Coincidences of lattices and beyond

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Brief historical overview

mid sixties: CSLs - grain boundaries

Ranganathan, Bollmann, Grimmer, . . .

mid ninties: quasicrystals $\longrightarrow$ CSM

Baake, Pleasants, Warrington, . . .

2002: Quantizing Using Lattice Intersections

Sloane, Beferull–Lozano

2005: Zou: Cartan-Dieudonné

1997-present: Aragón, Rodriguez et.al.: Clifford algebras

20xy: Baake, Grimm, Heuer, Moody, Pleasants, Scharlau, Loquias, Glied, Huck, PZ,. . .

Coincidence Site Modules

Modules and Lattices

- module M:

  $M =: \langle t_1, \ldots, t_r \rangle_{\mathbb{Z}} = \{n_1 t_1 + \ldots + n_r t_r\} \subseteq \mathbb{R}^d$

  with $t_1, \ldots, t_r \in \mathbb{R}^d$ rationally independent,

  $\langle t_1, \ldots, t_r \rangle_{\mathbb{R}} = \mathbb{R}^d$, $k \geq d$
- lattice Γ:= module with $k = d$
- submodule $M_1 \subseteq M$: full rank $k \iff [M : M_1]$ is finite

Commensurate Modules

Lemma

The following are equivalent:

- M_1 *and* M_2 *are commensurate.*
- $M_1 \cap M_2$ *is a submodule of both* M_1 *and* M_2.
- $M_1 \cap M_2$ *is a submodule of* M_1 *or* M_2.
- *There exists an* $m \in \mathbb{N}$ *such that* $mM_1 \subseteq M_2$ *and* $mM_2 \subseteq M_1$.
- *There exists an* $m \in \mathbb{N}$ *such that* $mM_1 \subseteq M_2$ *or* $mM_2 \subseteq M_1$.

Ordinary CSMs

Definition

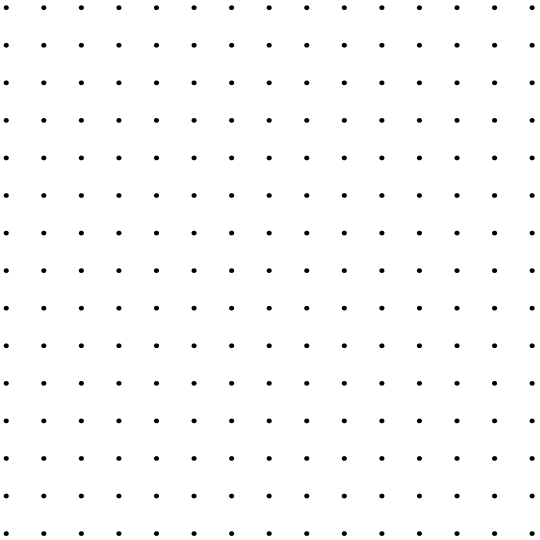
Let $M \subset \mathbb{R}^d$ be a module, $R \in O(d)$. Then

