Fluid/gravity correspondence

Veronika Hubeny

Durham University

Recent Developments in String/M Theory
KIAS, September 23, 2008

S Bhattacharyya, VH, S Minwalla, M Rangamani; arXiv:0712.2456

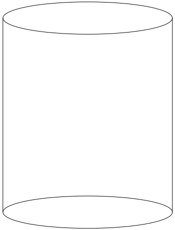
S Bhattacharyya, VH, R Loganayagam, G Mandal, S Minwalla, T Morita, M Rangamani, H Reall; arXiv:0803.2526

Motivation

Key question of quantum gravity:

What is the fundamental nature of spacetime?

Invaluable tool in recent years: AdS/CFT correspondence



string theory in $AdS \times S$

$\leftrightarrow$ gauge theory on boundary

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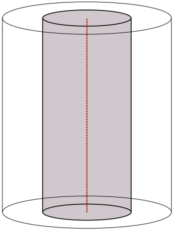

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Schwarzschild-AdS black hole

$\leftrightarrow$ (approximately) thermal state

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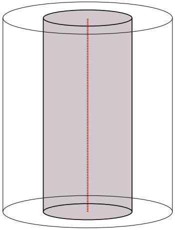

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$\leftrightarrow$ (approximately) thermal state

Need to probe AdS/CFT dictionary further:

- Which CFT configurations have a spacetime description?
- What types of spacetime singularities are allowed?
- Probe spacetime dynamics.

Motivation

Current interesting questions in QCD:

- Explore universal properties of non-abelian plasmas
- Understand quark gluon plasma (RHIC data)

Any strongly interacting field theory admits an effective description in terms of fluid dynamics.

Important questions in fluid dynamics:

- Physics away from thermodynamic equilibrium
- Global regularity of Navier-Stokes equation
- Turbulence
- Causality issues (Israel-Stewart)

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Outline

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References:

- Significant work over last 6 years in this subject pioneered by

  Policastro, Son, Starinets

  Herzog, Kovtun, Buchel

  many, many others . . .

- Inspiration for our work

  Janik, Peschanski

  Bhattacharyya, Lahiri, Loganayagam, Minwalla

- Related very interesting recent developments

  Baier, Romatschke, Son, Starinets, Stephanov; Loganayagam

  Van Raamsdonk; Natsuume, Okamura; Haack, Yarom

  Dutta; Buchel et.al.

  Bhattacharyya, Loganayagam, Minwalla, Nampuri, Trivedi, Wadia

Fluid dynamics

- Fluid dynamics is continuum effective description of any microscopic QFT valid when scales of variation are long compared to mean free path ℓ_{mfp}.
- The fluid description assumes that the system achieves local thermodynamic equilibrium.

Regime of validity: "long-wavelength approximation"

For local temperature of the fluid T
and scale of variation of the dynamical degrees of freedom L,
local equilibrium demands:

$$L\,T \equiv \frac{1}{\epsilon} \gg 1$$

Fluid dynamics

Dynamical degress of freedom:

- Local temperature T
- Fluid velocity u_μ (normalized $\eta^{\mu\nu}\, u_\mu\, u_\nu = -1$)
- Particle and charge densities ρ and q_i
- Pressure P and chemical potentials determined by equation of state
  - For conformal fluids $P = \frac{1}{d-1}\,\rho$
  - For convenience we will set all charges to zero
  - ρ can be expressed in terms of T

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$\Rightarrow$ This leaves d functions, $T(x^\mu)$ and $u_\nu(x^\mu)$, which specify our fluid configuration.

(x^μ are coordinates on the boundary spacetime on which the fluid lives.)

Conformal Fluid dynamics

The conformal fluid stress tensor $T^{\mu\nu}$

Encode all the fluid information by stress tensor $T^{\mu\nu}$, which is

- Traceless: $T^{\mu}_{\mu} = 0$
- Conserved: $\nabla_\mu T^{\mu\nu} = 0$

The conservation equation encapsulates the dynamical content of fluid dynamics.

Form of stress tensor is determined by symmetries,
order by order in derivative expansion;
fluid properties specified by finite # of undetermined coefficients.

