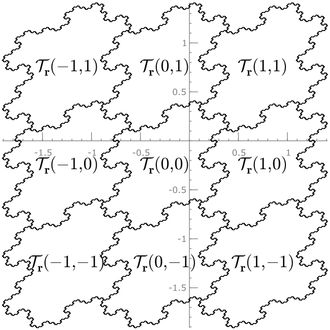
Theorem

For any $x \in \Lambda_\beta$, we have

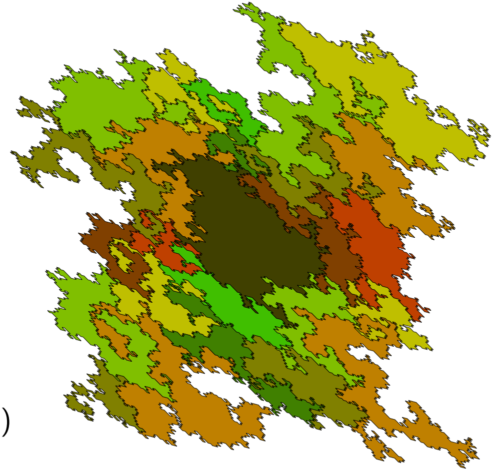
$$\mathcal{G}_\beta(x) = V \mathcal{T}_{\mathbf{r}}(\mathbf{x}),$$

where $x = \sum_{k=0}^{d-1} x_k W_k$, $\mathbf{x} = (x_0, \dots, x_{d-1})$, and V is the matrix realizing the base change from $\{W_0, \dots, W_{d-1}\}$ to $\{\beta^0, \dots, \beta^{d-1}\}$.

SRS tiles associated with CNS



$X^2 - X + 2$, $\mathbf{r} = (1/2, -1/2)$



$3X^2 + 4X + 4$, $\mathbf{r} = (3/4, 1)$

Relation with p-adic tiles

Let $\mathcal{O}$ bet the ring of integers of $\mathbb{Q}(\beta)$, write $\beta\mathcal{O} = \frac{\mathfrak{a}}{\mathfrak{b}}$ with ideals $\mathfrak{a}, \mathfrak{b}$ in $\mathcal{O}$ such that $(\mathfrak{a}, \mathfrak{b}) = \mathcal{O}$, and set

$$\mathbb{K}_\beta = \mathbb{R}^d \times \prod_{\mathfrak{p}|\mathfrak{b}} K_\mathfrak{p},$$

where $K_\mathfrak{p}$ is the completion of $\mathbb{Q}(\beta)$ with respect to $|\cdot|_\mathfrak{p}$. Let

$$\Phi_\beta : \ \mathbb{Q}(\beta) \to \mathbb{K}_\beta, \ x = \sum_{k=0}^{d-1} p_k \beta^k \mapsto (p_0, \ldots, p_{d-1}, x, \ldots, x),$$

$$\mathcal{F}_\beta = \left\{ \sum_{i=1}^{\infty} c_i \, \Phi_\beta(\beta^{-i}) \,\middle|\, c_i \in \mathcal{N} \right\},$$

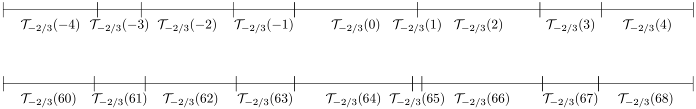
$$\mathcal{G}'_\beta(x) = \Phi_\beta(x) + \left\{ \sum_{i=1}^{\infty} c_i \, \Phi_\beta(\beta^{-i}) \,\middle|\, c_i \in \mathcal{N}, \beta^n x + \sum_{i=1}^{n} c_i \beta^{n-i} \in \Lambda_\beta \ \forall n \geq 0 \right\}.$$

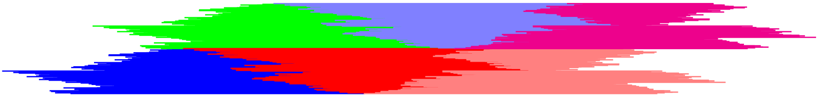
Theorem

For any $x \in \Lambda_\beta$, we have $\mathcal{G}'_\beta(x) = \mathcal{G}_\beta(x) \times \prod_{\mathfrak{p}|\mathfrak{b}} \{0\}$, and

$$\mathcal{F}_\beta = \overline{\bigcup_{x \in \Lambda_\beta} \left(\mathcal{G}'_\beta(x) - \Phi_\beta(x) \right)}.$$