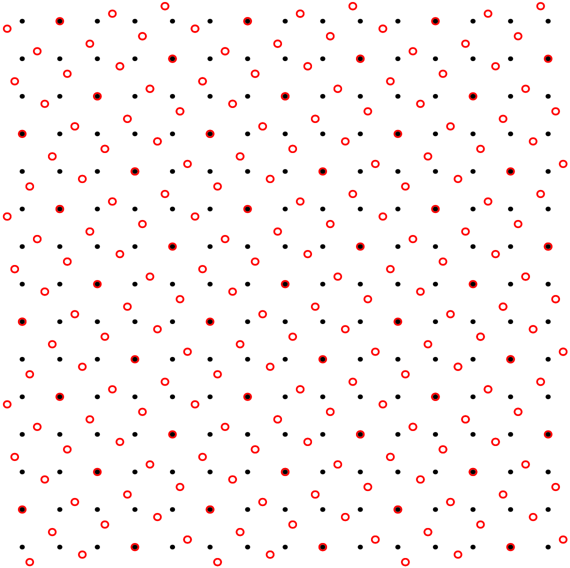
$$M(R) := M \cap RM$$

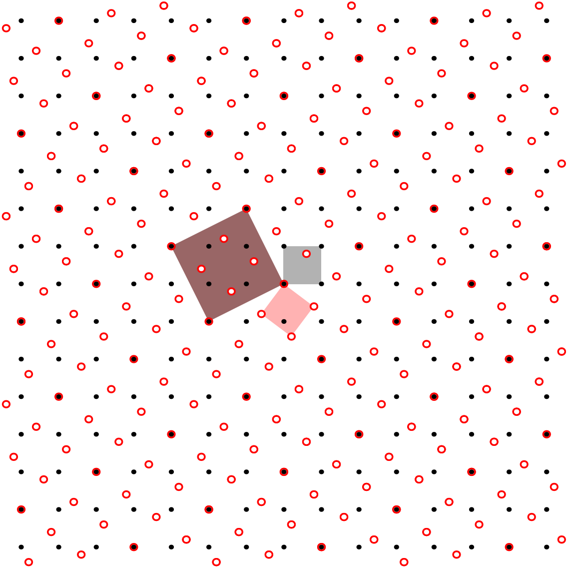
is called a (simple,ordinary) *coincidence site module* (CSM), if M and RM are commensurate. The index

$$\Sigma_M(R) := [M : M(R)] < \infty$$

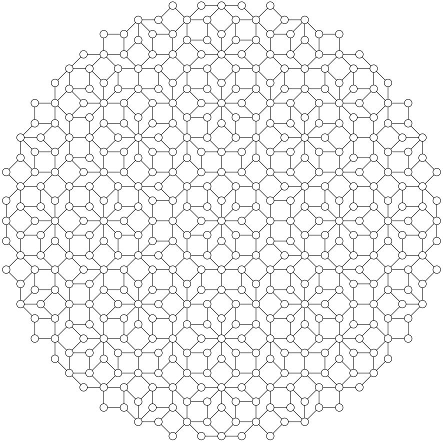
is called *coincidence index*.

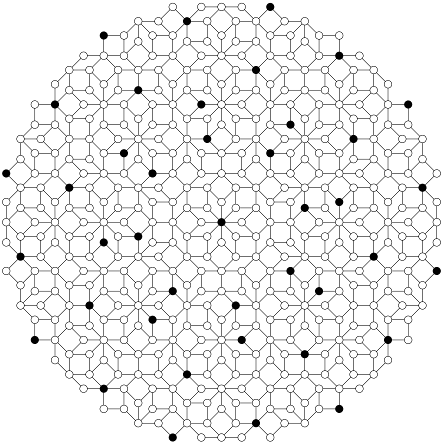






Example: Ammann-Beenker tiling





R the rotation about the center by $\theta = \tan^{-1}\left(-2\sqrt{2}\right) \approx 109.5^\circ$, $\Sigma(R) = 9$

Coincidence isometries

Lemma

The set of all coincidence isometries

$$OC(M) := \{R \in O(d) \big| \Sigma_M(R) < \infty\}$$

forms a group, a subgroup of $O(d)$.

Ordinary CSLs

If $M = \Gamma$ then

$$\Sigma_\Gamma(R) = \frac{\mathrm{vol}(\Gamma(R))}{\mathrm{vol}(\Gamma)} = \frac{\mathrm{dens}(\Gamma)}{\mathrm{dens}(\Gamma(R))}$$

$$OC(\Gamma) = OC(\Gamma^*)$$

$$\Sigma_\Gamma(R) = \Sigma_{\Gamma^*}(R)$$

Symmetry Operations

Lemma

The following are equivalent:

1. $R \in P(M)$
2. $\Sigma_M(R) = 1$

Corollary

$$P(M) = \{R \in OC(M) \,|\, \Sigma_M(R) = 1\} \subseteq OC(M)$$

Properties of the Coincidence Index

Assume

- $M = \Gamma$
- M satisfies $[M : M(R)] = [RM : M(R)]$ for all R

Lemma

For any coincidence isometry R

$$\Sigma_M(R) = \Sigma_M(R^{-1}).$$

Coincidences of Sublattices

Lemma

Let $M_1 \subseteq M$ with index $m := [M : M_1]$. Then

$$OC(M_1) = OC(M).$$

Let $\Sigma_1(R)$ be the coincidence index with respect to M_1. Then

$$\Sigma(R) \,\big|\, m\Sigma_1(R)$$
$$\Sigma_1(R) \,\big|\, m\Sigma(R).$$

Coincidence rotations of $\mathbb{Z}[i]$

coincidence rotations

$$e^{i\varphi} = \varepsilon \frac{z}{\bar{z}} = \varepsilon \prod_{p \equiv 1\,(4)} \left(\frac{\omega_p}{\bar{\omega}_p} \right)^{n_p}$$

ε unit, only finitely many $n_p \neq 0$

coincidence index

$$\Sigma(e^{i\varphi}) = \prod_{p \equiv 1\,(4)} p^{|n_p|}$$

spectrum

set of all integers that contain only prime factors $p \equiv 1 \pmod 4$.