In d dimensions:

$$T^{\mu\nu} = \alpha\, T^d\, (\eta^{\mu\nu} + d\, u^\mu\, u^\nu) + \pi^{\mu\nu}_{dissipative}$$

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Gravity in the bulk

- Consider any 2-derivative theory of 5-d gravity interacting with other fields with AdS_5 as a solution (e.g. IIB SUGRA on $\text{AdS}_5 \times S^5$)
- Solution space has a universal sub-sector: pure gravity with negative cosmological constant ($R_{AdS} = 1 \;\Rightarrow\; \Lambda = -6$):

$$E_{MN} \equiv R_{MN} - \frac{1}{2} R \, g_{MN} + \Lambda \, g_{MN} = 0$$

- We will focus on this sub-sector in long-wavelength limit.
- Our solutions "tubewise" approximate black branes in AdS.

A ‘Chiral Lagrangian’ for black branes

- Isometry group of AdS_5 is $SO(4,2)$.
- Distinguished subalgebra: Poincare + dilatations
  - $SO(3)$ rotations and translations
    leave black brane invariant
  - dilatation + boosts
    generate a 4-parameter family of black brane solutions
    (specified by temperature T and velocity u_ν of the brane)
- Our construction promotes these to ‘Goldstone fields’ (collective coordinate fields) $T(x^\mu)$, $u_\nu(x^\mu)$
- We determine the effective dynamics for these fields order by order in boundary derivative expansion.

Validity of semi-classical gravity and DOF truncation

Gravity dual to field theory

- The boundary stress tensor is related to the normalizable modes of the gravitational field in AdS.

$$ds^2 = \frac{dz^2 + \left(\eta_{\mu\nu} + \alpha\, z^d\, T_{\mu\nu}\right)\, dw^\mu\, dw^\nu}{z^2}$$

- Conversely, to a given a boundary stress tensor $T^{\mu\nu}$ there corresponds an asymptotically AdS solution.

Degrees of freedom counting

- A boundary conformally invariant stress tensor has $\frac{d(d+1)}{2} - 1$ degrees of freedom.
- ?: Can any such stress tensor give a regular bulk geometry?

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- **?**: Can any such stress tensor give a regular bulk geometry?

Regularity and dof truncation

- Claim: Regular solutions are given by stress tensors which are fluid dynamical.
- For pure gravity + cosmological constant, this is a reduction of degress of freedom, since fluid stress tensors have d dof (i.e. T and u_μ), rather than $\frac{d(d+1)}{2} - 1$.
- Uniqueness: In fact, the gravity solutions thus constructed are the most general *regular* long-wavelength solutions to Einstein's equations (gravity & -ve cc).
- i.e. the solutions admit a regular event horizon which shields a curvature singularity.

Overview

Dynamical picture:

- Start with generic high energy initial conditions
- System quickly settles down to local thermodynamic equilibrium $\Leftrightarrow$ non-uniform black brane in AdS
  (described by local velocity and temperature fields)
- Subsequent evolution: hydrodynamics $\Leftrightarrow$ Einstein's equations
- Late time behaviour: relaxation to global equilibrium state $\Leftrightarrow$ uniform black brane in AdS

Technical aspects:

- Long-wavelength regime of fluid dynamics: use perturbative expansion in boundary derivatives (exact in radial coordinate).
- We construct the stress-energy tensor $T^{\mu\nu}$ and corresponding metric $g_{\mu\nu}$ to second order in boundary derivative expansion.
- This yields a map between fluid dynamics and gravity.