SRS tiles, $\mathbf{r} = (-2/3)$, and $\mathcal{F}_\beta + \Phi_\beta(\mathbb{Z}[\beta])$, $\beta = 3/2$





Tiling theorem

Theorem (St–Thuswaldner)

$\{\Phi_\beta(x) + \mathcal{F}_\beta \mid x \in \mathbb{Z}[\beta]\}$ forms a tiling of $\mathbb{K}_\beta$.

Corollary

Let $\mathbf{r} = \left(\frac{a_d}{a_0}, \frac{a_{d-1}}{a_0}, \ldots, \frac{a_1}{a_0}\right) \in \mathbb{Q}^d$ be such that $\varrho(M_\mathbf{r}) < 1$ and $a_d X^d + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$ is irreducible.
Then $\{\mathcal{T}_\mathbf{r}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^d\}$ contains an exclusive point, thus it forms a weak tiling of $\mathbb{R}^d$.

Therefore, the set of $\mathbf{r} \in \mathbb{R}^d$ such that $\{\mathcal{T}_\mathbf{r}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^d\}$ forms a weak tiling of $\mathbb{R}^d$ is dense in $\{\mathbf{r} \in \mathbb{R}^d \mid \varrho(M_\mathbf{r}) < 1\}$.

Pisot numbers and β-transformation

A Pisot number is an algebraic integer $\beta > 1$ with $|\beta_j| < 1$ for every conjugate β_j of β. Write the minimal polynomial of β as

$$(X - \beta)(X^d + r_{d-1}X^{d-1} + \cdots + r_0X^0) \in \mathbb{Z}[X],$$

and let $\mathbf{r} = (r_0, \ldots, r_{d-1})$. Then $\varrho(M_\mathbf{r}) < 1$.

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The β-transformation is defined by

$$T_\beta : \ [0,1) \to [0,1), \ x \mapsto \{\beta x\} = \beta x - \lfloor \beta x \rfloor.$$

We have

$$T_\beta(\{\mathbf{r}\mathbf{x}\}) = \{\mathbf{r}\tau_\mathbf{r}(\mathbf{x})\},$$

and the map $\mathbb{Z}^d \to \mathbb{Z}[\beta] \cap [0,1)$, $\mathbf{x} \mapsto \{\mathbf{r}\mathbf{x}\}$ is a bijection. Hence, the restriction of T_β to $\mathbb{Z}[\beta] \cap [0,1)$ is conjugate to $\tau_\mathbf{r}$.

$(\mathbb{Z}^d, \tau_\mathbf{r})$ has the finiteness property iff β has the property

$$(\mathrm{F}) : \ \forall x \in \mathbb{Z}[\beta^{-1}] \cap [0,1) \ \exists n \geq 0 \text{ such that } T_\beta^n(x) = 0.$$

(Integral) β-tiles

Let $\beta_0 = \beta, \beta_1, \ldots, \beta_d$ be the Galois conjugates of β, $\beta_1, \ldots, \beta_r \in \mathbb{R}$, $\beta_{r+1} = \overline{\beta_{r+s+1}}, \ldots, \beta_{r+s} = \overline{\beta_{r+2s}} \in \mathbb{C}$, $d = r+2s$, $x^{(j)}$ be the corresponding conjugate of $x \in \mathbb{Q}(\beta)$, $1 \le j \le d$,

$$\Phi_\beta : \ \mathbb{Q}(\beta) \to \mathbb{R}^d, \ x \mapsto$$
$$\big(x^{(1)}, \ldots, x^{(r)}, \Re\big(x^{(r+1)}\big), \Im\big(x^{(r+1)}\big), \ldots, \Re\big(x^{(r+s)}\big), \Im\big(x^{(r+s)}\big)\big).$$

