

Convex Hull of n Planar Brownian Motions

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Collaborators:

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Ref: [Phys. Rev. Lett. 103, 140602 \(2009\)](#)

Extended Review: [J. Stat. Phys. 138, 955 \(2010\)](#)

Plan

Plan:

- Random Convex Hull $\Rightarrow$ definition

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- Motivation $\implies$ an ecological problem

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 - $\implies$ link to **Extreme Value Statistics**

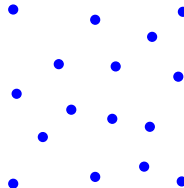
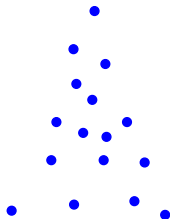
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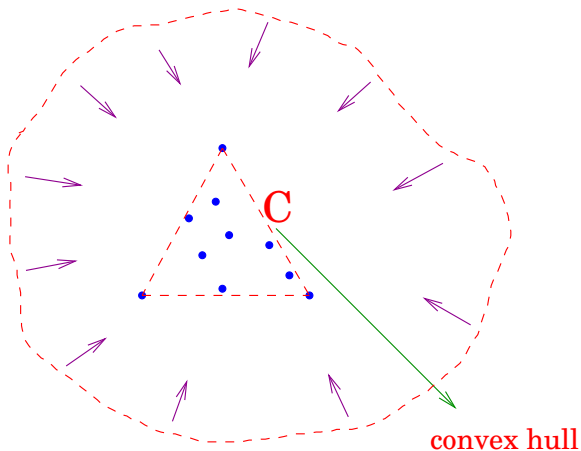
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- Summary and Conclusions

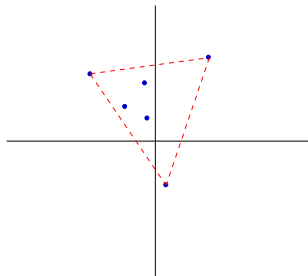
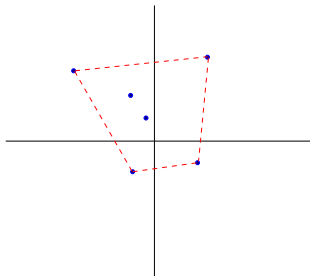
Shape of a set of Points



Shape of a set of Points: Convex Hull

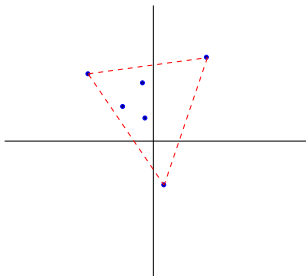
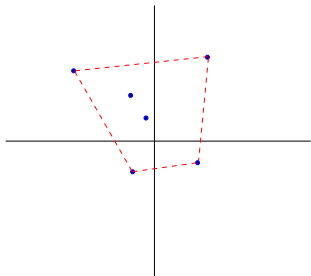


Random Convex Hull in a Plane



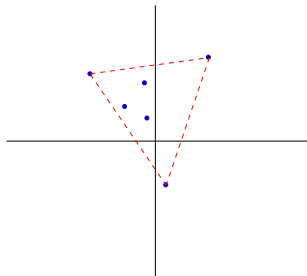
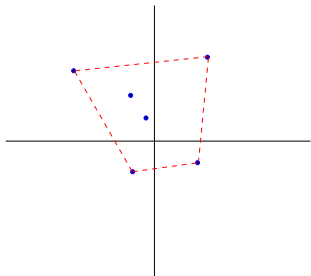
- Convex Hull $\implies$ Minimal convex polygon enclosing the set

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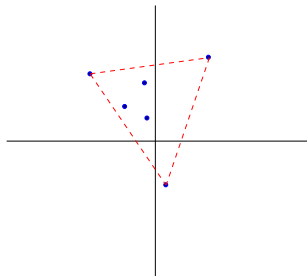
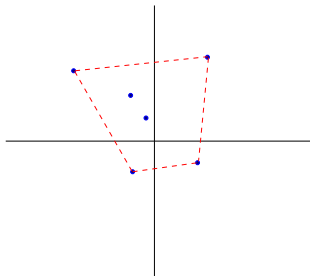
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Random Convex Hull in a Plane



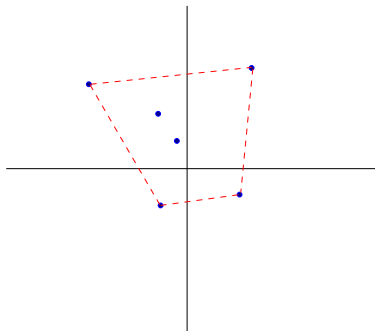
- Convex Hull $\implies$ Minimal convex polygon enclosing the set
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- Points drawn from a distribution $\rightarrow$ Independent or Correlated