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1. Motivation
2. Background
   - fluid dynamics
   - gravity
   - fluid/gravity map and DOF truncation
   - overview
3. Iterative construction of bulk $g_{\mu\nu}$ and boundary $T^{\mu\nu}$
   - 0th order
   - 1st order
4. Analysis of 2nd order solution
   - boundary stress tensor and transport coefficients
   - bulk geometry and event horizon
5. Summary & Remarks

0^{th} order: boosted Schwarzschild-AdS black hole

- Start with the well-known stationary solution: (derivation)
  boosted Schwarzschild-AdS$_5$ black hole (w/ planar symmetry)

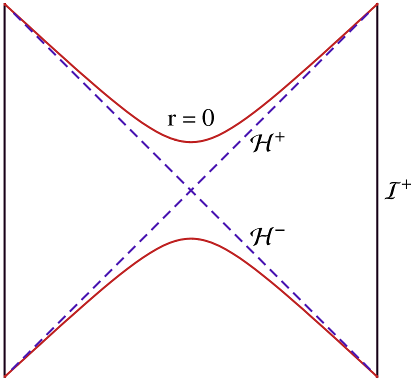
  $$ds^2 = -2\, u_\mu\, dx^\mu dr + r^2\, \left(\eta_{\mu\nu} + [1 - f(r/\pi T)]\, u_\mu\, u_\nu\right)\, dx^\mu dx^\nu \ ,$$

  with $f(r) \equiv 1 - \frac{1}{r^4}$
- It is parameterized by 4 parameters:
  temperature T and boosts u_i.
- The bulk black hole is dual to a bdy perfect fluid with

  $$T^{\mu\nu} = \pi^4\, T^4\, \left(\eta^{\mu\nu} + 4\, u^\mu\, u^\nu\right)$$

0^{th} order: boosted Schwarzschild-AdS black hole

Causal structure of this solution is:



- spacelike singularity
- regular event horizon
- timelike boundary (asymptotically AdS)

Deforming the 0^{th} order solution

- Now promote u_μ and T to fields depending on the boundary coordinates. Call such a metric $g^{(0)}$.
- Note: $g^{(0)}$ does NOT satisfy the equations of motion:

$$E_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R - 6\, g_{MN} = 0$$

- But starting from here we will construct an iterative solution.

A perturbation scheme for gravity

Assume that the variation in local temperature and velocities are slow

$$\frac{\partial_\mu \log T}{T} \sim \mathcal{O}(\epsilon) \ , \qquad \frac{\partial_\mu u}{T} \sim \mathcal{O}(\epsilon)$$

$\Rightarrow$ In local patches the solution is like a boosted black brane.

Basic idea

The perturbative scheme is aimed at constructing a regular bulk solution, by patching together pieces of the uniform boosted brane.

Use ϵ as a book-keeping parameter (counting # of x^μ derivatives), and expand:

$$g = \sum_{k=0}^{\infty} \epsilon^k \, g^{(k)} \ , \qquad T = \sum_{k=0}^{\infty} \epsilon^k \, T^{(k)} \ , \qquad u = \sum_{k=0}^{\infty} \epsilon^k \, u^{(k)}$$

A perturbation scheme for gravity

At a given order in the ϵ-expansion we find equations for $g^{(k)}$. These are ultra-local in the field theory directions and take the schematic form:

$$\mathbb{H}\left[g^{(0)}(u_\mu^{(0)}, T^{(0)})\right] g^{(k)}(x^\mu) = s_k$$

- $\mathbb{H}$ is a second order linear differential operator in r alone.
- s_k are regular source terms which are built out of $g^{(n)}$ with $n \leq k - 1$.

A perturbation scheme for gravity

Importantly the equations of motion split up into two kinds:

- Constraint equations: $E_{r\mu} = 0$, which implement stress-tensor conservation (at one lower order).
- Dynamical equations: $E_{\mu\nu} = 0$ and $E_{rr} = 0$ allow determination of $g^{(k)}$.

We solve the dynamical equations

$$g^{(k)} = \text{particular}(s_k) + \text{homogeneous}(\mathbb{H})$$

subject to

- regularity in the interior
- asymptotically AdS boundary conditions

(coordinate choice for g)

