Long-range correlations in models of driven systems

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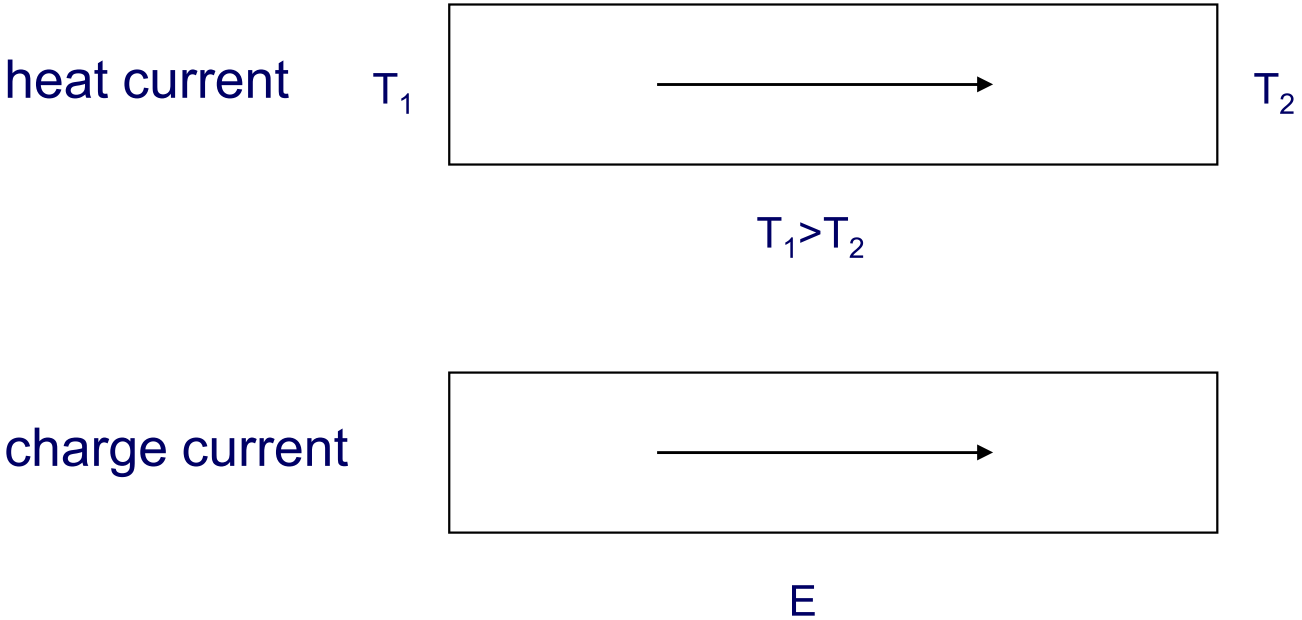
Main objective
some similarities between two classes of systems

- driven, non-equilibrium, systems with local dynamics
- Systems with long-range interactions in thermal equilibrium

Driven systems



- Local and stochastic dynamics
- No detailed balance (non-vanishing current)
- What is the nature of the steady state?

drive in conserving systems result in many cases in long-range correlations leading, in some cases, to spontaneous symmetry breaking and condensation transition even in one dimension.

What can be learned from long-range interacting systems at equilibrium on steady state properties of driven systems?

Systems with long-range interactions at equilibrium

two-body interaction

$$v(r) \propto \frac{1}{r^{d+\sigma}}$$

in d dimensions

for σ<0

$$E \propto VR^{-\sigma} \propto V^{1-\sigma/d}$$

and the energy is not extensive

self gravitating systems	(1/r)	σ=-2
ferromagnets	$(1/r^3)$	σ=0
2d vortices	log(r)	σ=-2
mean-field	1 / V	σ=-d

energy-entropy balance

Free Energy: $$F = E - TS$$

since $$E \propto V^{1-\sigma/d} \quad , \quad S \propto V$$

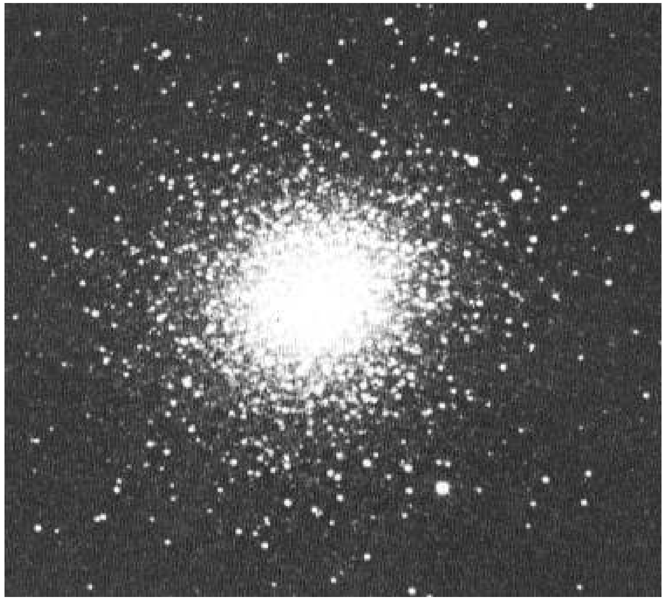
$S << E$ the entropy may be neglected in the thermodynamic limit.

In finite systems, although E>>S, if T is high enough E may be comparable to TS, and the full free energy need to be considered, (Self gravitating systems, e.g. globular clusters).

Alternatively, one may rescale the Hamiltonian

$$H \to V^{\sigma/d} H \quad \Rightarrow \quad E \propto V$$

Globular clusters are gravitationally bound concentrations of approximately ten thousand to one million stars, spread over a volume of several tens to about 200 light years in diameter.



For a typical cluster (M2)

N=150,000 stars

R= 175 light years

$M = 2 \cdot 10^{30}$Kg

$$F = E - TS$$

$$E \sim \frac{GN^2M^2}{R} \qquad S \sim k_B N$$

$$\frac{E}{S} \sim \frac{1}{k_B}\frac{GNM^2}{R} \sim 10^{61}\ \mathrm{K}$$

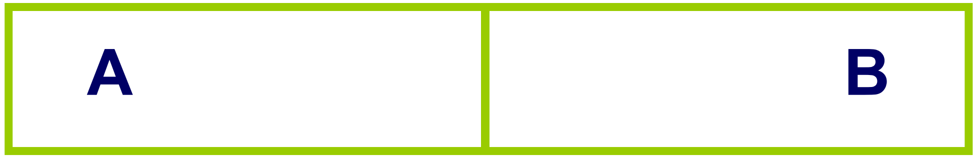
$$v \approx 10\ \ \mathrm{km/sec}$$

$$\frac{1}{2}k_B T = \frac{1}{3}Mv^2$$

Thus although $E \propto V^{5/3}$ and $S \propto V$

E may be comparable to TS

These systems are non-additive (even after rescaling)

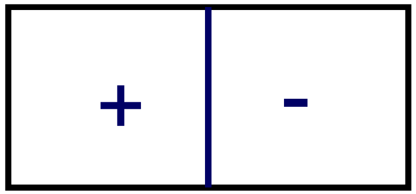


$$F \neq F_A + F_B$$

Take for example the Ising model:

$$H = -\frac{J}{2N}\left(\sum_{i=1}^{N} S_i\right)^2 \qquad S_i = \pm 1$$

The energy is non-additive:



$$N_+ = N_- \qquad E = 0$$

$$E_+ = E_- = -JN/4 \qquad E \neq E_1 + E_2$$

As a result, many of the common properties of typical systems with short range interactions are not shared by these systems.

Features which result from non-additivity

Thermodynamics

- Negative specific heat in microcanonical ensemble
- Inequivalence of microcanonical (MCE) and canonical (CE) ensembles
- Temperature discontinuity in MCE

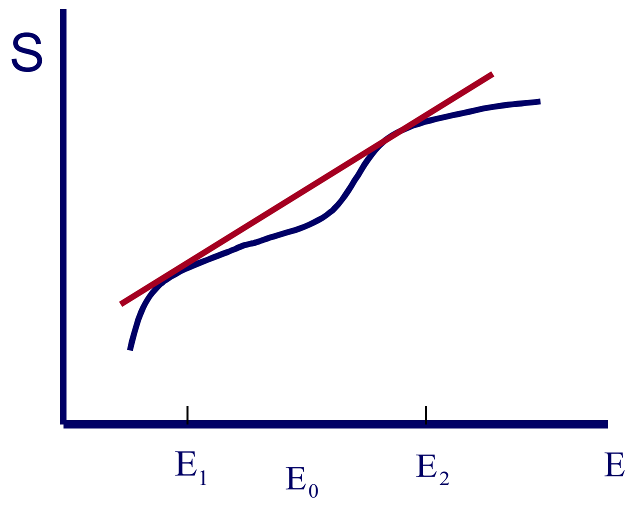
Dynamics

- Breaking of ergodicity in microcanonical ensemble
- Slow dynamics, diverging relaxation time

Some general considerations

Negative specific heat in microcanonical ensemble
of non-additive systems.