Random Convex Hull in a Plane



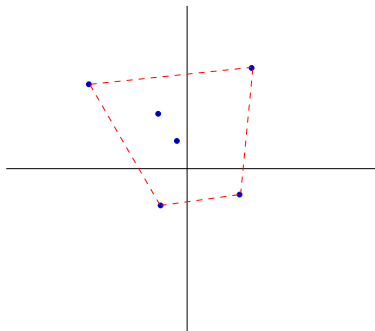
- Convex Hull $\implies$ Minimal convex polygon enclosing the set
- The shape of the convex hull $\rightarrow$ different for each sample
- Points drawn from a distribution $\rightarrow$ Independent or Correlated
- Question: Statistics of observables: perimeter, area and no. of vertices

Independent Points in a Plane



Each point chosen **independently** from the same distribution

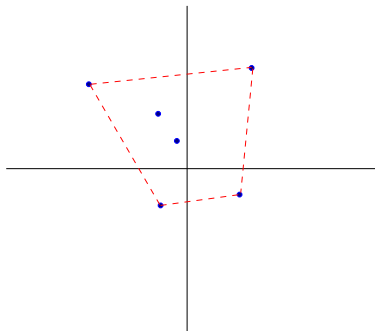
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Associated **Random Convex Hull** $\rightarrow$ well studied by **diverse** methods

Independent Points in a Plane



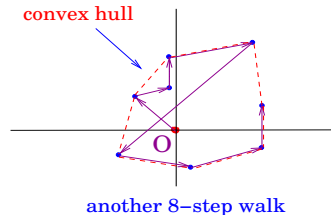
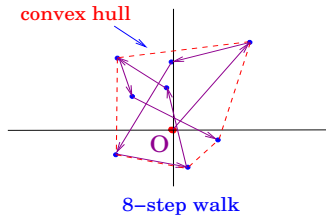
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Associated **Random Convex Hull** → well studied by **diverse** methods

P. Lévy ('48), J. Geffroy ('59), Spitzer & Widom ('59), Baxter ('59)..

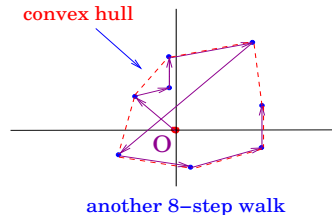
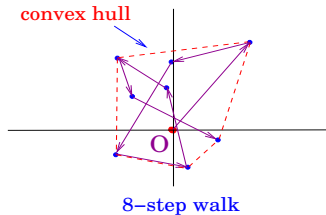
Rényi & Sulanke ('63), Efron ('65),many others

Correlated Points: Vertices of an Open Random Walk



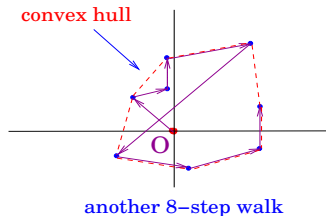
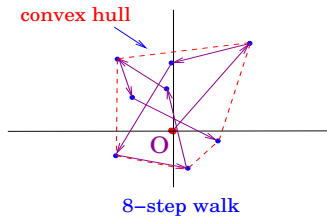
- Continuous-time limit: **Brownian path** of duration T

Correlated Points: Vertices of an Open Random Walk



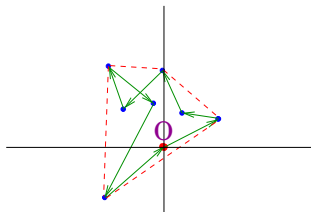
- Continuous-time limit: Brownian path of duration T
- mean perimeter and mean area of the associated Convex hull?

Correlated Points: Vertices of an Open Random Walk

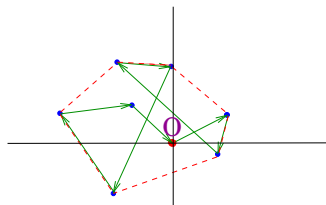


- Continuous-time limit: **Brownian path** of duration T
- **mean perimeter** and **mean area** of the associated **Convex hull**?
- **mean perimeter**: $\langle L_1 \rangle = \sqrt{8\pi T}$ (Takács, '80)
- **mean area**: $\langle A_1 \rangle = \frac{\pi}{2} T$ (El Bachir, '83, Letac '93)

Correlated Points: Vertices of a Closed Random Walk



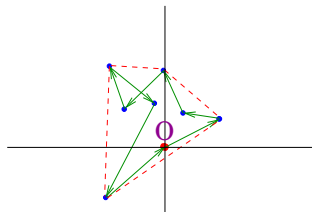
8 step random bridge



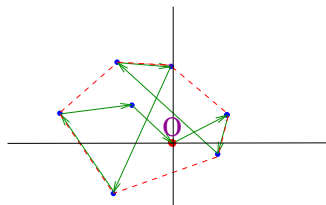
another 8 step bridge

- Continuous-time limit: **Brownian bridge** of duration T : starting at O and returning to it after time T

Correlated Points: Vertices of a Closed Random Walk



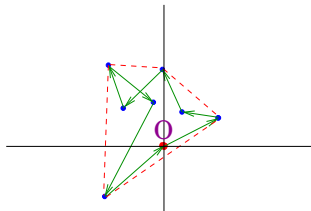
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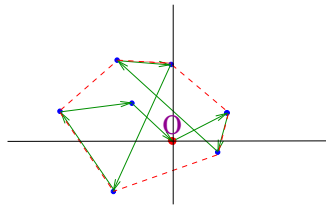
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- **mean perimeter**: $\langle L_1 \rangle = \sqrt{\frac{\pi^3}{2}} T$ (Goldman, '96).