CSLs of $\mathbb{Z}[i]$

$$\omega(\varphi) := \prod_{\substack{p\equiv 1\,(4)\\ n_p>0}} \omega_p^{n_p} \prod_{\substack{p\equiv 1\,(4)\\ n_p<0}} \bar{\omega}_p^{n_p}$$

CSLs

$$\mathbb{Z}[i] \cap e^{i\varphi}\mathbb{Z}[i] = \omega(\varphi)\mathbb{Z}[i]$$

Example – Square lattice

number of CSLs

$$\Phi(s) = \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \prod_{p \equiv 1(4)} \frac{1+p^{-s}}{1-p^{-s}}$$
$$= 1 + \frac{2}{5^s} + \frac{2}{13^s} + \frac{2}{17^s} + \frac{2}{25^s} + \frac{2}{29^s} + \frac{2}{37^s} + \frac{2}{41^s}$$
$$+ \frac{2}{53^s} + \frac{2}{61^s} + \frac{4}{65^s} + \frac{2}{73^s} + \dots$$

Known CSLs

- Square lattice, hexagonal lattice
- certain planar modules with N–fold symmetry
- cubic lattices and related modules
- hypercubic lattices
- A_4–lattice, ring of icosians

Affine Coincidences and Shifted Lattices

Affine Coincidences of Modules

Definition

Let $M \subset \mathbb{R}^d$ be a module, $R \in O(d)$, $v \in \mathbb{R}^d$. Then

$$M(v,R) := M \cap (v,R)M$$

is called an *affine coincidence site module* (CSM),

if $M(v,R)$ is an (affine) submodule of full rank.

(v,R) is called an affine coincidence isometry.

Affine Coincidences of Modules

Theorem

$$AC(M) = \{(v, R) : R \in OC(M) \text{ and } v \in M + RM\}$$

Remark

$AC(M)$ is not a group in general.

Affine Coincidences of Lattices

Grimmer 1974

$$AC(\Gamma) = \{(v, R) : R \in OC(\Gamma) \text{ and } v \in \Gamma + R\Gamma\}$$

$\Gamma + R\Gamma$ ... DSC lattice

Coincidences of shifted lattices

Linear coincidences of shifted lattices:

$$(x + \Gamma) \cap R(x + \Gamma)$$

Theorem

$$OC(x + \Gamma) = \{R \in OC(\Gamma) : Rx - x \in \Gamma + R\Gamma\}$$

- In general, $OC(x + \Gamma)$ is not a group.
- Problem: Product of coincidence isometries need not be a coincidence isometry

Coincidences of shifted lattices

Linear coincidences of shifted lattices:

$$(x+\Gamma)\cap R(x+\Gamma)$$

Theorem

$$OC(x+\Gamma) = \{R \in OC(\Gamma) : Rx - x \in \Gamma + R\Gamma\}$$

- In general, $OC(x+\Gamma)$ is not a group.
- Problem: Product of coincidence isometries need not be a coincidence isometry

Groupoid

Definition

$(G, ^{-1}, *)$, with $^{-1} : G \to G$ and $* : G \times G \to G$ a partial function, is called a *groupoid*, if

- $(a * b) * c = a * (b * c)$ if $a * b$ and $b * c$ are defined.
- $a^{-1} * a$ and $a * a^{-1}$ are defined.
- $a * b * b^{-1} = a$, $a^{-1} * a * b = b$, if $a * b$ is defined.

Theorem

$OC(x+\Gamma)$ *is a groupoid* $\Longleftrightarrow$ $OC(x+\Gamma)$ *is a group*

Coincidence isometries of $x + \mathbb{Z}[i]$

Theorem

Let $\Gamma = \mathbb{Z}[i]$ and $x \in \mathbb{C}$.

1. *$SOC(x + \Gamma)$ is a subgroup of $SOC(\Gamma)$*
2. *$OC(x + \Gamma)$ is a subgroup of $OC(\Gamma)$ if and only if for any T_1, $T_2 \in OC(x + \Gamma) \setminus SOC(x + \Gamma)$, $T_1 T_2 \in SOC(x + \Gamma)$*

Coincidence isometries of $x + \mathbb{Z}[i]$

- $x = \dfrac{r}{q}$ where $r, q \in \mathbb{Z}[i]$, r and q relatively prime

Lemma

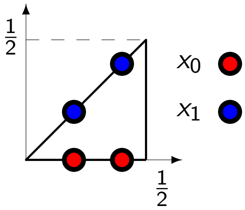
If q has no prime factor ω_p, then $OC(x + \Gamma)$ is a group.