First order computation

Explicit solution to first order

Bulk metric:

$$ds^2 = -2\,u_\mu\,dx^\mu dr + r^2\,\left(\eta_{\mu\nu} + [1 - f(r/\pi T)]\,u_\mu\,u_\nu\right)\,dx^\mu dx^\nu$$
$$+\,2r\left[\frac{r}{\pi T}\,F(r/\pi T)\,\sigma_{\mu\nu} + \frac{1}{3}\,u_\mu u_\nu\,\partial_\lambda u^\lambda - \frac{1}{2}\,u^\lambda \partial_\lambda\left(u_\nu u_\mu\right)\right]\,dx^\mu dx^\nu,$$

with

$$F(r) = \int_r^\infty dx\,\frac{x^2+x+1}{x(x+1)\,(x^2+1)} = \frac{1}{4}\left[\ln\left(\frac{(1+r)^2(1+r^2)}{r^4}\right) - 2\,\arctan(r) + \pi\right]$$

Boundary stress tensor:

$$T^{\mu\nu} = \pi^4\,T^4\,(4\,u^\mu u^\nu + \eta^{\mu\nu}) - 2\,\pi^3\,T^3\,\sigma^{\mu\nu}.$$

with $\sigma^{\mu\nu}$ = transverse traceless symmetric part of $\partial^\mu u^\nu$

Viscosity/entropy ratio

Boundary stress tensor:

$$T^{\mu\nu} = \pi^4 \, T^4 \, (4 \, u^\mu u^\nu + \eta^{\mu\nu}) - 2 \, \pi^3 \, T^3 \, \sigma^{\mu\nu} \, .$$

Note: shear of the fluid is defined by

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \, \partial_{(\alpha} u_{\beta)} - \frac{1}{3} P^{\mu\nu} \, \partial_\alpha u^\alpha$$

where $P^{\mu\nu} = \eta^{\mu\nu} + u^\mu \, u^\nu$ is a co-moving spatial projector.

The coeff of $\sigma^{\mu\nu}$ gives the viscosity; here

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

in agreement with well-known results.

The 4-dimensional conformal fluid from AdS$_5$

The stress tensor to second order

$$T^{\mu\nu} = (\pi T)^4 \left(\eta^{\mu\nu} + 4\, u^\mu u^\nu\right) - 2\, (\pi T)^3\, \sigma^{\mu\nu} +$$
$$+(\pi T)^2 \left((\ln 2)\ T^{\mu\nu}_{2a} + 2\, T^{\mu\nu}_{2b} + (2 - \ln 2) \left[\frac{1}{3}\, T^{\mu\nu}_{2c} + T^{\mu\nu}_{2d} + T^{\mu\nu}_{2e} \right] \right)$$

$$T^{\mu\nu}_{2a} = \epsilon^{\alpha\beta\gamma(\mu}\, \sigma^{\nu)}{}_{\gamma}\, u_\alpha\, \ell_\beta\ , \qquad T^{\mu\nu}_{2b} = \sigma^{\mu\alpha}\sigma^{\nu}{}_{\alpha} - \frac{1}{3}\, P^{\mu\nu}\, \sigma^{\alpha\beta}\sigma_{\alpha\beta}$$
$$T^{\mu\nu}_{2c} = \partial_\alpha u^\alpha\, \sigma^{\mu\nu}\ , \qquad T^{\mu\nu}_{2d} = \mathcal{D}u^\mu\, \mathcal{D}u^\nu - \frac{1}{3}\, P^{\mu\nu}\, \mathcal{D}u^\alpha\, \mathcal{D}u_\alpha$$
$$T^{\mu\nu}_{2e} = P^{\mu\alpha}\, P^{\nu\beta}\, \mathcal{D}\left(\partial_{(\alpha} u_{\beta)}\right) - \frac{1}{3}\, P^{\mu\nu}\, P^{\alpha\beta}\, \mathcal{D}\left(\partial_\alpha u_\beta\right)$$

with $\mathcal{D} = u^\mu\, \partial_\mu$ and $\ell_\mu = \epsilon_{\alpha\beta\gamma\mu}\, u^\alpha \partial^\beta u^\gamma$.

more compact definitions

The 4-dimensional conformal fluid from AdS$_5$

Fluid description of $\mathcal{N} = 4$ Super-Yang Mills: It is useful to write the second order stress tensor in a different basis of operators $\mathcal{T}_k^{\mu\nu}$:

$$T^{\mu\nu}_{(2)} = \tau_\pi \, \eta \, \mathcal{T}_1^{\mu\nu} + \kappa \, \mathcal{T}_2^{\mu\nu} + \lambda_1 \, \mathcal{T}_3^{\mu\nu} + \lambda_2 \, \mathcal{T}_4^{\mu\nu} + \lambda_3 \, \mathcal{T}_5^{\mu\nu}$$

which manifest the conformal properties.