Antonov (1962); Lynden-Bell & Wood (1968); Thirring (1970), Thirring & Posch



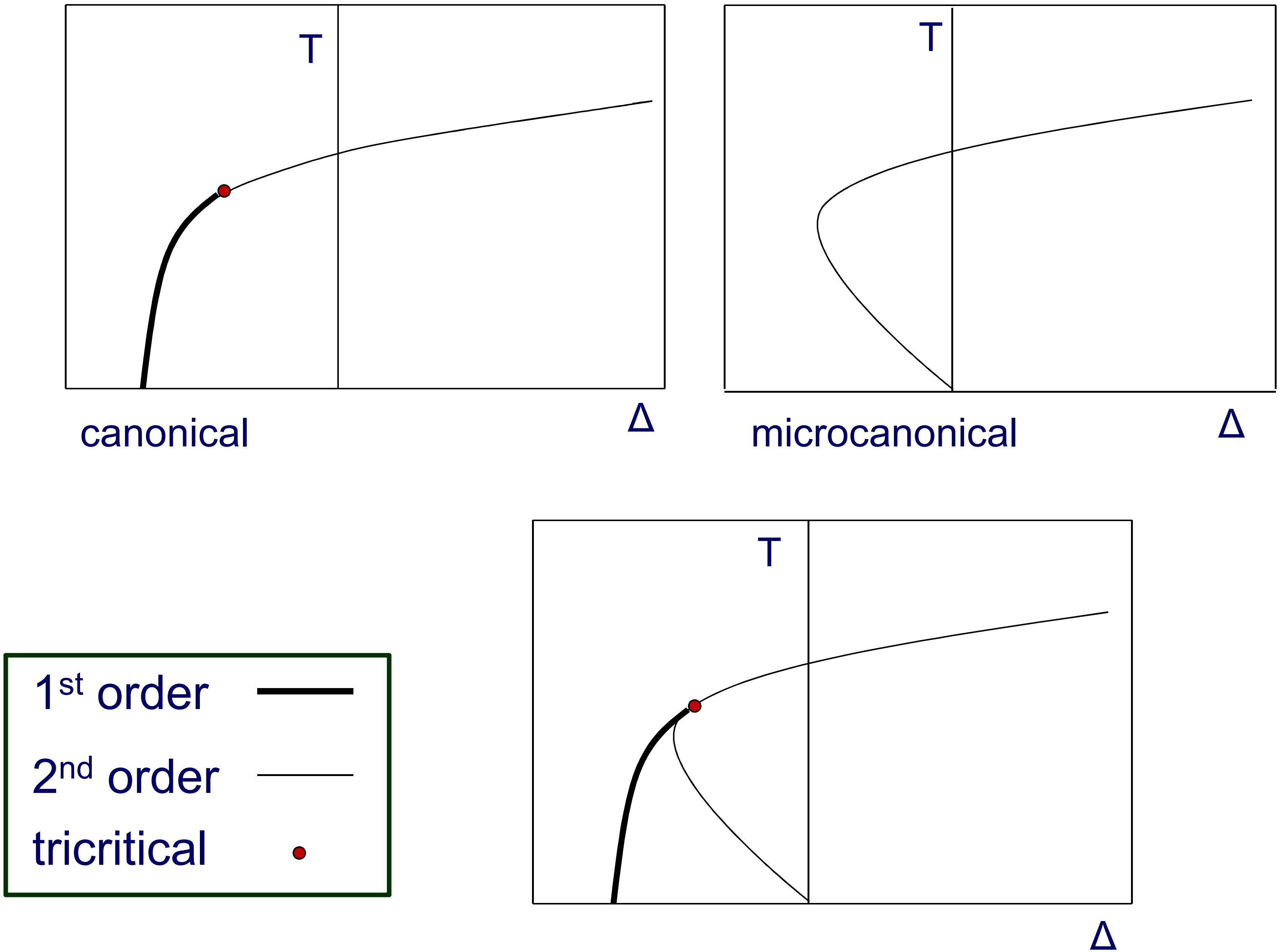
coexistence region
in systems with short range interactions

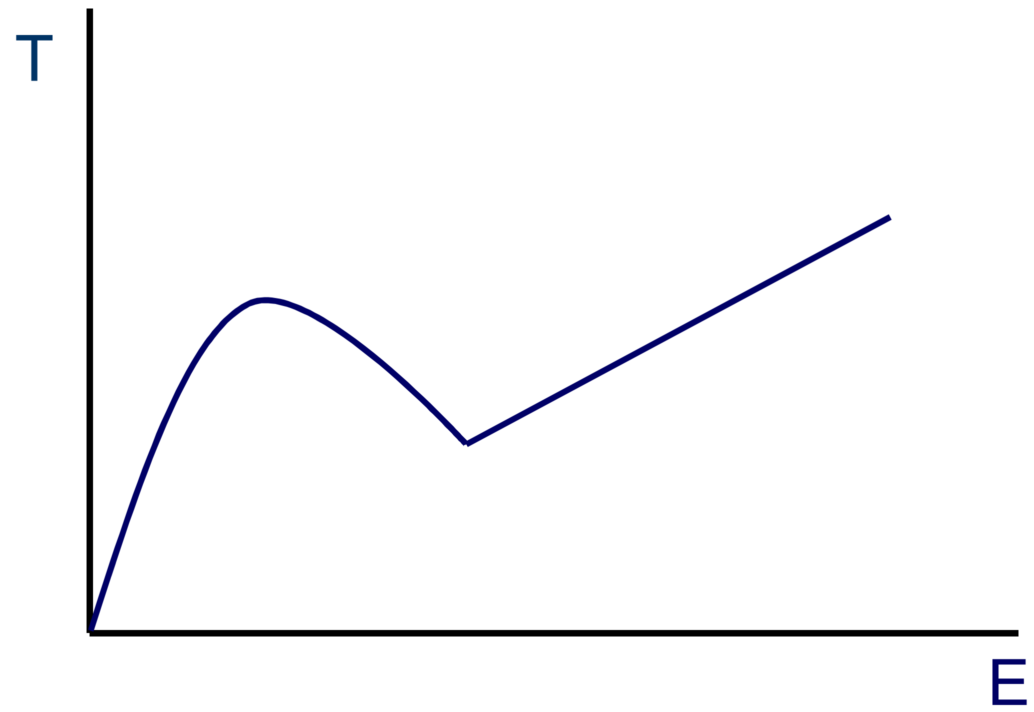
$$E_0 = xE_1 + (1-x)E_2$$
$$S_0 = xS_1 + (1-x)S_2$$