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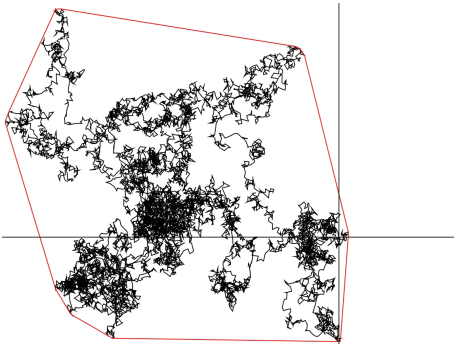
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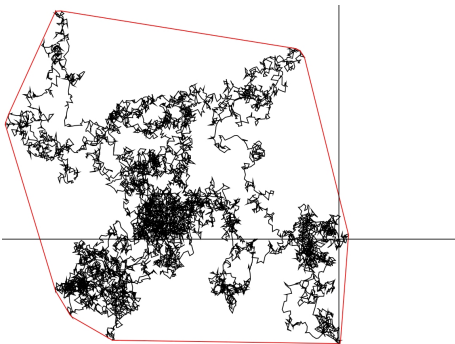
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Home Range Estimate via Convex Hull



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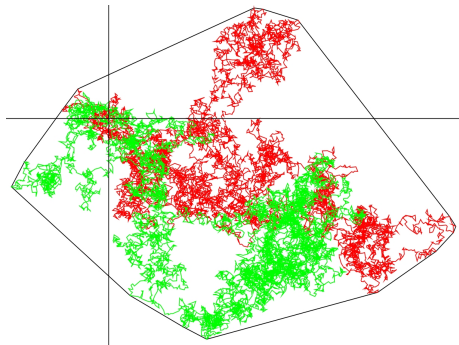
Models of home range for animal movement, Worton (1987)

Integrating Scientific Methods with Habitat Conservation Planning, Murphy and Noon (1992)

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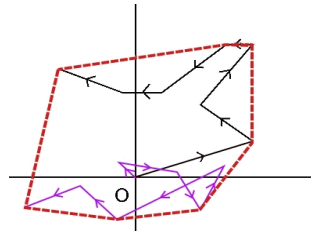
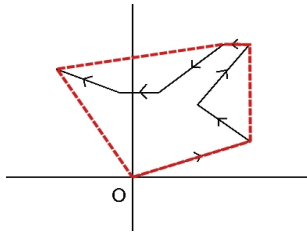
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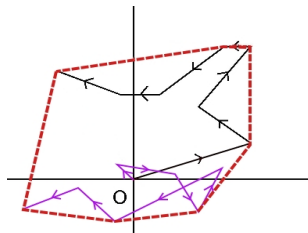
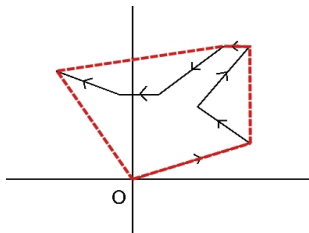
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Global Convex Hull of n Independent Brownian Paths

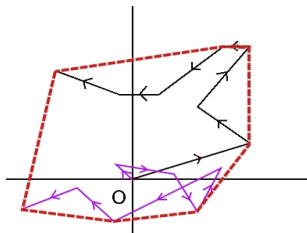
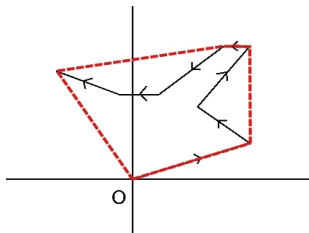


Global Convex Hull of n Independent Brownian Paths



- Mean **perimeter** $\langle L_n \rangle$ and mean **area** $\langle A_n \rangle$ of n independent Brownian paths (bridges) each of duration T ?

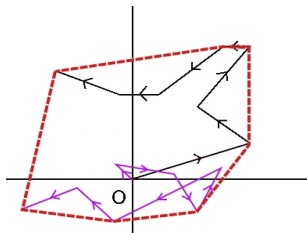
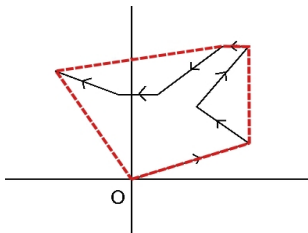
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Global Convex Hull of n Independent Brownian Paths



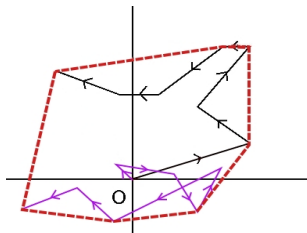
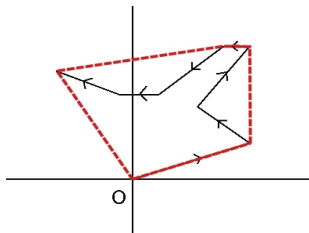
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- $\langle L_n \rangle = \alpha_n \sqrt{T}$; $\langle A_n \rangle = \beta_n T$