Baier, Romatschke, Son, Starinets, Stephanov; Loganayagam

The fluid parameters (shear viscosity, relaxation timescales, ...) are

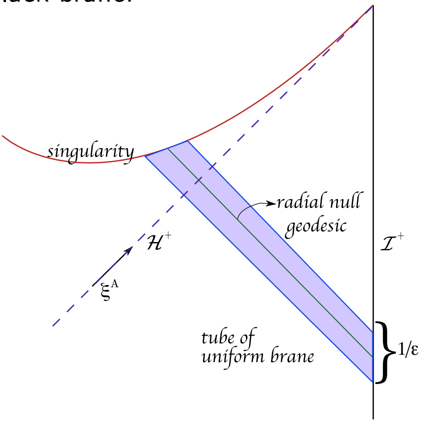
$$\eta = \frac{N^2}{8\pi} \, (\pi \, T)^3$$

$$\tau_\pi = \frac{2 - \ln 2}{\pi \, T} \, , \qquad \lambda_1 = \frac{2 \, \eta}{\pi \, T} \, , \qquad \lambda_2 = \frac{2 \, \eta \, \ln 2}{\pi \, T} \, , \qquad \lambda_3 = 0.$$

which agrees with the results of Baier, Romatschke, Son, Starinets, Stephanov.
They also derive the curvature coupling term: $\kappa = \frac{\eta}{\pi T}$

The spacetime geometry dual to fluids

The bulk solution thus constructed is tubewise approximated by a black brane!



Bulk causal structure; in each "tube" metric approximates uniformly boosted Schwarzschild-AdS black hole.

The event horizon

The background has a regular event horizon.

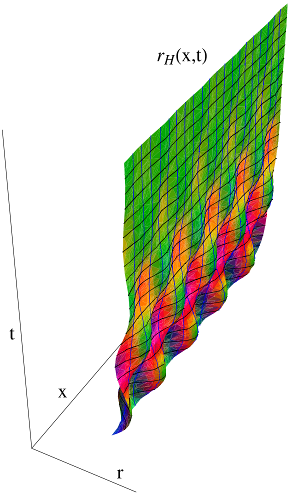
- One can determine the event horizon locally using the fact that the solution settles down at late times to an uniformly boosted black brane.
- The horizon location can be determined within the perturbation scheme

$$r = r_H(x) = \pi\, T(x) + \sum_{k=1}^{\infty} \epsilon^k\, r_{(k)}(x)$$

- In fact, $r_{(k)}(x)$ is determined algebraically by demanding that the surface given by $r = r_H(x)$ be null.

simpler analogy

Cartoon of the event horizon



Note:

- Horizon is null everywhere
- Late time approach to uniform black brane
- Horizon area increases

The Entropy current

- Given a bulk geometry with a horizon we can determine the Bekenstein-Hawking entropy.
- Bulk construction of entropy: using area-form A of spatial slices of the event horizon in Planck units.

Fluid entropy current

The area-form A on event horizon can be pulled back to the boundary to define a fluid entropy current J_S^μ

$$J_S = *_\eta A$$

with non-negative divergence

$$\partial_\mu J_S^\mu \geq 0$$

Entropy details

Loganayagam

Properties of Entropy current

- The bulk-boundary pull-back is facilitated by our coordinates: pull-back along radial ingoing geodesics (const r)

$$x^\mu(\mathcal{H}) \quad \rightarrow \quad x^\mu(\mathrm{bdy})$$

- Fluid entropy current consistent with second law and equations of motion has a 5 parameter ambiguity. details
- Bulk construction of entropy current is ambiguous, but less so: (i) ability to add total derivative terms without changing area (ii) pull-back is ambiguous to boundary diffeomorphisms. At second order this results in a two parameter ambiguity for Weyl covariant current with positive divergence.