hence S is concave and the microcanonical specific heat is non-negative

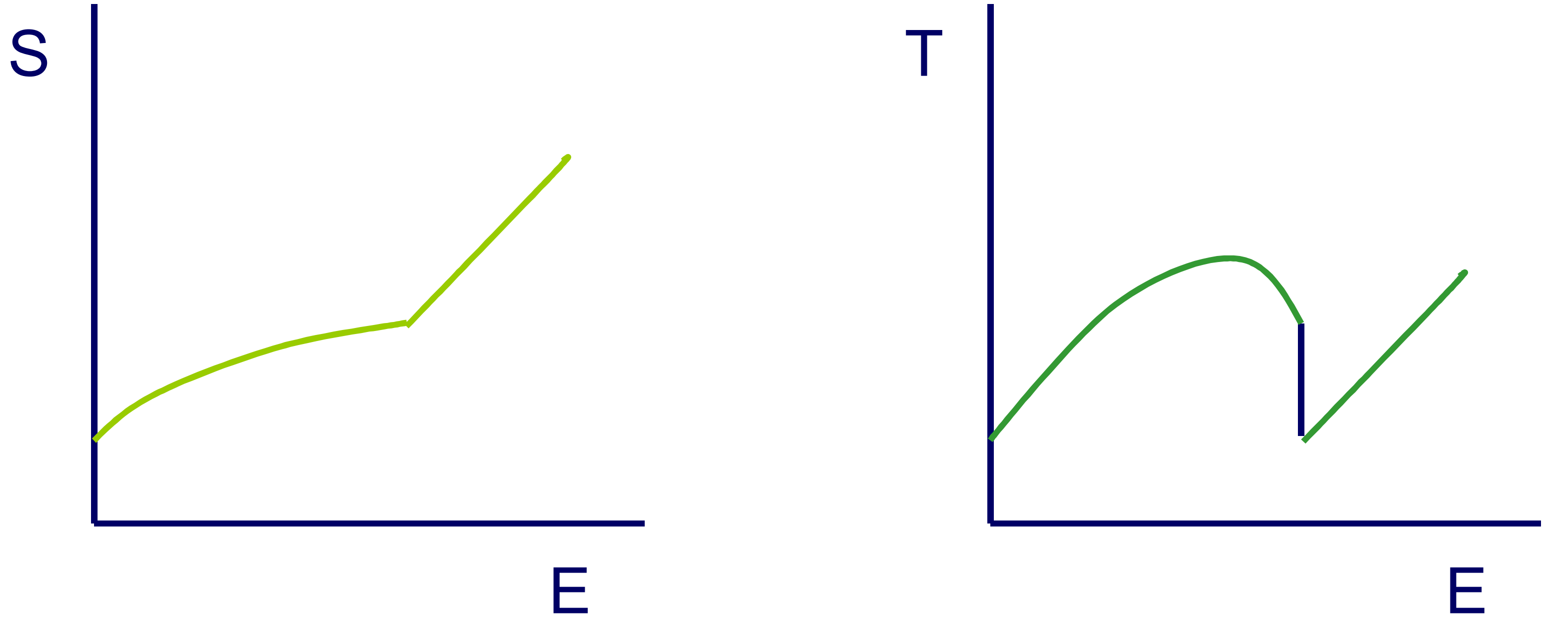
In canonical ensemble $T^2 C_V = \langle E^2 \rangle - \langle E \rangle^2 \geq 0$

Typical (but not exclusive) resulting phase diagrams





- continuous microcanonical transition
- negative microcanonical specific heat
- multivalued E(T) curve
- first order canonical transition
- similarly for ρ(μ) curve in canonical vs grand canonical



Temperature discontinuity at a first order microcanonical transition

Driven systems

Long range correlations in driven systems

- Conserved variables tend to produce long-range correlations in driven systems, sometimes resulting in LRO even in d=1.

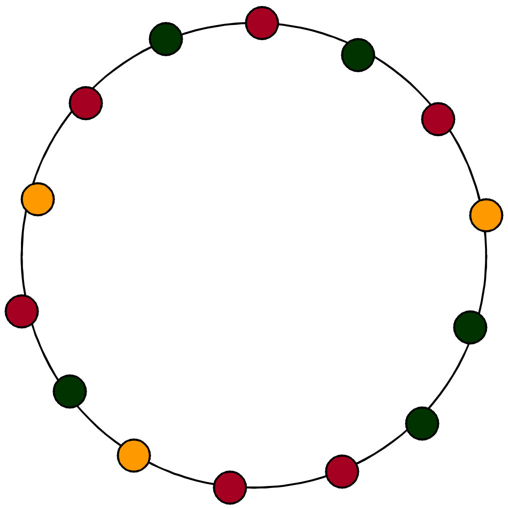
- Can these correlations be viewed as resulting from effective long-range interactions, even when the dynamics is local?

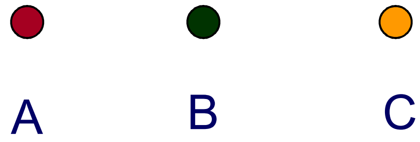
- Features like ensemble inequivalence etc.

The ABC model

One dimensional driven model with stochastic local dynamics which results in phase separation (long range order) where the steady state can be expressed as a Boltzmann distribution of an effective energy with long-range interactions.

ABC Model





dynamics

$$AB \underset{1}{\overset{q}{\rightleftarrows}} BA$$

$$BC \underset{1}{\overset{q}{\rightleftarrows}} CB$$

$$CA \underset{1}{\overset{q}{\rightleftarrows}} AC$$

Evans,Kafri, Koduvely, Mukamel PRL 80, 425 (1998)
A model with similar features was discussed by Lahiri, Ramaswamy PRL 79, 1150 (1997)

Simple argument:

ACCCC → CCCCA

CBBBB → BBBBC

BAAAA → AAAAB

$$AB \underset{1}{\overset{q}{\rightleftharpoons}} BA$$

$$BC \underset{1}{\overset{q}{\rightleftharpoons}} CB$$

$$CA \underset{1}{\overset{q}{\rightleftharpoons}} AC$$

…AACBBBCCAAACBBBCCC…

↓ fast rearrangement

…AABBBCCCAAABBBCCCC…

↓ slow coarsening

…AAAAABBBBBCCCCCCAA…

The model reaches a phase separated steady state

- logarithmically slow coarsening

…AAAAABBBBBCCCCCCAA…

$$t \propto q^{-l} \qquad l \propto \ln t$$

- needs n>2 species to have phase separation
- Phase separation takes place for any q (except q=1)
- Phase separation takes place for any density N_A, N_B, N_C
- strong phase separation: no fluctuation in the bulk;
  only at the boundaries.

…AAAAAAAAAABBBBBBBBBBBBCCCCCCCCCCCC…

Special case $N_A = N_B = N_C$

The argument presented before is general, independent of densities.

For the equal densities case the model has detailed balance for arbitrary q.

For any microscopic configuration {X} one can define an energy E({X})
such that the steady state distribution is

$$P(\{X\}) \propto q^{E(\{X\})}$$

AAAAAABBBBBBCCCCC   E=0

……AB….. $\longrightarrow$ ……BA…..   E $\rightarrow$ E+1

……BC….. $\longrightarrow$ ……CB…..   E $\rightarrow$ E+1

……CA….. $\longrightarrow$ ……AC…..   E $\rightarrow$ E+1

With this weight one has:

$$W(AB \to BA)P(\ldots AB\ldots) = W(BA \to AB)P(\ldots BA\ldots)$$

=q   =1

$$P(\ldots BA\ldots)/P(\ldots AB\ldots) = q \qquad P(\{X\}) \propto q^{E(\{X\})}$$

This definition of energy is possible only for $N_A = N_B = N_C$

AAAAABBBBBCCCCC → AAAABBBBBCCCCCA

$E \longrightarrow E + N_B - N_C$

$N_B = N_C$

Thus such energy can be defined only for $N_A=N_B=N_C$

Partition sum

Excitations near a single interface: AAAAAAABBBBBB

$$Z_1(q) = \sum p(n) q^n$$

P(n)= degeneracy of the excitation with energy n

P(0)=1
P(1)=1
P(2)=2 (2, 1+1)
P(3)=3 (3, 2+1, 1+1+1)
P(4)=5 (4, 3+1, 2+2, 2+1+1, 1+1+1+1)

P(n)= no. of partitions of an integer n

$$p(n) \sim \frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}} \qquad n \to \infty$$

$$Z_1(q) = \sum p(n) q^n$$

$$Z_1(q) = \frac{1}{(1-q)(1-q^2)....}$$

$$(1 + q + q^2 + q^3 + ..)(1 + q^2 + q^4 + ..)(1 + q^3 + q^6 + ..)..$$

$$\Phi(q) = \prod_{k=1}^{\infty} (1 - q^k)$$ (Euler's function)

Partition sum: $$Z(q) = N\left[\frac{1}{(1-q)(1-q^2)......}\right]^3$$

Correlation function: $$\langle A_1 A_r \rangle \approx 1/3$$

with $$\langle A_1 \rangle \langle A_r \rangle = 1/9$$

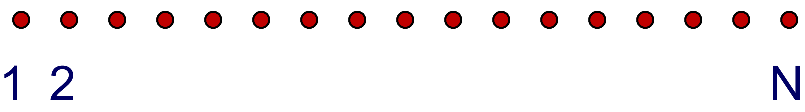
for $$-1/\ln q < r < N/3$$

$$N_A = N_B = N_C$$

$$P(\{x\}) = q^{E(\{x\})}$$

The energy E({X}) may be written as

$$E(\{x\}) = \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right) - (N/3)^2$$



(mean-field like interaction with σ=-d)

Alternatively, in a manifestly translational invariant form:

$$E(\{x\}) = \sum_{i=1}^{N} \sum_{k=1}^{N-1} \left(1 - \frac{k}{N} \right) \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right)$$

$$P(\{x\}) = q^{E(\{x\})}$$

$$E(\{x\}) = \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right) - (N/3)^2$$

$$E(\{x\}) = \sum_{i=1}^{N} \sum_{k=1}^{N-1} \left(1 - \frac{k}{N} \right) \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right)$$

- Local dynamics
- Long range interaction
- Long range order at any q≠1

Weakly asymmetric ABC model

q=1 - homogeneous

q<1 - phase separation

consider $$q = e^{-\beta / N}$$

the model exhibits a phase transition at $\beta_c = 2\pi\sqrt{3}$
for the case of equal densities

$\beta < \beta_c$ homogeneous

$\beta > \beta_c$ phase separated

This feature persists at non-equal densities.