Coincidence rotations of $x + \mathbb{Z}[i]$

Lemma

- $SOC(x+\Gamma) = SOC\left(\frac{1}{q} + \Gamma\right)$
- $SOC\left(\frac{1}{q_2} + \Gamma\right) \subseteq SOC\left(\frac{1}{q_1} + \Gamma\right)$ *if* $q_1 \mid q_2$
- $SOC\left(\frac{1}{q_1 q_2} + \Gamma\right) = SOC\left(\frac{1}{q_1} + \Gamma\right) \cap SOC\left(\frac{1}{q_2} + \Gamma\right)$ *if* q_1 *and* q_2 *are relatively prime*
- $SOC\left(\frac{1}{q} + \Gamma\right) = SOC\left(\frac{1}{\bar{q}} + \Gamma\right) = SOC\left(\frac{1}{\mathrm{lcm}(q,\bar{q})} + \Gamma\right)$

Example:

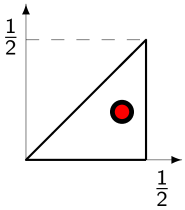


- $x_0 = \frac{1}{5}, \frac{2}{5}$ and $x_1 = \frac{1}{5} + \frac{1}{5}i, \frac{2}{5} + \frac{2}{5}i \Rightarrow q = 5$
- $SOC(x_0 + \Gamma) = SOC(x_1 + \Gamma) = SOC\left(\frac{1}{5} + \Gamma\right)$
- $OC(x_0 + \Gamma)$ and $OC(x_1 + \Gamma)$ are groups

Example:

$$
\begin{aligned}
\Phi_x(s) &= \frac{1-5^{-s}}{1+5^{-s}}\Phi(s) \\
&= 1 + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s} + \tfrac{2}{41^s} + \tfrac{2}{53^s} + \tfrac{2}{61^s} + \tfrac{2}{73^s} + \dots \\
\Phi(s) &= 1 + \tfrac{2}{5^s} + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{2}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s} + \tfrac{2}{41^s} + \tfrac{2}{53^s} \\
&\quad + \tfrac{2}{61^s} + \tfrac{4}{65^s} + \tfrac{2}{73^s} + \dots \\
&= \prod_{p\equiv 1(4)} \frac{1+p^{-s}}{1-p^{-s}}
\end{aligned}
$$

Example:



- $x = \frac{2}{5} + \frac{1}{5}i = \frac{i}{1+2i} \Rightarrow q = 1 + 2i$
- $SOC(x + \Gamma) = SOC\left(\frac{1}{1+2i} + \Gamma\right) = SOC\left(\frac{1}{5} + \Gamma\right)$
- $OC(x + \Gamma)$ is **NOT** a group!

Example:

$$
\begin{aligned}
\Phi_x(s) &= \frac{1}{1+5^{-s}}\Phi(s) \\
&= 1 + \tfrac{1}{5^s} + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{1}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s} + \tfrac{2}{41^s} + \tfrac{2}{53^s} \\
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&\quad + \tfrac{2}{61^s} + \tfrac{4}{65^s} + \tfrac{2}{73^s} + \dots \\
\Psi_x(s) &= 1 + \tfrac{4}{5^s} + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{4}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s} + \tfrac{2}{41^s} + \tfrac{2}{53^s} \\
&\quad + \tfrac{2}{61^s} + \tfrac{8}{65^s} + \tfrac{2}{73^s} + \dots \\
&= \frac{1+3\cdot 5^{-s}}{1+5^{-s}}\Phi(s)
\end{aligned}
$$

Coincidence rotations of $x + \mathbb{Z}[\xi_n]$

$M_n = \mathbb{Z}[\xi_n]$ with class number 1

Lemma

- $SOC\left(\frac{r}{q} + M_n\right) = SOC\left(\frac{1}{q} + M_n\right)$
- $SOC\left(\frac{1}{q_2} + M_n\right) \subseteq SOC\left(\frac{1}{q_1} + M_n\right)$ *if* $q_1 \mid q_2$
- $SOC\left(\frac{1}{q_1 q_2} + M_n\right) = SOC\left(\frac{1}{q_1} + M_n\right) \cap SOC\left(\frac{1}{q_2} + M_n\right)$ *if* q_1 *and* q_2 *are relatively prime*
- $SOC\left(\frac{1}{q} + M_n\right) = SOC\left(\frac{1}{\bar{q}} + M_n\right) = SOC\left(\frac{1}{\mathrm{lcm}(q,\bar{q})} + M_n\right)$