- Recall $\boxed{\alpha_1 = \sqrt{8\pi}, \beta_1 = \pi/2}$ (open path)

$$\boxed{\alpha_1 = \sqrt{\pi^3/2}, \beta_1 = ?} \text{ (closed path)}$$

Global Convex Hull of n Independent Brownian Paths



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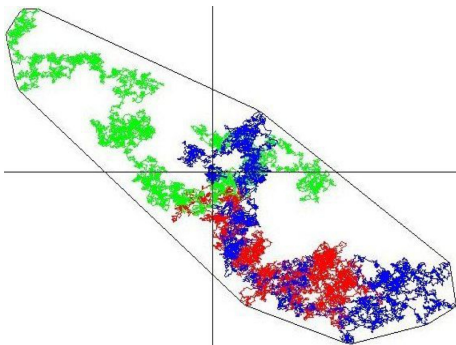
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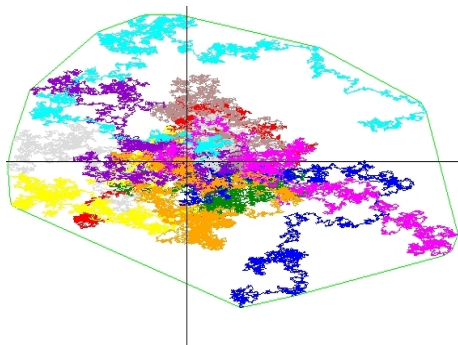
$$\alpha_1 = \sqrt{\pi^3/2}, \quad \beta_1 = ? \quad (\text{closed path})$$

- $\alpha_n, \beta_n = ? \rightarrow$ both for open and closed paths $\rightarrow n$ -dependence?

Global Convex Hull of n Independent Brownian Paths

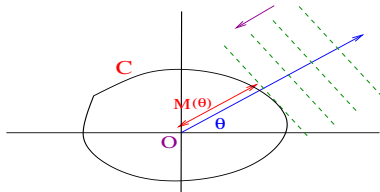


$n = 3$ closed paths



$n = 10$ open paths

Cauchy's Formulae for a Closed Convex Curve

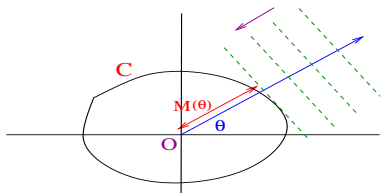


C : CLOSED CONVEX CURVE

- For any point $[X(s), Y(s)]$ on C define:

Support function: $M(\theta) = \max_{s \in C} [X(s) \cos(\theta) + Y(s) \sin(\theta)]$

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- Perimeter:

$$L = \int_0^{2\pi} d\theta M(\theta)$$

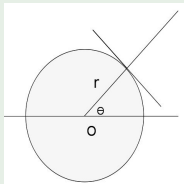
- Area:

$$A = \frac{1}{2} \int_0^{2\pi} d\theta [M^2(\theta) - [M'(\theta)]^2]$$

Examples

a circle centered at the origin:

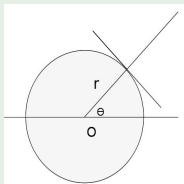
$$M(\theta) = r$$



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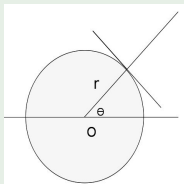
$$L = \int_0^{2\pi} d\theta M(\theta) = 2\pi r$$

$$A = \frac{1}{2} \int_0^{2\pi} d\theta \left[M^2(\theta) - [M'(\theta)]^2 \right] = \pi r^2$$

Examples

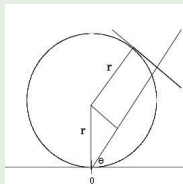
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a circle touching the origin:

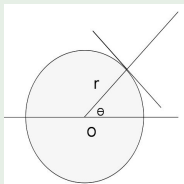
$$M(\theta) = r(1 + \sin \theta)$$



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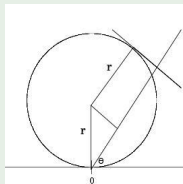
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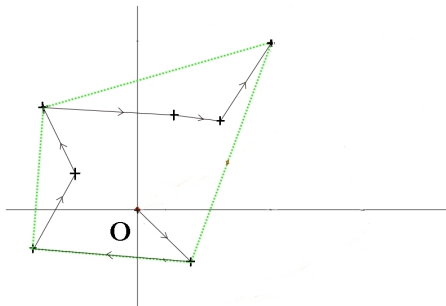
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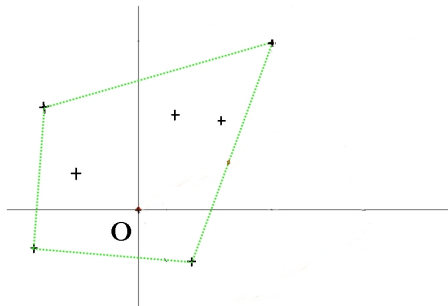
Cauchy's formulae Applied to Convex Polygon



Let $(x_k, y_k) \in I \Rightarrow$ vertices of an N -step random walk starting at O