Clincy, Derrida, Evans, PRE 67, 066115 (2003)

The choice $q = e^{-\beta / N}$ amounts to rescaling the energy by 1/N

$$E(\{x\}) = \frac{1}{N} \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right) - N/9$$

effective rescaled “energy”

without rescaling:
energy is dominates the entropy, no transition $q = e^{-\beta}$

with rescaling:
energy and entropy are comparable, resulting in a transition $q = e^{-\beta / N}$

A brief summary of the ABC model

- Driven model with local dynamics
- Exhibits long range correlation (phase separation)
- It exhibits a phase transition in the weak bias limit
- In the case of equal densities its steady state may be expressed by an energy with long range interactions

Outline

- Generalize the model to study non-conserving processes
- Compare steady states of conserving and non-conserving dynamics
- The existence of effective long range interactions may lead to different steady states in both cases for equal densities
- Use this as a starting point to move into non-equal densities (where there is no detailed balance)

Generalized ABC model

- Add vacancies: A , B, C, 0; $N_A + N_B + N_C = N \quad , \quad N \le L$

Dynamics

$$AB \underset{1}{\overset{q}{\rightleftarrows}} BA$$

$$BC \underset{1}{\overset{q}{\rightleftarrows}} CB$$

$$CA \underset{1}{\overset{q}{\rightleftarrows}} AC$$

$$0X \underset{1}{\overset{1}{\rightleftarrows}} 0X$$

X=A,B,C

Vacancies are “inert”

A. Lederhendler, O. Cohen, D. Mukamel ; A. Lederhendler, D. Mukamel, arXive:1006.2715;

For $N_A=N_B=N_C$ there is detailed balance

$$P(\{x\}) = q^{E(\{x\})} \qquad q = e^{-\beta / L}$$

$$E(\{x\}) = \sum_{i=1}^{L-1} \sum_{k=1}^{L-i} \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right) - N^2 / 9$$

not important

Non conserving processes

$$AB \underset{1}{\overset{q}{\rightleftharpoons}} BA$$

$$BC \underset{1}{\overset{q}{\rightleftharpoons}} CB$$

$$CA \underset{1}{\overset{q}{\rightleftharpoons}} AC$$

$$0X \underset{1}{\overset{1}{\rightleftharpoons}} X0 \qquad X=A,B,C$$

$$ABC \underset{p}{\overset{pq^{\theta}}{\rightleftharpoons}} 000 \qquad \theta = 3L\mu$$

…A000ACBABCCA00AACBBB00000CCC…

ABC ↔ 000

000 ↔ ABC

For $N_A=N_B=N_C$: there is detailed balance with respect to

$$E(\{x\}) = \left[\sum_{i=1}^{L-1}\sum_{k=1}^{L-i}\left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k}\right) - \frac{N^2}{6}\right] - \mu L N$$

$$P(\{X\}) \propto q^{E(\{X\})} \qquad q = e^{-\beta/L}$$

- $E(\{X\}, N+3) = E(\{X\}, N) - 3L\mu$

  irrespective of {X} and of where the deposition is made

  …A000ACBABCCA00AACBBB00000CCC…

  E(…B ABC…) = E(… ABC B…)

- The dynamics is local

continuum limit

$$F(\rho(x),T) = E - TS \ , \ q = e^{-\beta/L} \quad , \ \beta = 1/k_B T$$

$$E = \int_0^1 dx \int_0^{1-x} dz \left[\rho_B(x)\rho_A(x+z) + \rho_A(x)\rho_C(x+z) + \rho_C(x)\rho_B(x+z)\right] - \frac{1}{6}\left[\int_0^1 dx \rho(x)\right]^2$$

$$S = \int_0^1 dx \left[\rho_A(x)\ln \rho_A(x) + \rho_B(x)\ln \rho_B(x) + \rho_C(x)\ln \rho_C(x) + +\rho_0(x)\ln \rho_0(x)\right]$$

$$\rho(x) = \rho_A(x) + \rho_B(x) + \rho_C(\mathrm{x}) \quad , \quad \rho_0(x) = 1 - \rho(x)$$

Conserving dynamics corresponds to the canonical ensemble:

minimize $F(\rho(x),T)$ and then determine μ by taking the derivative.

$$\rho_A(x) = \frac{r}{3} + a_1 \cos 2\pi x + \ldots \qquad r = N/L$$

$$\rho_B(x) = \frac{r}{3} + a_1 \cos 2\pi (x - \frac{1}{3}) + \ldots$$

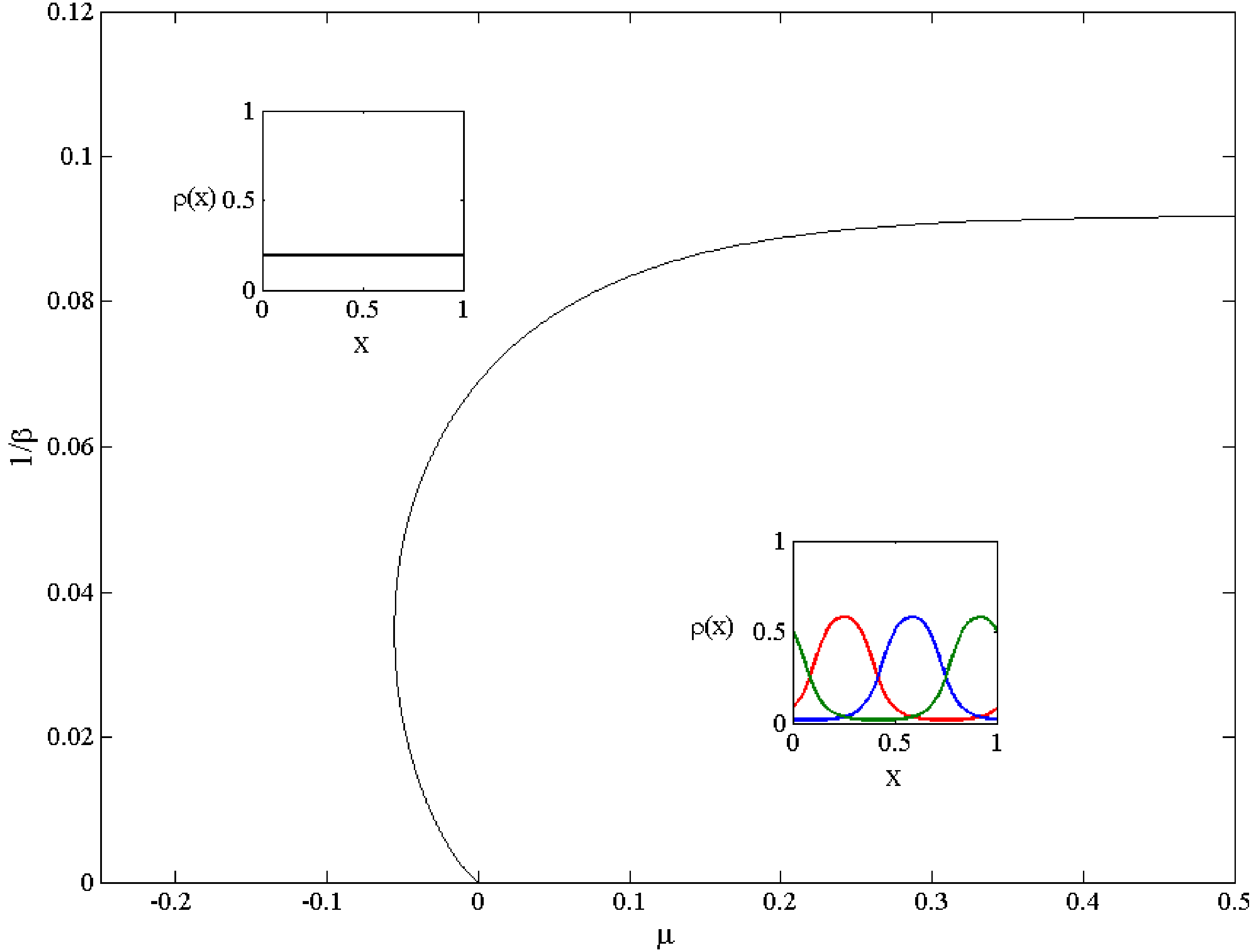
$$\rho_C(x) = \frac{r}{3} + a_1 \cos 2\pi (x - \frac{2}{3}) + \ldots$$

$$F = f_2(\beta, r) a_1^2 + f_4(\beta, r) a_1^4 + \ldots$$

$f_2(\beta, r) = 0$ ⇒ The model exhibits a transition from homogeneous to modulated structure at

$$\beta_c = 2\pi\sqrt{3}/r, \quad \mu = \frac{\partial F}{\partial r} = \frac{1}{\beta} \ln \frac{r}{3(1-r)}$$