For $x \in \mathbb{Z}[\beta] \cap [0,1)$, the β-tile is the (compact) set

$$\mathcal{R}_\beta(x) = \lim_{n\to\infty} \Phi_\beta\big(\beta^n T_\beta^{-n}(x)\big)$$

(cf. Thurston 1989, Akiyama 1999).
We have $\mathbf{t} \in \mathcal{R}_\beta(x)$ if and only if there exist $c_i \in \mathbb{Z}$ with

$$\mathbf{t} = \Phi_\beta(x) + \sum_{i=0}^{\infty} \Phi_\beta(\beta^i c_i), \ \frac{c_{n-1}}{\beta} + \cdots + \frac{c_0}{\beta^n} + \frac{x}{\beta^n} \in [0,1) \ \forall n \ge 0.$$

(Integral) β-tiles

Let $\beta_0 = \beta, \beta_1, \ldots, \beta_d$ be the Galois conjugates of β, $\beta_1, \ldots, \beta_r \in \mathbb{R}$, $\beta_{r+1} = \overline{\beta_{r+s+1}}, \ldots, \beta_{r+s} = \overline{\beta_{r+2s}} \in \mathbb{C}$, $d = r + 2s$, $x^{(j)}$ be the corresponding conjugate of $x \in \mathbb{Q}(\beta)$, $1 \le j \le d$,

$$\Phi_\beta : \ \mathbb{Q}(\beta) \to \mathbb{R}^d, \ x \mapsto$$
$$\big(x^{(1)}, \ldots, x^{(r)}, \Re\big(x^{(r+1)}\big), \Im\big(x^{(r+1)}\big), \ldots, \Re\big(x^{(r+s)}\big), \Im\big(x^{(r+s)}\big)\big).$$

For $x \in \mathbb{Z}[\beta] \cap [0,1)$, the integral β-tile is the (compact) set

$$\mathcal{S}_\beta(x) = \lim_{n\to\infty} \Phi_\beta\big(\beta^n\big(T_\beta^{-n}(x) \cap \mathbb{Z}[\beta]\big)\big).$$

We have $\mathbf{t} \in \mathcal{S}_\beta(x)$ if and only if there exist $c_i \in \mathbb{Z}$ with

$$\mathbf{t} = \Phi_\beta(x) + \sum_{i=0}^{\infty} \Phi_\beta(\beta^i c_i), \ \frac{c_{n-1}}{\beta} + \cdots + \frac{c_0}{\beta^n} + \frac{x}{\beta^n} \in [0,1) \cap \mathbb{Z}[\beta] \ \forall n.$$

Relation between SRS tiles and integral β-tiles

Theorem

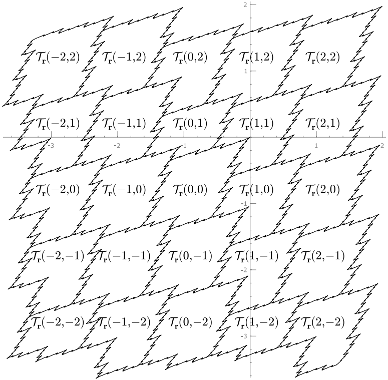
Let $X^d + r_{d-1}X^{d-1} + \cdots + r_0 = (X - \beta_j)(X^{d-1} + q_{d-2}^{(j)}X^{d-2} + \cdots + q_0^{(j)})$, $1 \le j \le d$,

$$U = \begin{pmatrix} q_0^{(1)} & q_1^{(j)} & \cdots & q_{d-2}^{(1)} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ q_0^{(r)} & q_1^{(j)} & \cdots & q_{d-2}^{(r)} & 1 \\ \Re(q_0^{(r+1)}) & \Re(q_1^{(r+1)}) & \cdots & \Re(q_{d-2}^{(r+1)}) & 1 \\ \Im(q_0^{(r+1)}) & \Im(q_1^{(r+1)}) & \cdots & \Im(q_{d-2}^{(r+1)}) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \Re(q_0^{(r+s)}) & \Re(q_1^{(r+s)}) & \cdots & \Re(q_{d-2}^{(r+s)}) & 1 \\ \Im(q_0^{(r+s)}) & \Im(q_1^{(r+s)}) & \cdots & \Im(q_{d-2}^{(r+s)}) & 0 \end{pmatrix} \in \mathbb{R}^{d \times d},$$