Let C (green) be the associated **Convex Hull**

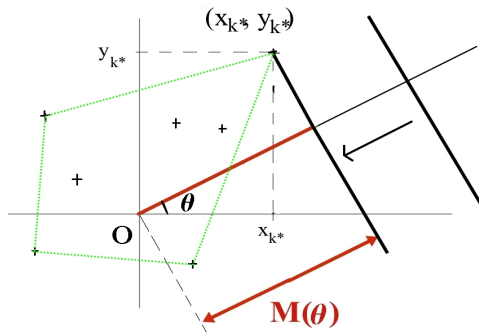
Cauchy's formulae Applied to Convex Polygon



$(x_k, y_k) \in I \implies$ vertices of the walk

$C \rightarrow$ Convex Hull with coordinates $\{X(s), Y(s)\}$ on C

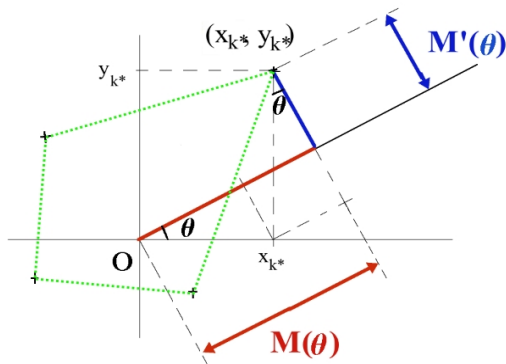
Cauchy's formulae Applied to Convex Polygon



$$\begin{aligned} M(\theta) &= \max_{s \in C} [X(s) \cos \theta + Y(s) \sin \theta] \\ &= \max_{k \in I} [x_k \cos \theta + y_k \sin \theta] \\ &= x_{k^*} \cos \theta + y_{k^*} \sin \theta \end{aligned}$$

$k^* \rightarrow$ label of the point with largest projection along θ

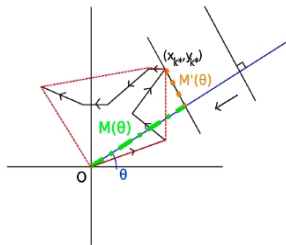
Support Function of a Convex Hull



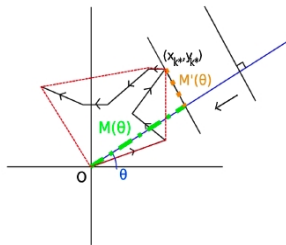
$$M(\theta) = x_{k*} \cos \theta + y_{k*} \sin \theta$$

$$M'(\theta) = -x_{k*} \sin \theta + y_{k*} \cos \theta$$

Cauchy's Formulae Applied to Random Convex Hull



Cauchy's Formulae Applied to Random Convex Hull



Mean perimeter of a random convex polygon

$$\langle L \rangle = \int_0^{2\pi} d\theta \langle M(\theta) \rangle$$

$$\text{with } M(\theta) = x_k^* \cos \theta + y_k^* \sin \theta$$

Mean area of a random convex polygon

$$\langle A \rangle = \frac{1}{2} \int_0^{2\pi} d\theta \left[\langle M^2(\theta) \rangle - \langle [M'(\theta)]^2 \rangle \right]$$

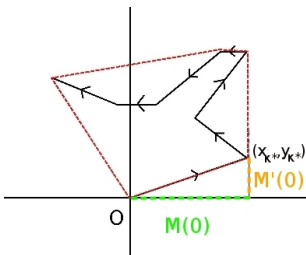
$$\text{with } M'(\theta) = -x_k^* \sin \theta + y_k^* \cos \theta$$

Isotropically Distributed Points

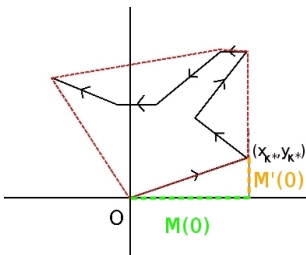
Mean Perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

$$\text{with } M(\theta = 0) = \max_{k \in I} \{x_k\} = x_{k^*}$$



Isotropically Distributed Points



Mean Perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

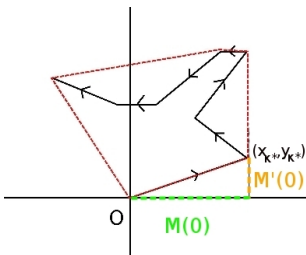
$$\text{with } M(\theta = 0) = \max_{k \in I} \{x_k\} = x_{k*}$$

Mean Area

$$\langle A \rangle = \pi \left[\langle M^2(0) \rangle - \langle [M'(0)]^2 \rangle \right]$$

$$\text{with } M'(\theta = 0) = y_{k*}$$

Isotropically Distributed Points



Mean Perimeter

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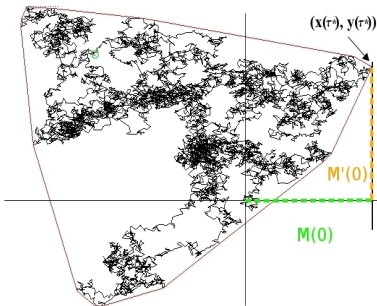
Mean Area

$$\langle A \rangle = \pi \left[\langle M^2(0) \rangle - \langle [M'(0)]^2 \rangle \right]$$

$$\text{with } M'(\theta = 0) = y_{k^*}$$

⇒ Link to **Extreme Value Statistics**

Cauchy's Formulae Applied to the Convex Hull of a Brownian Path (n=1)