Conserving dynamics



Density profiles

$$\frac{d\rho_A}{dt} = \frac{d}{dx}\left[\beta\rho_A(\rho_B - \rho_C) + \frac{d\rho_A}{dx}\right] \qquad \left[\beta\rho_A(\rho_B - \rho_C) + \frac{d\rho_A}{dx}\right] = J_A$$

$$\frac{d\rho_B}{dt} = \frac{d}{dx}\left[\beta\rho_B(\rho_C - \rho_A) + \frac{d\rho_B}{dx}\right] \qquad \left[\beta\rho_B(\rho_C - \rho_A) + \frac{d\rho_B}{dx}\right] = J_B$$

$$\frac{d\rho_C}{dt} = \frac{d}{dx}\left[\beta\rho_C(\rho_A - \rho_B) + \frac{d\rho_C}{dx}\right] \qquad \left[\beta\rho_C(\rho_A - \rho_B) + \frac{d\rho_C}{dx}\right] = J_C$$

At equilibrium: $N_A = N_B = N_C$ ⇒ $J_A(x) = J_B(x) = J_C(x) = 0$

$$\rho_A(x) = \frac{1 + \mathrm{sn}[2\beta x/\chi, k]}{\alpha_+ + \alpha_- \mathrm{sn}[2\beta x/\chi, k]}$$

$$\rho_B(x) = \rho_A(x - \frac{1}{3}), \rho_C(x) = \rho_A(x + \frac{1}{3})$$

where $\chi, k, \alpha_+, \alpha_-$ are functions of β, and sn[x,k] is the Jacobi elliptic function

A. Ayyer, E. A. Carlen, J. L. Lebowitz, P. K. Mohanty, D. Mukamel, and E. R. Speer
J. Stat. Phys. , 137(5-6):1166–1204, 2009.

Non-conserving dynamics

$$AB \underset{1}{\overset{q}{\rightleftharpoons}} BA$$

$$BC \underset{1}{\overset{q}{\rightleftharpoons}} CB$$

$$CA \underset{1}{\overset{q}{\rightleftharpoons}} AC$$

$$0X \underset{1}{\overset{1}{\rightleftharpoons}} 0X \qquad X=A,B,C$$

$$ABC \underset{1}{\overset{q^{\theta}}{\rightleftharpoons}} 000 \qquad \theta = 3L\mu$$

$$G(\mu, T; \rho(x)) = F(\rho(x), T) - \mu\rho$$

Grand canonical ensemble: minimize G with respect to ρ(x) at a given μ

$$G(\mu, T; \rho(x)) = F(\rho(x), T) - \mu\rho$$

Grand canonical ensemble: minimize G with respect to ρ(x) at a given μ

$$G = g_2(\beta, \mu)a_1^2 + g_4(\beta, \mu)a_1^4 + g_6(\beta, \mu)a_1^6 \dots$$

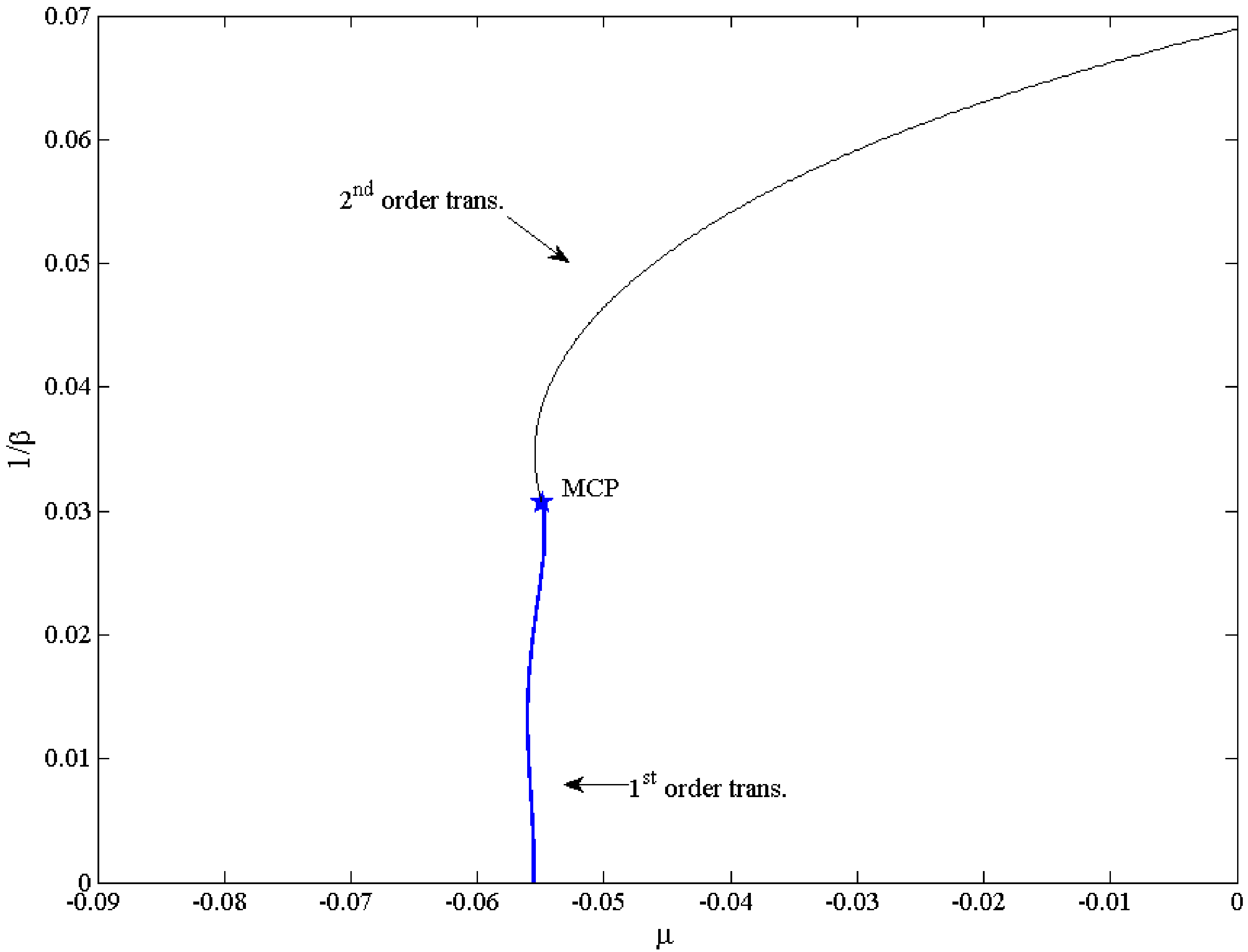
$g_2(\beta, \mu) = 0$ ⇒ One finds the same critical line as in the canonical ensemble

$$\beta_c = 2\pi\sqrt{3}/\rho, \quad \mu = \frac{1}{\beta}\ln\frac{\rho}{3(1-\rho)}$$