Summary

- $\exists$ a map between conformal fluid configurations on $\mathbb{R}^{d-1,1}$ & regular asymp. AdS_{d+1} non-uniform black branes
  $\Rightarrow$ gain insight into *generic* behaviour of gravity
- Bulk spacetime solutions
  - naturally uphold Cosmic Censorship
  - imply a new variant of Uniqueness Theorem
- Long-wavelength regime of fluid dynamics allows this construction to any order in a boundary derivative expansion.
- This yields *local* determination of the event horizon.
- The solutions satisfy the Area increase theorem & corresponding entropy current satisfies the 2^{nd} law.
- Recovered the well-known value of viscosity: $\frac{\eta}{s} = \frac{1}{4\pi}$
- Predicted second order fluid parameters $(\tau_\pi, \lambda_1, \lambda_2, \lambda_3)$

Horizon physics described by fluid dynamics...

Where does the fluid live?

- On the event horizon?
  (null hypersurface, defined globally...)
- On the dynamical horizon? Gourgoulhon & Jaramillo
  (spacelike hypersurface)
- On the stretched horizon? (a la Membrane Paradigm)

Membrane Paradigm
Thorne, Macdonald, Price

Horizon interpreted as a fluid membrane with certain dissipative properties: (e.g. electrical conductivity, shear & bulk viscosity, etc.)

- On the spacetime boundary. (AdS/CFT)
  Fluid dynamics describes the full spacetime, not just horizon.

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Beyond the horizon...

Consequence of ultra-locality:

- The fluid configuration on the boundary determines the full radial form of the metric in the bulk.
- $\Rightarrow$ The fluid encodes the geometry past the horizon and
- it "knows" about the black hole singularity.
- (However, geometrically the near-singularity structure mimics that of the uniform black brane.)

Puzzles & future directions

- Role of non-long-wavelength bulk semiclassical solutions
- More detailed bulk analysis: horizon topology, nature of curvature singularity, Cosmic Censorship
- $\exists$ striking difference between turbulence in 3+1 and 2+1 nonrelativistic fluids (eg. inverse cascade) $\stackrel{?}{\Longrightarrow}$ qualitative difference in gravitational dynamics (eg. equilibration time of AdS_4 vs. AdS_5 BHs) Van Raamsdonk
- Relation to the black hole Membrane Paradigm
- Generalisations: forced fluids, fluids on curved background, charged fluids, extremal fluids (superfluids), non-conformal fluids, non-relativistic fluids, ...
- Gravity dual of turbulence

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Boosted Schwarzschild-AdS black hole

- Static Schwarzschild-AdS black hole in planar limit:

$$ds^2 = r^2 \left(-f(r)\, dt^2 + \sum_i (dx^i)^2 \right) + \frac{dr^2}{r^2\, f(r)}$$

with $f(r) = 1 - \frac{r_+^4}{r^4}$. ($\rightsquigarrow$ temperature $T = r_+/\pi$.)

- To avoid coordinate singularity at the horizon $r = r_+$, use ingoing coordinates: $v = t + r_*$ where $dr_* = \frac{dr}{r^2\, f(r)}$:

$$ds^2 = -r^2\, f(r)\, dv^2 + 2\, dv\, dr + r^2 \sum_i (dx^i)^2$$

- Now 'covariantize' by boosting: $v \to u_\mu\, x^\mu$, $x_i \to P_{i\mu}\, x^\mu$.

back

Choice of coordinates

For dealing with regularity issues etc., it is simplest to work in an analog of ingoing Eddington-Finkelstein coordinates.

$$ds^2 = -2\, u_\mu(x)\, \mathcal{S}(r,x)\, dr\, dx^\mu + \chi_{\mu\nu}(r,x)\, dx^\mu\, dx^\nu$$

- The choice of coordinates is such that $x^\mu = constant$ are ingoing null geodesics.
- It is well adapted to discuss features of horizon, such as entropy in the fluid language.

back

Computation at first order

Details of first order computation:

- To solve the equations to first order we need to ensure conservation of the perfect fluid stress tensor

$$\partial_\mu T^{\mu\nu}_{(0)} = 0$$

  which needs to be solved only locally (at say $x^\mu = 0$).
- This can be used to eliminate derivatives of $T^{(0)}$ in terms of those of $u_i^{(0)}$.

$$\partial_v (\pi\, T^{(0)})^{-1} = \frac{1}{3}\, \partial_i\, u_i^{(0)} \;, \qquad \partial_i\, (\pi\, T^{(0)})^{-1} = \partial_v\, u_i^{(0)}$$

- Then we solve $\mathbb{H} g^{(1)} = s_1$ where the operators and sources are given as follows:

Computation at first order

The operator $\mathbb{H}$: Useful to decompose metric perturbations into $SO(3)$ representations: scalars **1**, vectors **3** and symmetric traceless tensors **5**. For instance, we find:

$$\mathbb{H}_{\mathbf{3}} \# = \frac{d}{dr}\left(\frac{1}{r^3}\,\frac{d}{dr}\#\right)$$

$$\mathbb{H}_{\mathbf{5}} \# = \frac{d}{dr}\left(r^5\, f(r)\,\frac{d}{dr}\#\right)$$

The source terms: These differ at various orders in perturbation theory. At first order:

$$s_1^{\mathbf{3}} = -\frac{3}{r^2}\,\partial_v\, u_i^{(0)}$$

$$s_1^{\mathbf{5}} = -6\, r^2\, \sigma_{ij}^{(0)}$$

solution

Useful fluid velocity gradient quantities

Velocity field u^μ naturally decomposes spacetime $\to$ space + time, w/ induced metric on spatial slices $P^{\mu\nu} \equiv g^{\mu\nu} + u^\mu\, u^\nu$.

We can decompose 4-velocity gradient $\nabla^\nu u^\mu$ as follows:

$$\nabla^\nu u^\mu = -a^\mu\, u^\nu + \sigma^{\mu\nu} + \omega^{\mu\nu} + \frac{1}{3}\,\theta\, P^{\mu\nu} \ ,$$

where expansion, acceleration, shear, and vorticity, are defined as:

$$\begin{aligned}
\theta &= \nabla_\mu u^\mu = P^{\mu\nu}\, \nabla_\mu u_\nu \\
a^\mu &= u^\nu\, \nabla_\nu u^\mu \equiv \mathcal{D}\, u^\mu \\
\sigma^{\mu\nu} &= \nabla^{(\mu} u^{\nu)} + u^{(\mu}\, a^{\nu)} - \frac{1}{3}\,\theta\, P^{\mu\nu} = P^{\mu\alpha}\, P^{\nu\beta}\, \nabla_{(\alpha} u_{\beta)} - \frac{1}{3}\,\theta\, P^{\mu\nu} \\
\omega^{\nu\mu} &= \nabla^{[\mu} u^{\nu]} + u^{[\mu}\, a^{\nu]} = P^{\mu\alpha}\, P^{\nu\beta}\, \nabla_{[\alpha} u_{\beta]}
\end{aligned}$$

In terms of these, and adopting the notation (used in Baier et.al.) $A^{\langle\mu\nu\rangle}$ to denote the symmetric, transverse, traceless part of $A^{\mu\nu}$,

$$
\begin{aligned}
T_{2a}^{\mu\nu} &= -2\,\omega^{\rho\langle\mu}\,\sigma_\rho{}^{\nu\rangle} \\
T_{2b}^{\mu\nu} &= \sigma^{\rho\langle\mu}\sigma_\rho{}^{\nu\rangle} \\
T_{2c}^{\mu\nu} &= \theta\,\sigma^{\mu\nu} \\
T_{2d}^{\mu\nu} &= a^{\langle\mu}\,a^{\nu\rangle}
\end{aligned}
$$

and

$$
\frac{1}{3}\,T_{2c}^{\mu\nu} + T_{2d}^{\mu\nu} + T_{2e}^{\mu\nu} = {}^{\langle}\mathcal{D}\sigma^{\mu\nu\rangle} + \frac{1}{3}\,\theta\,\sigma^{\mu\nu}
$$

back

Event horizon in Vaidya-AdS

Vaidya = spher. sym. black hole with ingoing null matter:

$$ds^2 = -\left(1 - \frac{2\,m(v)}{r}\right)\,dv^2 + 2\,dv\,dr + r^2\,d\Omega^2$$