$x(\tau), y(\tau) \rightarrow$ a pair of independent **one-dimensional** Brownian motions: $0 \leq \tau \leq T$

$$\frac{dx}{d\tau} = \eta_x(\tau)$$

$$\frac{dy}{d\tau} = \eta_y(\tau)$$

Mean Perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

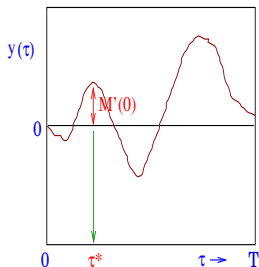
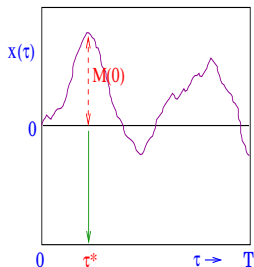
$$\text{with } M(0) = \max_{0 \leq \tau \leq T} \{x(\tau)\} \equiv x(\tau^*)$$

Mean Area

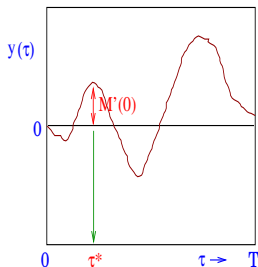
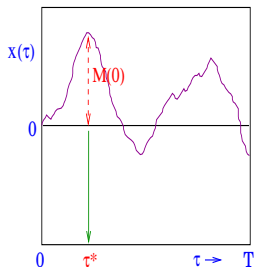
$$\langle A \rangle = \pi \left[\langle M^2(0) \rangle - \langle [M'(0)]^2 \rangle \right]$$

$$\text{with } M'(0) = y(\tau^*)$$

$M'(0) \rightarrow$ value of y at the special time τ^* when $x(\tau)$ is maximal

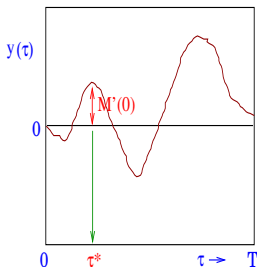
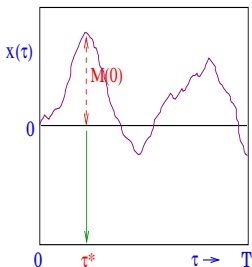


$M'(0) \rightarrow$ value of y at the special time τ^* when $x(\tau)$ is maximal



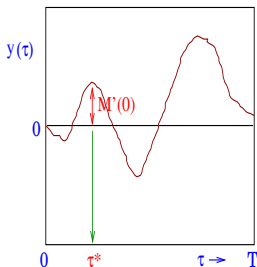
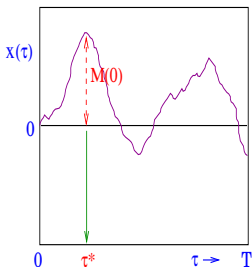
- $\langle M(0) \rangle = \int_0^\infty dM M \sigma_1(M|T); \quad \langle M^2(0) \rangle = \int_0^\infty dM M^2 \sigma_1(M|T)$

$M'(0) \rightarrow$ value of y at the special time τ^* when $x(\tau)$ is maximal



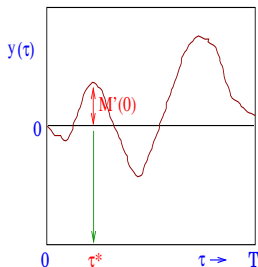
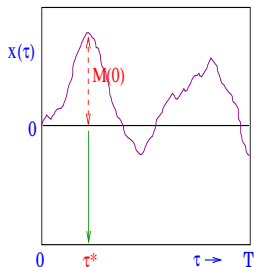
- $\langle M(0) \rangle = \int_0^\infty dM M \sigma_1(M|T)$; $\langle M^2(0) \rangle = \int_0^\infty dM M^2 \sigma_1(M|T)$
- $\sigma_1(M|T) \rightarrow$ prob. density of maximum $M(0)$ of $x(\tau)$ in $[0, T]$

$M'(0) \rightarrow$ value of y at the special time τ^* when $x(\tau)$ is maximal



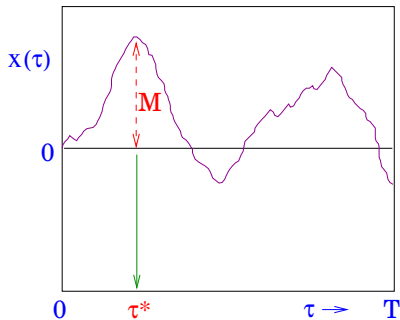
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- $\langle [M'(0)]^2 \rangle = \int_0^T d\tau^* \rho_1(\tau^*|T) \langle y^2(\tau^*) \rangle$