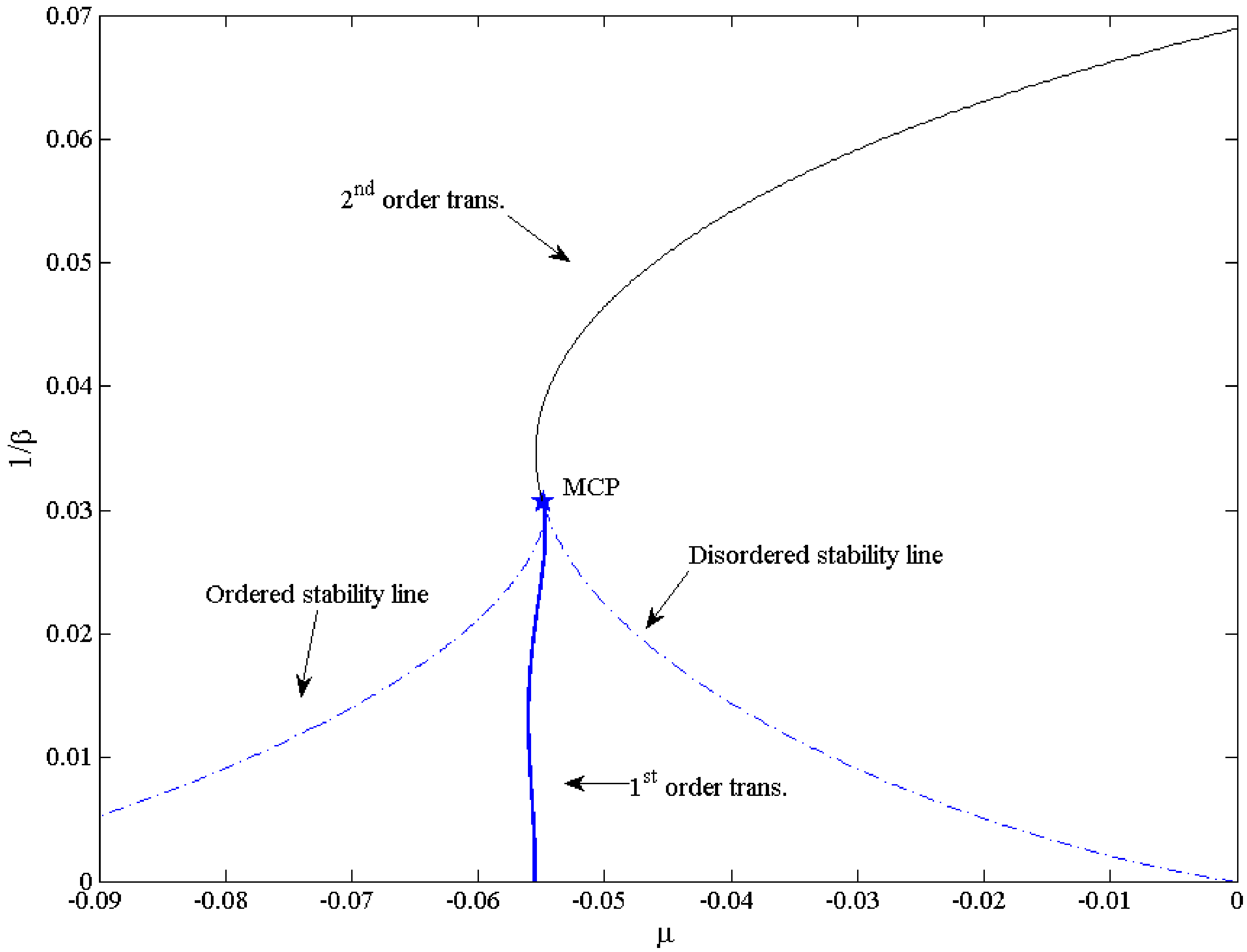
$g_4(\beta, \mu) = 0$ ⇒ But at $\rho_{MCP} = 1/3$ this transition becomes first order

Surprisingly, $g_6(\beta, \mu) = 0$. Hence this is a fourth order critical point.