- suppose horizon is at $r = r_H(v)$
- normal $n = dr - \dot{r}\,dv$ is null when

$$r_H(v) = 2\,m(v) + 2\,r_H(v)\,\dot{r}_H(v)$$

- Exact solution gives horizon *non-locally* in terms of $m(v)$.
- But for $m(v)$ slowly varying, $\dot{m}(v) = \mathcal{O}(\epsilon)$, $m\,\ddot{m} = \mathcal{O}(\epsilon^2)$, use ansatz

$$r_H = 2\,m + a\,m\,\dot{m} + b\,m\,\dot{m}^2 + c\,m^2\,\ddot{m} + \ldots$$

- Iterative solution gives $a = 8$, $b = 64$, $c = 32$, ...

Expression for entropy current

The gravitational entropy current:

$$(4\pi\,\eta)^{-1}\,J^\mu_S = \Big[1 + b^2\left(A_1\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} + A_2\,\omega_{\alpha\beta}\,\omega^{\alpha\beta} + A_3\,\mathcal{R}\right)\Big]\,u^\mu$$
$$+\, b^2\,\Big[B_1\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + B_2\,\mathcal{D}_\lambda\omega^{\mu\lambda}\Big]$$
$$+\, C_1\;b\,\ell^\mu + C_2\;b^2 u^\lambda\,\mathcal{D}_\lambda\ell^\mu + \ldots$$

with

$$A_1 = \frac{1}{4} + \frac{\pi}{16} + \frac{\ln 2}{4}; \qquad A_2 = -\frac{1}{8}; \qquad A_3 = \frac{1}{8}$$
$$B_1 = \frac{1}{4}; \qquad B_2 = \frac{1}{2}$$
$$C_1 = C_2 = 0$$

◂ Entropy current

Divergence of entropy current

Entropy current:

$$
\begin{aligned}
(4\pi\,\eta)^{-1}\, J^\mu_S = & \left[1 + b^2 \left(A_1\, \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} + A_2\, \omega_{\alpha\beta}\, \omega^{\alpha\beta} + A_3\, \mathcal{R}\right)\right] u^\mu \\
& + b^2 \left[B_1\, \mathcal{D}_\lambda \sigma^{\mu\lambda} + B_2\, \mathcal{D}_\lambda \omega^{\mu\lambda}\right] \\
& + C_1\; b\; \ell^\mu + C_2\; b^2 u^\lambda\, \mathcal{D}_\lambda \ell^\mu + \ldots
\end{aligned}
$$

Divergence of entropy current:

$$
\begin{aligned}
4\, G_N^{(5)}\, b^3\, \mathcal{D}_\mu J^\mu_S = \frac{b}{2} & \left[\sigma_{\mu\nu} + b\left(2\,A_1 + 4\,A_3 - \frac{1}{2} + \frac{1}{4}\ln 2\right) u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu}\right. \\
& \left. + 4\, b\,(A_2 + A_3)\, \omega^{\mu\alpha} \omega_\alpha{}^\nu + b\,(4\,A_3 - \frac{1}{2})(\sigma^{\mu\alpha}\, \sigma_\alpha{}^\nu) + b\, C_2\, \mathcal{D}^\mu \ell^\nu\right]^2 \\
& + (B_1 - 2A_3)\, b^2\, \mathcal{D}_\mu \mathcal{D}_\lambda \sigma^{\mu\lambda} + (C_1 + C_2)\, b^2\, \ell_\mu\, \mathcal{D}_\lambda \sigma^{\mu\lambda} + \ldots
\end{aligned}
$$

Non-negativity of divergence: $\mathcal{D}_\mu\, J^\mu_S \geq 0$ (when $\sigma^{\mu\nu} = 0$) demands

$$
B_1 = 2\,A_3\;, \qquad C_1 + C_2 = 0
$$

back