$M'(0) \rightarrow$ value of y at the special time τ^* when $x(\tau)$ is maximal

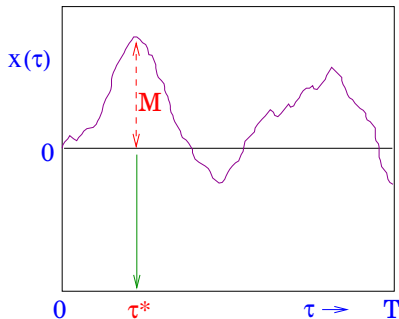


- $\langle M(0) \rangle = \int_0^\infty dM M \sigma_1(M|T)$; $\langle M^2(0) \rangle = \int_0^\infty dM M^2 \sigma_1(M|T)$
- $\sigma_1(M|T) \rightarrow$ prob. density of maximum $M(0)$ of $x(\tau)$ in $[0, T]$
- $\langle [M'(0)]^2 \rangle = \int_0^T d\tau^* \rho_1(\tau^*|T) \langle y^2(\tau^*) \rangle = \langle \tau^* \rangle$

Distribution of M and τ^* for a single Brownian Path

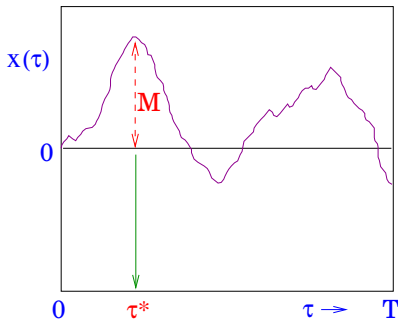


Distribution of M and τ^* for a single Brownian Path



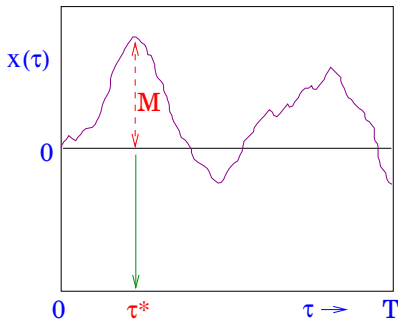
- Joint Distribution: $P_1(M, \tau^* | T) = \frac{M}{\pi \tau^{*3/2} \sqrt{T - \tau^*}} e^{-M^2/2\tau^*}$

Distribution of M and τ^* for a single Brownian Path



- Joint Distribution: $P_1(M, \tau^* | T) = \frac{M}{\pi \tau^{*3/2} \sqrt{T - \tau^*}} e^{-M^2/2\tau^*}$
- Marginals: $\sigma_1(M | T) = \sqrt{\frac{2}{\pi T}} e^{-M^2/2T}$

Distribution of M and τ^* for a single Brownian Path



- Joint Distribution: $P_1(M, \tau^* | T) = \frac{M}{\pi \tau^{*3/2} \sqrt{T - \tau^*}} e^{-M^2/2\tau^*}$

- Marginals: $\sigma_1(M | T) = \sqrt{\frac{2}{\pi T}} e^{-M^2/2T}$

$$\rho_1(\tau^* | T) = \frac{1}{\pi \sqrt{\tau^*(T - \tau^*)}} \rightarrow \text{Lévy's arcsine law}$$

Distribution of the time τ^* at which a Brownian Motion is maximal over $[0, T]$

Lévy's Arcsine Law: $\rho_1(\tau^* | T) = \frac{1}{T} f_1\left(\frac{\tau^*}{T}\right)$

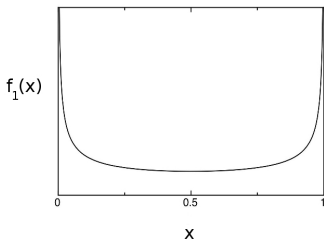
$$f_1(x) = \frac{1}{\pi \sqrt{x(1-x)}}$$

Distribution of the time τ^* at which a Brownian Motion is maximal over $[0, T]$

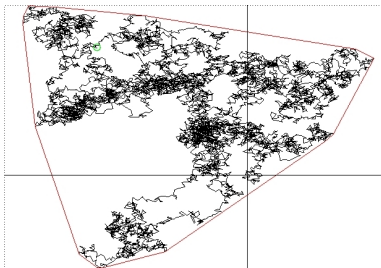
Lévy's Arcsine Law: $\rho_1(\tau^*|T) = \frac{1}{T} f_1\left(\frac{\tau^*}{T}\right)$

$$f_1(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

Cumulative distribution: $\text{Prob}(\tau^* \leq t|T) = \frac{2}{\pi} \arcsin\left(\sqrt{t}\right)$



Results for $n=1$ Open Brownian Path



$x(\tau), y(\tau) \rightarrow$ a pair of
independent one-dimensional
Brownian motions over
 $0 \leq \tau \leq T$