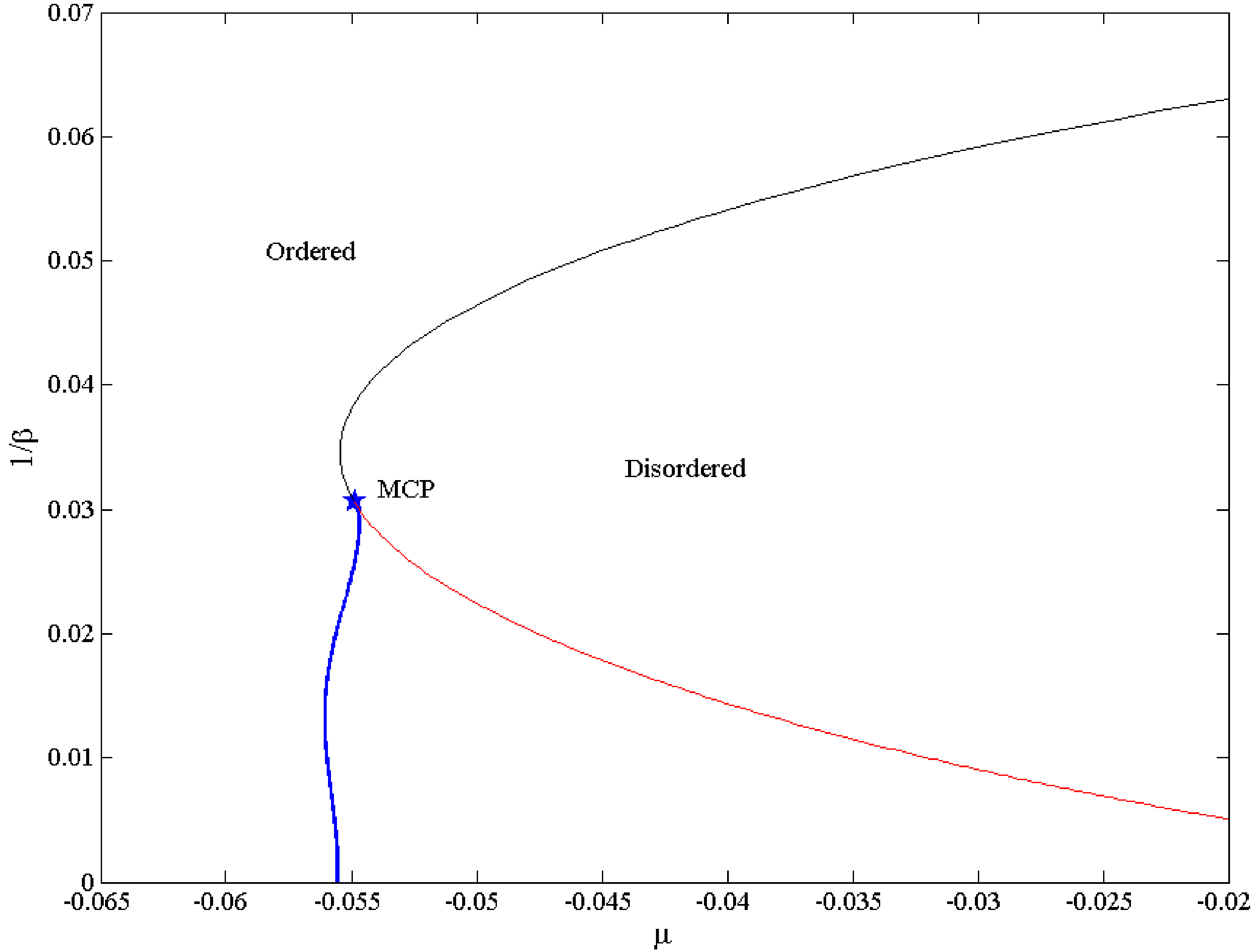
Non-conserving dynamics



Non-conserving dynamics



Canonical vs. grand-canonical phase diagrams



0000000 A..AB..BC..C

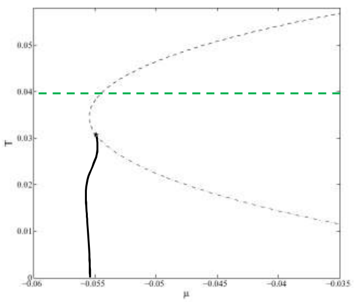
T=0

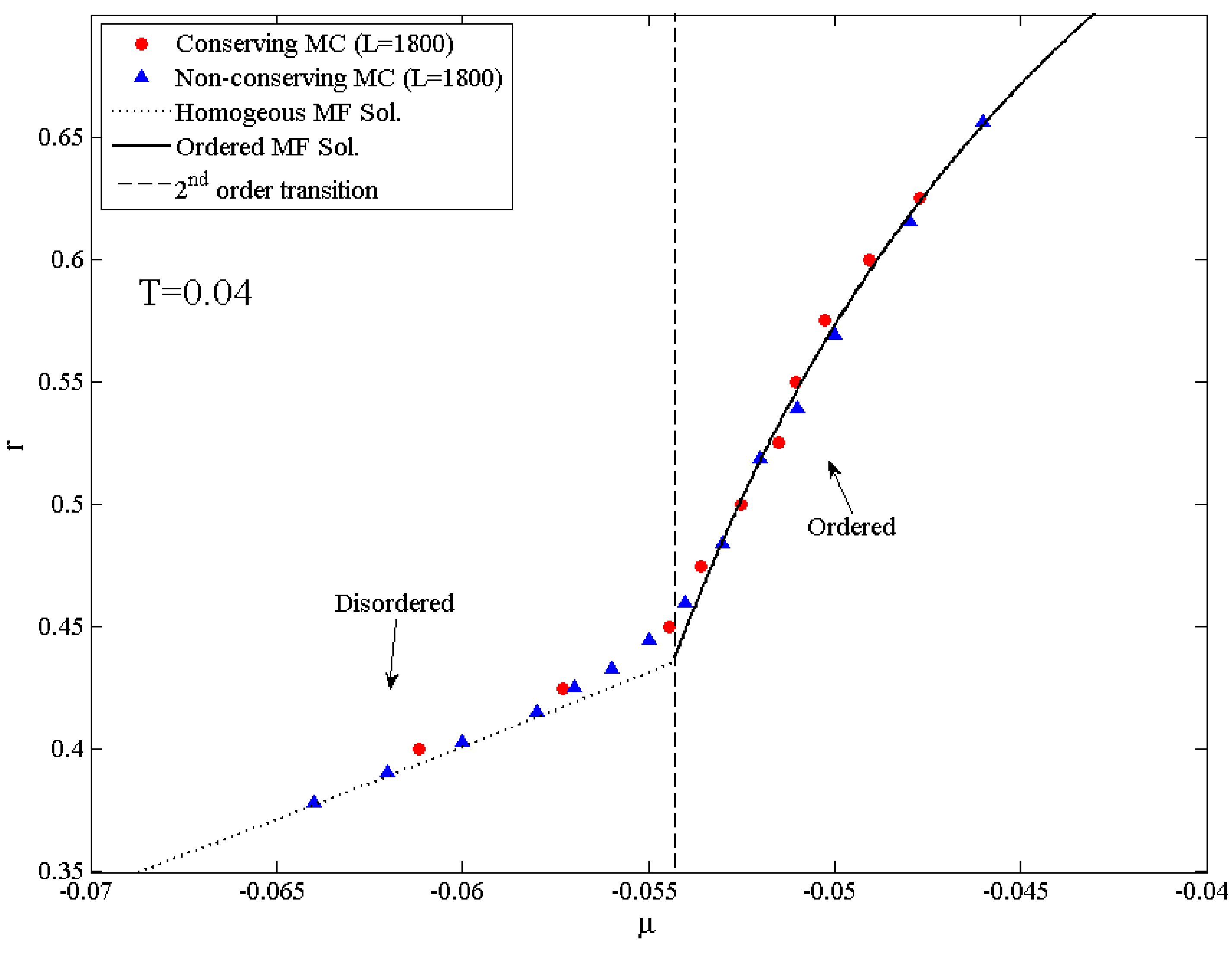
-1/18

μ

Conserving vs. non-conserving dynamics: 2^{nd} order line

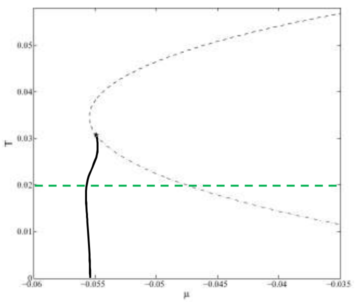
T=0.04

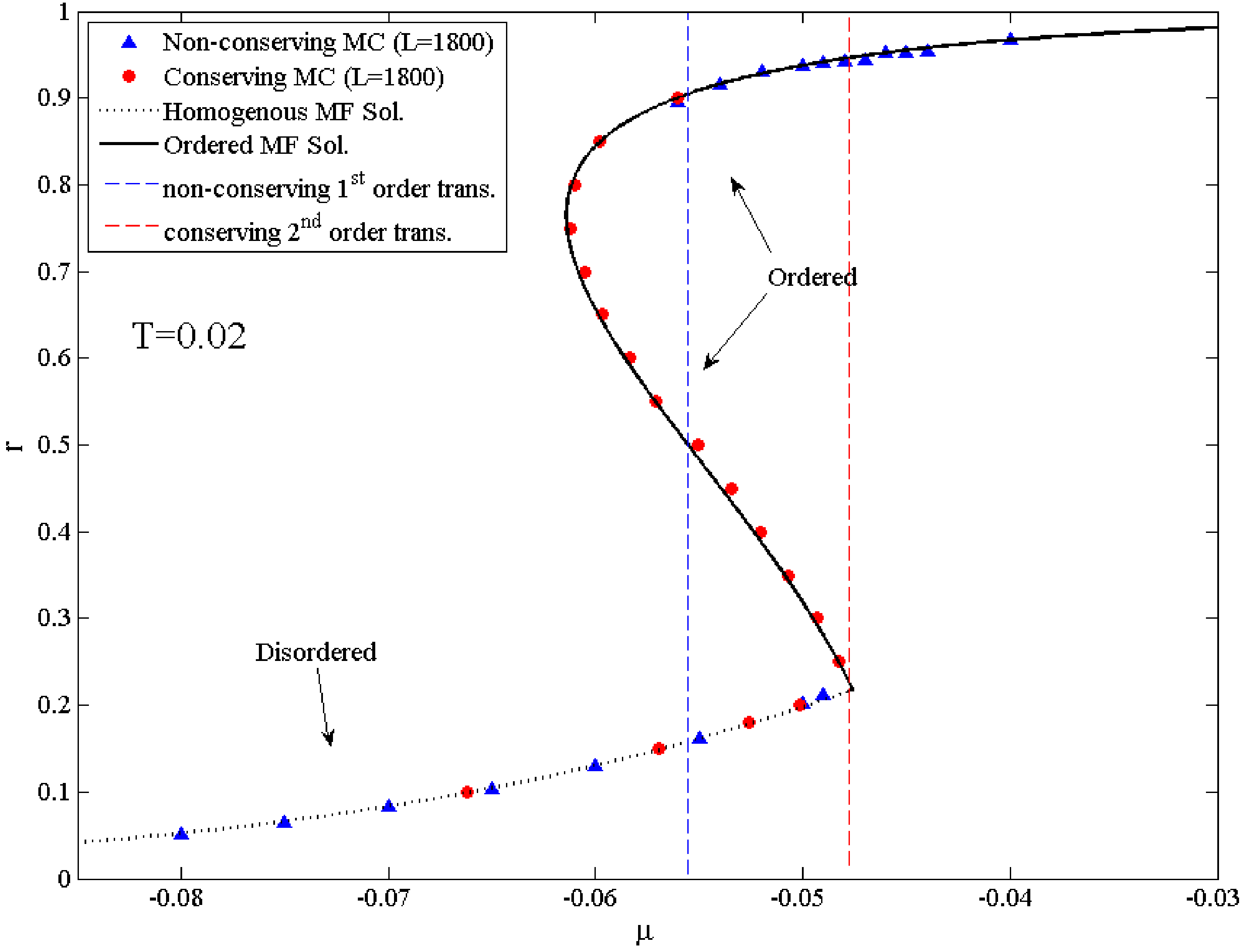




Conserving vs. non-conserving dynamics: 1^{st} order line

T=0.02





out of Equilibrium – unequal densities

$$\frac{d\rho_A}{dt} = \frac{d}{dx}\left[\beta\rho_A(\rho_B - \rho_C) + \frac{d\rho_A}{dx}\right] + L^2 p\left[\rho_0^3 - e^{-3\beta\mu}\rho_A\rho_B\rho_C\right]$$

$$\frac{d\rho_B}{dt} = \frac{d}{dx}\left[\beta\rho_B(\rho_C - \rho_A) + \frac{d\rho_B}{dx}\right] + L^2 p\left[\rho_0^3 - e^{-3\beta\mu}\rho_A\rho_B\rho_C\right]$$

$$\frac{d\rho_C}{dt} = \frac{d}{dx}\left[\beta\rho_C(\rho_A - \rho_B) + \frac{d\rho_C}{dx}\right] + L^2 p\left[\rho_0^3 - e^{-3\beta\mu}\rho_A\rho_B\rho_C\right]$$