Coincidences of Multilattices

Multilattices

Multilattice

$L = \bigcup_{i=0}^{n-1} (x_i + \Gamma)$ with $x_0 = 0$

Coincidences

$L(R) := L \cap RL$

R coincidence isometry $\iff \Sigma_L(R) := \frac{\mathrm{dens}(L)}{\mathrm{dens}(L(R))}$ is finite

$OC(L) := \{R \in O(d) \big| \Sigma_L(R) < \infty\}$

Lemma

- $(x_i + \Gamma) \cap R(x_j + \Gamma)$ *is an affine sublattice of* $x_i + \Gamma$ *(or* $R(x_j + \Gamma)$*) if and only if* $R \in OC(\Gamma)$ *and* $Rx_j - x_i \in \Gamma + R\Gamma$
- $(x_i + \Gamma) \cap R(x_j + \Gamma) = x_i + t_{ij} + \Gamma(R)$ *with* $t_{ij} \in \Gamma$

Coincidences of Multilattices

Theorem

- $OC(L) = OC(\Gamma)$
- *Let* $K := \{(i,j) : Rx_j - x_i \in \Gamma + R\Gamma\}$. *Then:*

  $\Sigma_L(R) = \frac{n}{|K|}\Sigma_\Gamma(R)$

  $L(R) = \bigcup_{(i,j)\in K} (x_i + t_{ij} + \Gamma(R))$

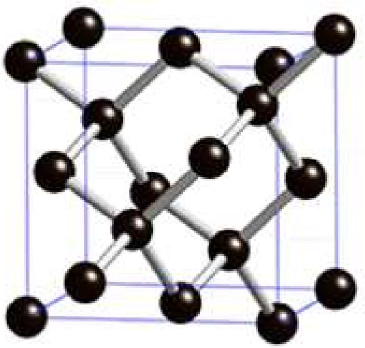
Example $n = 2$

$L = \Gamma \cup (x + \Gamma)$

1. $\Sigma_L(R) = 2\Sigma_\Gamma(R) \iff x, Rx, Rx - x \notin \Gamma + R\Gamma$
2. $\Sigma_L(R) = \frac{1}{2}\Sigma_\Gamma(R) \iff x, Rx, Rx - x \in \Gamma + R\Gamma$
3. $\Sigma_L(R) = \Sigma_\Gamma(R) \iff$ exactly one of $x, Rx, Rx - x$ in $\Gamma + R\Gamma$

Example: diamond packing

$$L = \Gamma_{fcc} \cup \left(\tfrac{1}{4}(1,1,1) + \Gamma_{fcc}\right)$$



Example: diamond packing

$$\Sigma_L(R) = \Sigma_{fcc}(R), 2\Sigma_{fcc}(R)$$

$$\Phi_L(s) = (1+2^{-s})\Phi_{fcc}(s) = (1-2^{1-s})\prod_p \frac{1+p^{-s}}{1-p^{1-s}}$$

$$= 1 + \tfrac{1}{2^s} + \tfrac{4}{3^s} + \tfrac{6}{5^s} + \tfrac{4}{6^s} + \tfrac{8}{7^s} + \tfrac{12}{9^s} + \tfrac{6}{10^s} + \tfrac{12}{11^s} + \tfrac{14}{13^s} + \dots$$

$$\Phi_{fcc}(s) = \frac{1-2^{1-s}}{1+2^{-s}} \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)} = \prod_{p\neq 2} \frac{1+p^{-s}}{1-p^{1-s}}$$

$$= 1 + \tfrac{4}{3^s} + \tfrac{6}{5^s} + \tfrac{8}{7^s} + \tfrac{12}{9^s} + \tfrac{12}{11^s} + \tfrac{14}{13^s} + \tfrac{24}{15^s} + \tfrac{18}{17^s} + \dots$$

Example: lattice - sublattice relations

$$\Gamma_2 \subseteq \Gamma_1$$

$$\Gamma_1 = \bigcup_{i=0}^{m-1} (x_i + \Gamma_2)$$

$$D = \{j : Rx_j \in \Gamma_2 + R\Gamma_2\} \qquad I = \{i : \exists x_j : Rx_j - x_i \in \Gamma_2 + R\Gamma_2\}$$

$$E = \{i : x_i \in \Gamma_2 + R\Gamma_2\} \qquad J = \{j : \exists x_i : Rx_j - x_i \in \Gamma_2 + R\Gamma_2\}$$