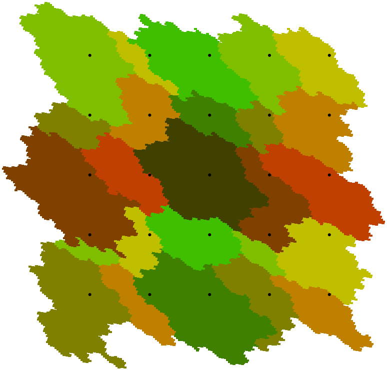
I_d *be the identity matrix. For every* $\mathbf{x} \in \mathbb{Z}^d$, *we have*

$$\mathcal{S}_\beta(\{\mathbf{r}\mathbf{x}\}) = U(M_\mathbf{r} - \beta I_d)\mathcal{T}_\mathbf{r}(\mathbf{x}).$$

SRS tiles associated with Pisot numbers



$\beta^3 = 3\beta^2 - 1, \mathbf{r} = (-1/\beta, \beta - 3)$



$\beta^3 = 2\beta^2 + 2\beta + 2, \mathbf{r} = (2/\beta, \beta - 2)$

The integral β-tiles are given by $\mathcal{S}_\beta(\{\mathbf{rx}\}) = U(M_\mathbf{r} - \beta I_d)\mathcal{T}_\mathbf{r}(\mathbf{x})$, but the "centers" of the integral β-tiles are given by $\Phi_\beta(\{\mathbf{rx}\}) = U(\tau_\mathbf{r}(\mathbf{x}) - \beta\mathbf{x}) = U(M_\mathbf{r} - \beta I_d)\mathbf{x} + U(0, \ldots, 0, \{\mathbf{rx}\})^t$.

Properties of β-tiles

If β is a Pisot unit ($\beta^{-1} \in \mathbb{Z}[\beta]$), then

- $\mathcal{R}_\beta(x) = \mathcal{S}_\beta(x)$ for every $x \in \mathbb{Z}[\beta] \cap [0,1)$,
- we have only finitely many tiles up to translation,
- the boundary of each tile has zero Lebesgure measure,
- each tile is the closure of its interior,
- $\{\mathcal{S}_\beta(x) \mid x \in \mathbb{Z}[\beta] \cap [0,1)\}$ forms a multiple tiling of $\mathbb{R}^d$,
- $\{\mathcal{S}_\beta(x) \mid x \in \mathbb{Z}[\beta] \cap [0,1)\}$ forms a tiling if (F) holds,
- $\{\mathcal{S}_\beta(x) \mid x \in \mathbb{Z}[\beta] \cap [0,1)\}$ forms a tiling iff (W) holds: for every $x \in \mathbb{Z}[\beta] \cap [0,1)$ and every $\varepsilon > 0$, there exists some $y \in [0,\varepsilon)$ with finite β-expansion such that $x + y$ has finite β-expansion,

see Akiyama 1999, 2002, Berthé–Siegel 2005.

Pisot conjecture

Conjecture

If β is a Pisot unit of degree $d+1$, then $\{\mathcal{R}_\beta(x) \mid x \in \mathbb{Z}[\beta] \cap [0,1)\}$ forms a tiling of $\mathbb{R}^d$.

Proved for several classes of Pisot units.
(Frougny–Solomyak 1992, Hollander 1996,
Akiyama–Rao–St 2004, Barge–Kwapisz 2006)

Conjecture

If $\varrho(M_{\mathbf{r}}) < 1$, then $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^d\}$ forms a weak tiling of $\mathbb{R}^d$.

Proved for a dense set of **r**, see above.

α-Shift Radix Systems

For $\mathbf{r} = (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d$, the α-SRS $(\mathbb{Z}^d, \tau_{\mathbf{r},\alpha})$ is defined by

$$\tau_{\mathbf{r},\alpha}: \ \mathbb{Z}^d \to \mathbb{Z}^d, \ \mathbf{x} = (x_0, \ldots, x_{d-1}) \mapsto (x_1, \ldots, x_{d-1}, -\lfloor \mathbf{r}\mathbf{x} + \alpha \rfloor).$$

For every $\mathbf{x} \in \mathbb{Z}^d$, the α-SRS tile is defined by

$$\mathcal{T}_{\mathbf{r},\alpha}(\mathbf{x}) = \lim_{n\to\infty} M_{\mathbf{r}}^n \tau_{\mathbf{r},\alpha}^{-n}(\mathbf{x}).$$

A $1/2$-SRS is also called symmetric SRS.