$$ABC \underset{p}{\overset{pq^{\theta}}{\rightleftharpoons}} 000$$

For small p, $L^2 p \to 0$ the second term in the RHS disappears

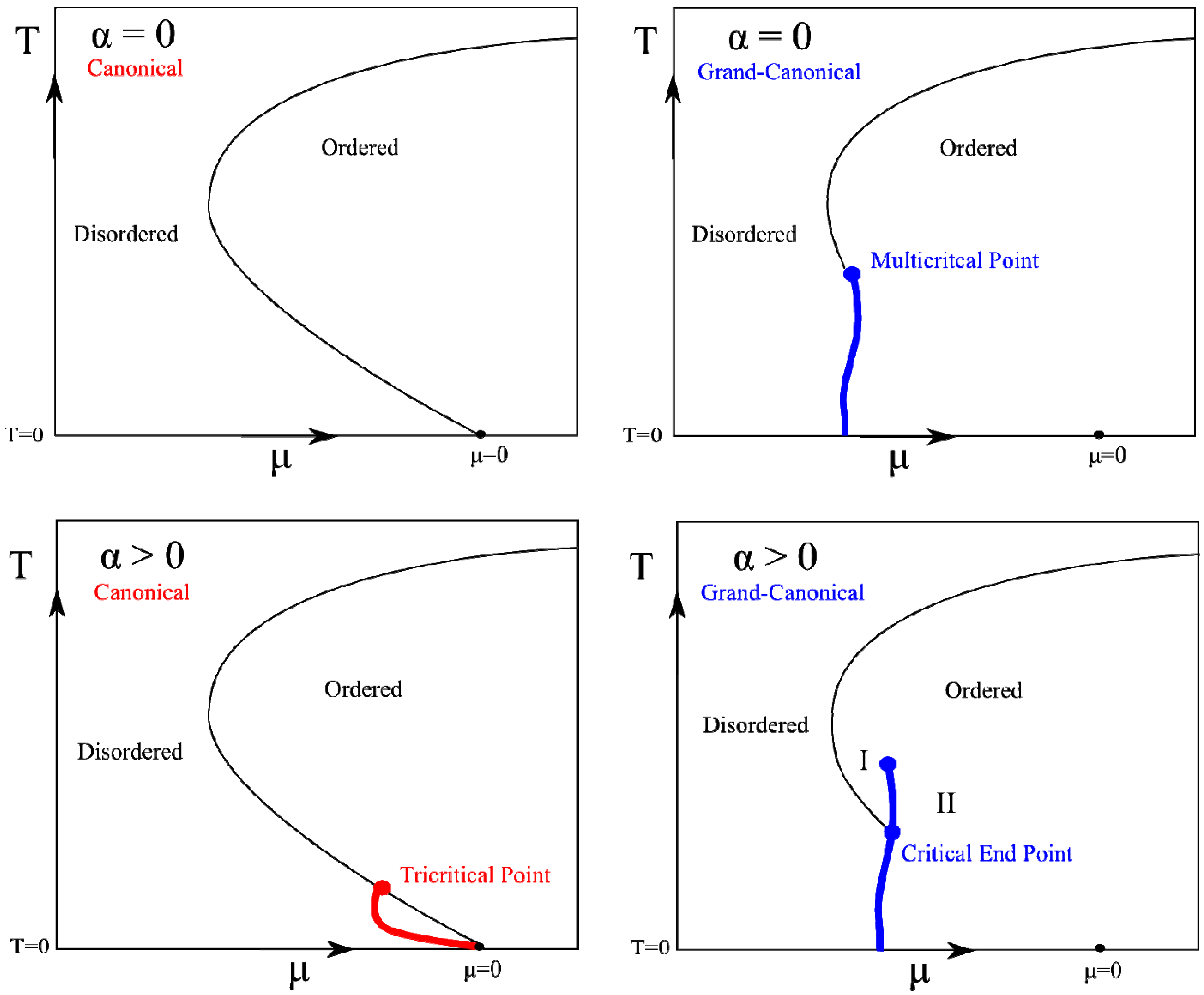
$$\left[\beta\rho_A(\rho_B - \rho_C) + \frac{d\rho_A}{dx}\right] = J_A$$

$$\left[\beta\rho_B(\rho_C - \rho_A) + \frac{d\rho_B}{dx}\right] = J_B$$

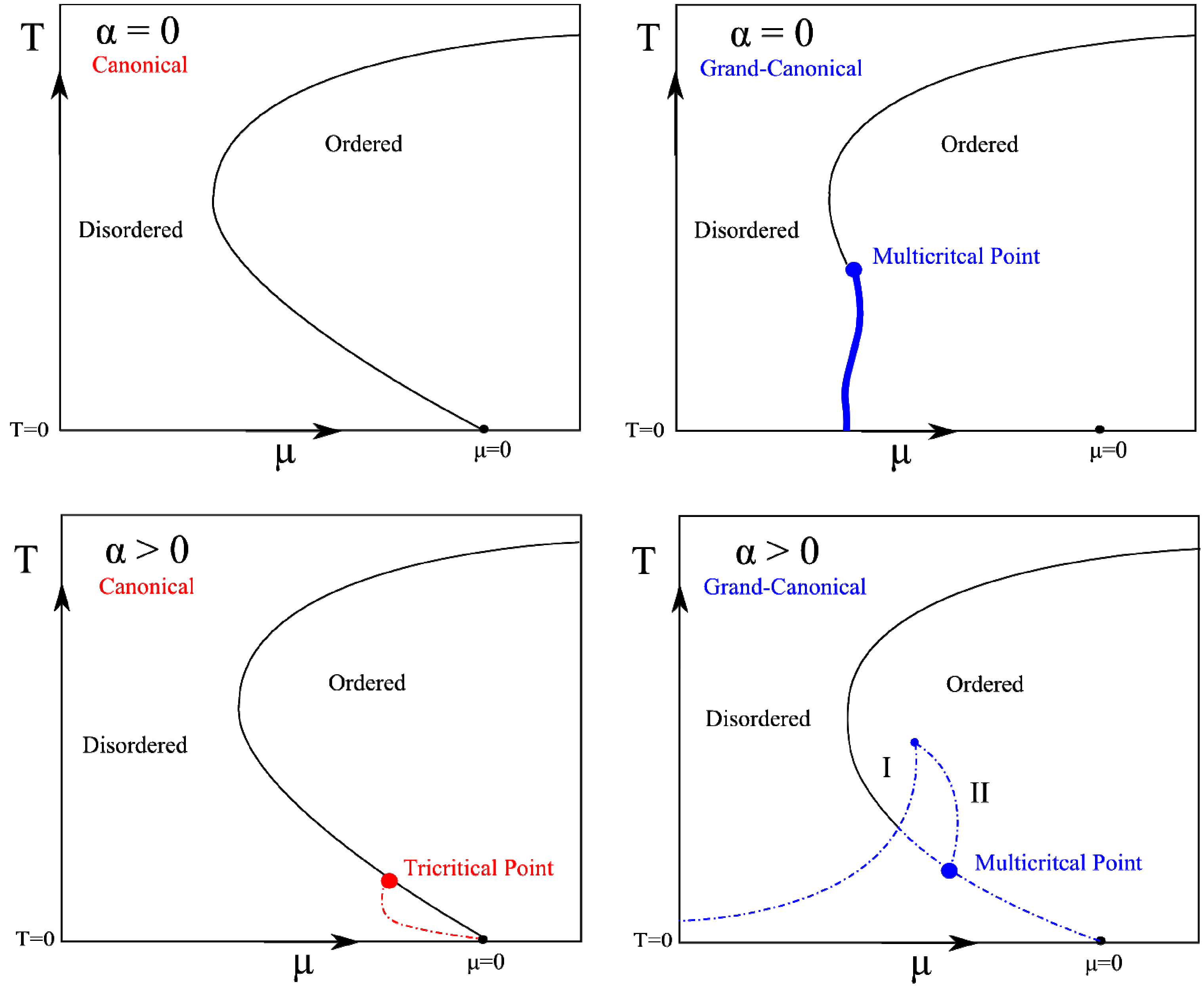
$$\left[\beta\rho_C(\rho_A - \rho_B) + \frac{d\rho_C}{dx}\right] = J_C$$

- The profile can still be calculated as a function of J's (J≠0)
- Critical lines obtain by expansion near homogeneous phase

Out of equilibrium $N_A = N_B \neq N_C$



$N_A = N_B = (r/3-\alpha) \quad N_C = (r/3+2\alpha)$



$$N_A = N_B = (r/3-\alpha) \quad N_C = (r/3+2\alpha)$$

Summary

- Local stochastic dynamics may result in effective long-range interactions in driven systems.
- This is manifested in the existence of phase transitions in one dimensional driven models.
- Existence of effective long range interactions can be explicitly demonstrated in the ABC model.
- The model exhibits phase separation for any drive $q \neq 1$
- Phase separation is a result of effective long-range interactions generated by the local dynamics.
- Inequivalence of ensembles in the driven model.