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Example: lattice - sublattice relations

Theorem

$$\Sigma_2(R) = \frac{|D||I|}{m}\Sigma_1(R) = \frac{|E||J|}{m}\Sigma_1(R)$$

Lemma

$u := |D| = [\Gamma_2 \cap \Gamma_1(R) : \Gamma_2(R)]$ $\quad t := |I| = [\Gamma_1(R) : \Gamma_2 \cap \Gamma_1(R)]$

$v := |E| = [R\Gamma_2 \cap \Gamma_1(R) : \Gamma_2(R)]$ $\quad s := |J| = [\Gamma_1(R) : R\Gamma_2 \cap \Gamma_1(R)]$

$u \mid s,\ v \mid t,\ s \mid m,\ t \mid m$

Example: lattice - sublattice relations

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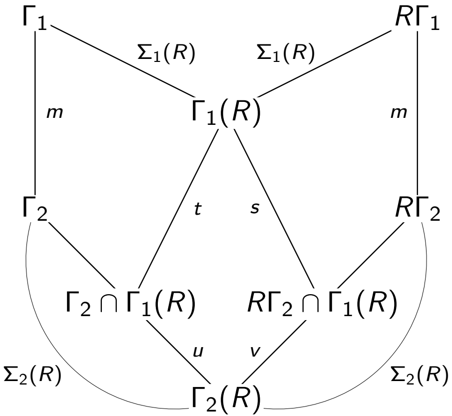
$u \mid s,\ v \mid t,\ s \mid m,\ t \mid m$

sublattice diagram



Example: rectangular lattice $\mathbb{Z} \times 4\mathbb{Z}$

$$\Gamma_1 = \frac{1}{4}\mathbb{Z} \times \mathbb{Z}$$

$$\Gamma_2 = \mathbb{Z}^2$$

$$\Phi_{\mathbb{Z}\times 4\mathbb{Z}}(s) = (1 + 4^{-s})\Phi_{\mathbb{Z}^2} =$$

$$= 1 + \tfrac{1}{4^s} + \tfrac{2}{5^s} + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{2}{20^s} + \tfrac{2}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s}$$

$$+ \tfrac{2}{41^s} + \tfrac{2}{52^s} + \tfrac{2}{53^s} + \tfrac{2}{61^s} + \tfrac{4}{65^s} + \tfrac{2}{68^s} + \tfrac{2}{73^s} + \dots$$

$$\Phi_{\mathbb{Z}^2}(s) = 1 + \tfrac{2}{5^s} + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{2}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s} + \tfrac{2}{41^s} + \tfrac{2}{53^s}$$

$$+ \tfrac{2}{61^s} + \tfrac{4}{65^s} + \tfrac{2}{73^s} + \dots$$

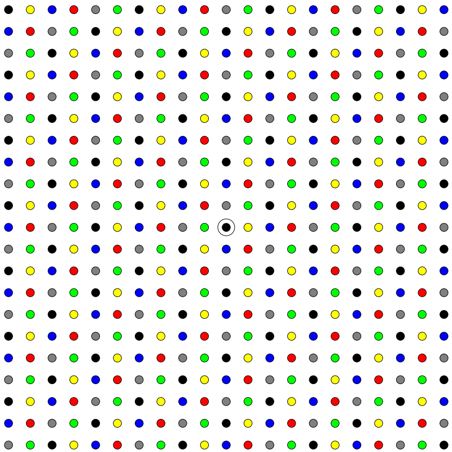
Example: rectangular lattice $\mathbb{Z} \times 4\mathbb{Z}$