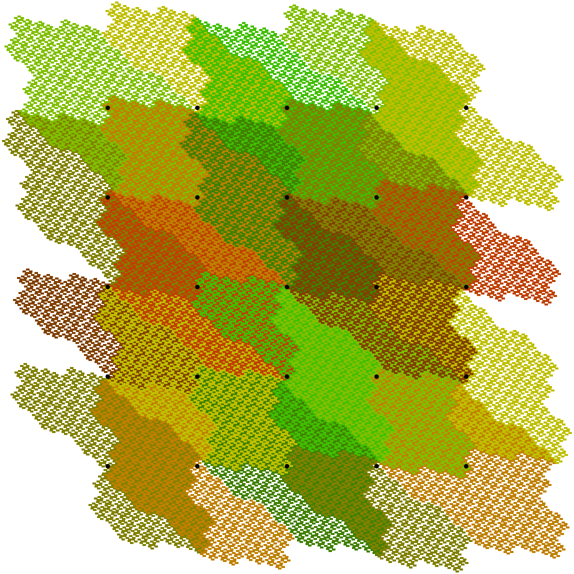
Theorem (St–Thuswaldner)

Let $\mathbf{r} = \left(\frac{a_d}{a_0}, \frac{a_{d-1}}{a_0}, \ldots, \frac{a_1}{a_0}\right) \in \mathbb{Q}^d$ be such that $\varrho(M_{\mathbf{r}}) < 1$ and $a_d X^d + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$ is irreducible.
Then $\{\mathcal{T}_{\mathbf{r},1/2}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^d\}$ forms a weak tiling of $\mathbb{R}^d$, and these $\mathbf{r}$ are dense in $\{\mathbf{r} \in \mathbb{R}^d \mid \varrho(M_{\mathbf{r}}) < 1\}$.

Theorem (Kalle–St)

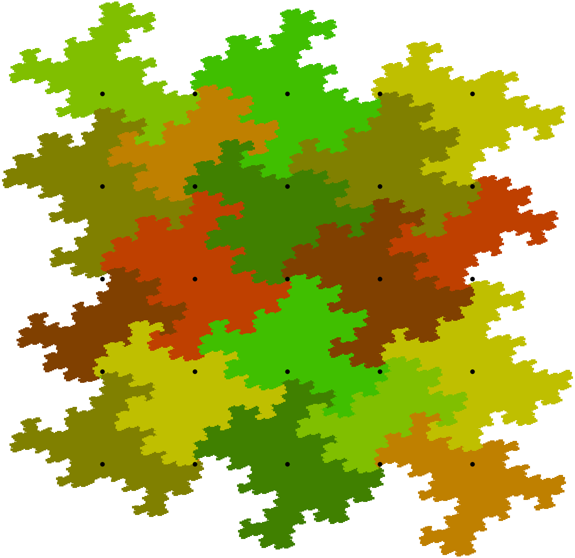
Let β be the smallest Pisot number ($\beta^3 = \beta + 1$), $\mathbf{r} = (1/\beta, \beta)$, or the Tribonacci number ($\beta^3 = \beta^2 + \beta + 1$), $\mathbf{r} = (1/\beta, \beta - 1)$, then $\{\mathcal{T}_{\mathbf{r},1/2}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Z}^2\}$ forms a 2-tiling of $\mathbb{R}^2$.

Double tiling for a symmetric SRS



$\beta^3 = \beta^2 + \beta + 1$, $\mathbf{r} = (1/\beta, \beta - 1)$, $\alpha = 1/2$

Tiling for a symmetric SRS



$$\beta^3 = 2\beta^2 - \beta + 1,\ \mathbf{r} = (1/\beta, \beta - 2),\ \alpha = 1/2$$