Totally Real Immersions

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A smooth immersion $f : M^n \to \mathbf{C}^N$ is called totally real if $f_*(TM)$ does not contain any complex line. This is equivalent to requiring that

$$f_*(TM) \cap i(f_*(TM)) = \{0\}.$$

Theorem 1 *There exists a totally real immersion of M^n into $\mathbf{C}^N$ provided $N > \frac{3}{2}n - 1$.*

This is proved by a straight-forward transversality argument.

Theorem 2 (Gromov) *M^n has a totally real immersion into $\mathbf{C}^n$ if and only if $\mathbf{C} \otimes (TM)$ is trivial.*

Gromov's proof may be modified to yield necessary and sufficient conditions for all values of N between n and $\frac{3n}{2}$.

Theorem 3 *M^n has a totally real immersion into $\mathbf{C}^N$ if and only if there exists a bundle Q over M of rank $N - n$ for which the bundle*

$$(\mathbf{C} \otimes (TM)) \oplus Q$$

is trivial.

Theorem 1 is not optimal, at least when restricted to compact and orientable manifolds. It is known that every such M^2 has a totally real immersion into $\mathbf{C}^2$ and every such M^3 has a totally real immersion into $\mathbf{C}^3$, rather than requiring $N = 3$ and $N = 4$, respectively. This is in contrast with the case $n = 4$ where the value $N = 6$ cannot be improved.

Theorem 4 *There exists a compact and orientable manifold M^4 which does not have a totally real immersion into $\mathbf{C}^5$.*