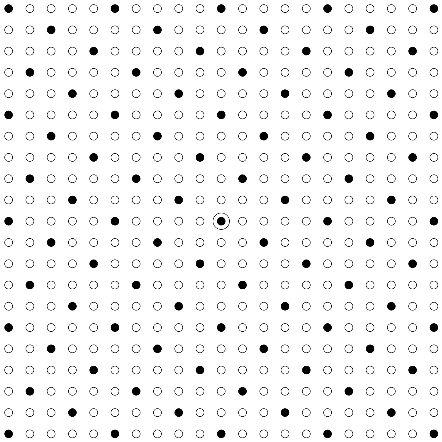
$$\Gamma_1 = \frac{1}{4}\mathbb{Z} \times \mathbb{Z}$$

$$\Gamma_2 = \mathbb{Z}^2$$

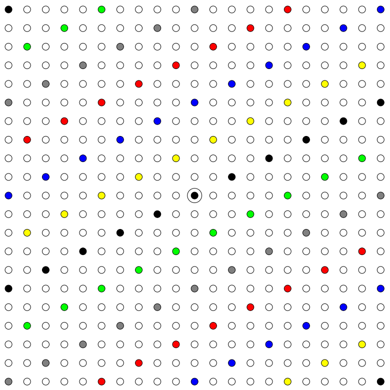
$$\begin{aligned}\Phi_{\mathbb{Z}\times 4\mathbb{Z}}(s) &= (1+4^{-s})\Phi_{\mathbb{Z}^2} = \\ &= 1 + \tfrac{1}{4^s} + \tfrac{2}{5^s} + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{2}{20^s} + \tfrac{2}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s} \\ &\quad + \tfrac{2}{41^s} + \tfrac{2}{52^s} + \tfrac{2}{53^s} + \tfrac{2}{61^s} + \tfrac{4}{65^s} + \tfrac{2}{68^s} + \tfrac{2}{73^s} + \ldots \\ \Phi_{\mathbb{Z}^2}(s) &= 1 + \tfrac{2}{5^s} + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{2}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s} + \tfrac{2}{41^s} + \tfrac{2}{53^s} \\ &\quad + \tfrac{2}{61^s} + \tfrac{4}{65^s} + \tfrac{2}{73^s} + \ldots\end{aligned}$$

Example Coulorings

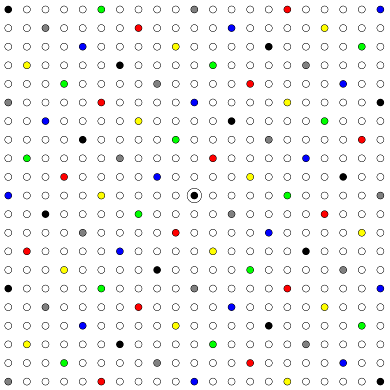




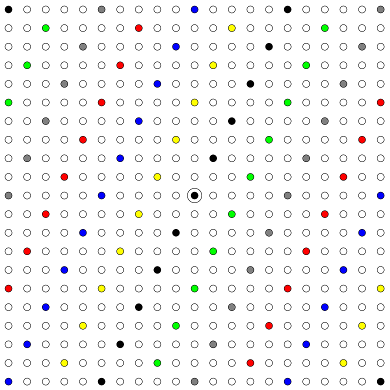
rotation about the origin (counterclockwise) by $\theta = \arctan\left(\frac{3}{4}\right)$



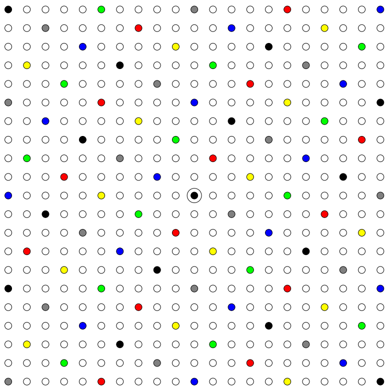
colouring of $\Gamma_1(R^{-1})$



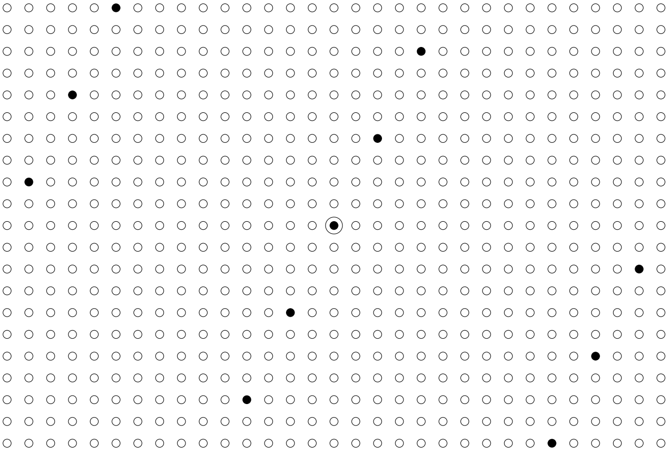
colouring of $\Gamma_1(R)$



colouring of $\Gamma_1(R^{-1})$ rotated by R



colouring of $\Gamma_1(R)$



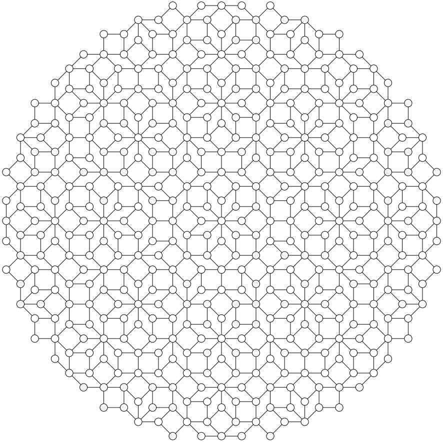
$\Gamma_2(R)$

In our example:

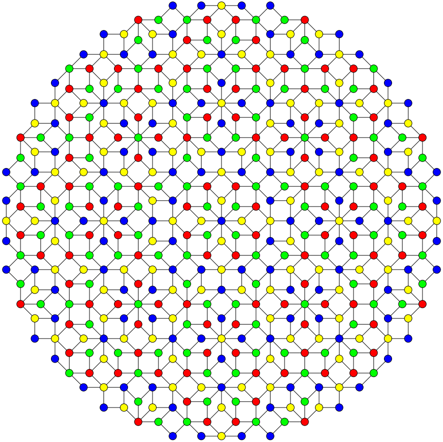
$\Sigma_1(R) = 5$, $m = t = s = 6$, and $u = v = 2$

$\Rightarrow \Sigma_2(R) = 10$.

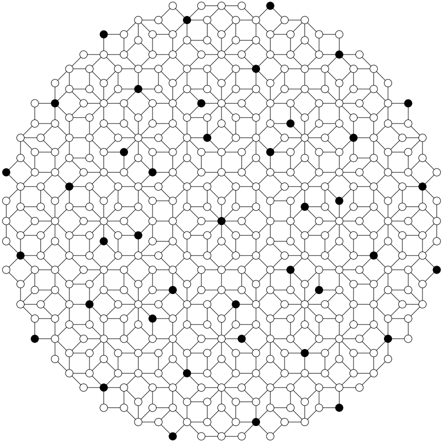
Example: Ammann-Beenker tiling



Example: Ammann-Beenker tiling

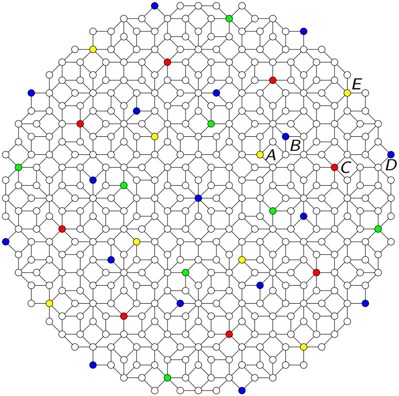


Example: Ammann-Beenker tiling

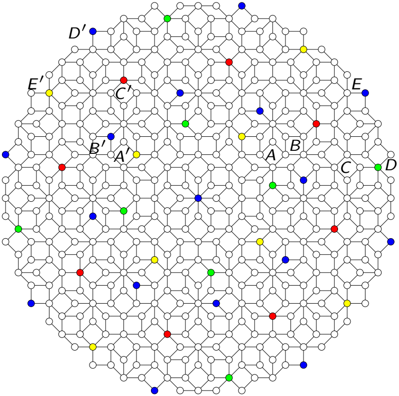


R the rotation about the center by $\theta = \tan^{-1}\left(-2\sqrt{2}\right) \approx 109.5^\circ$, $\Sigma(R) = 9$

Example: Ammann-Beenker tiling

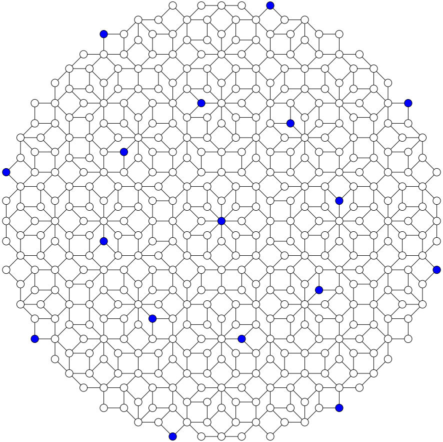


$T_2 \cap T_1(R^{-1})$



$T_2 \cap T_1(R)$

Example: Ammann-Beenker tiling



$T_2(R)$

Thank you!