Mean Perimeter

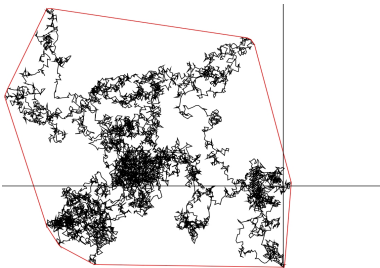
$$\langle L \rangle = \sqrt{8\pi T}$$

Mean Area

$$\langle A \rangle = \frac{\pi T}{2}$$

Takács, *Expected perimeter length*, Amer. Math. Month., **87** (1980)
El Bachir, (1983)

Results for $n=1$ Closed Brownian Path



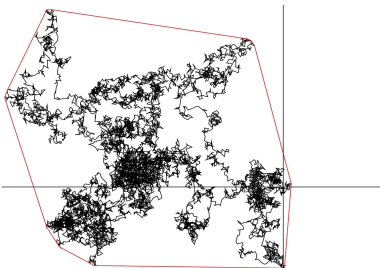
Mean Perimeter

$$\langle L \rangle = \sqrt{\frac{\pi^3 T}{2}}$$

Goldman, '96

$x(\tau), y(\tau) \rightarrow$ a pair of
independent one-dimensional
Brownian bridges over $0 \leq \tau \leq T$

Results for $n=1$ Closed Brownian Path



$x(\tau), y(\tau) \rightarrow$ a pair of independent one-dimensional Brownian bridges over $0 \leq \tau \leq T$

Mean Perimeter

$$\langle L \rangle = \sqrt{\frac{\pi^3 T}{2}}$$

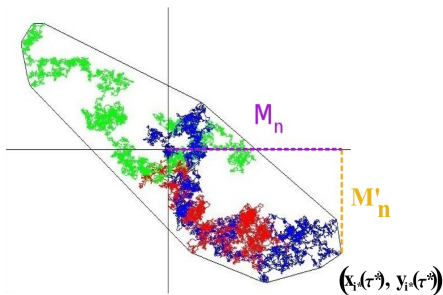
Goldman, '96

Mean Area

$$\langle A \rangle = \frac{\pi T}{3}$$

$\rightarrow$ New Result

Convex Hull of n Independent Brownian Paths



$x_i(\tau), y_i(\tau) \rightarrow 2n$ independent one-dimensional Brownian paths each of duration T

Mean Perimeter

$$\langle L_n \rangle = 2\pi \langle M_n \rangle$$

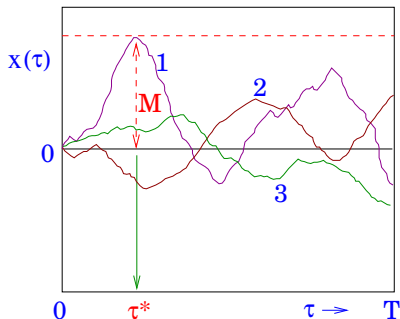
$$\text{with } M_n = \max_{\tau, i} \{x_i(\tau)\} \equiv x_{i^*}(\tau^*)$$

Mean Area

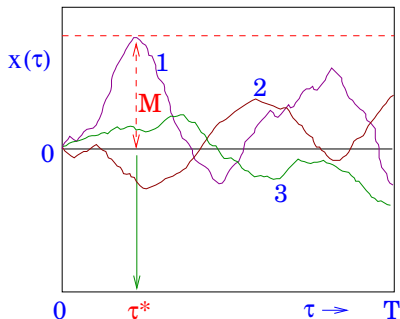
$$\langle A_n \rangle = \pi \left[\langle M_n^2 \rangle - \langle [M'_n]^2 \rangle \right]$$

$$\text{with } M'_n = y_{i^*}(\tau^*)$$

Distribution of the global maximum M and τ^* for n paths



Distribution of the global maximum M and τ^* for n paths



- Joint Distribution: $P_n(M, \tau^* | T) = n P_1(M, \tau^* | T) \left[\operatorname{erf} \left(\frac{M}{\sqrt{2T}} \right) \right]^{n-1}$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z du e^{-u^2}$$

Marginals of M and τ^* for arbitrary n

- Marginals: $\sigma_n(M|T) = \sqrt{\frac{2}{\pi T}} n e^{-M^2/2T} \left[\operatorname{erf} \left(\frac{M}{\sqrt{2T}} \right) \right]^{n-1}$

Marginals of M and τ^* for arbitrary n

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 $\rho_n(\tau^*|T) = \frac{1}{T} f_n(\tau^*/T)$

Marginals of M and τ^* for arbitrary n

- Marginals: $\sigma_n(M|T) = \sqrt{\frac{2}{\pi T}} n e^{-M^2/2T} \left[\operatorname{erf} \left(\frac{M}{\sqrt{2T}} \right) \right]^{n-1}$

$$\rho_n(\tau^*|T) = \frac{1}{T} f_n(\tau^*/T)$$

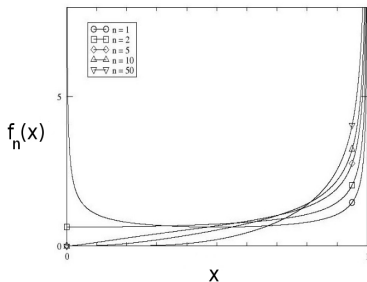
$$f_n(x) = \frac{2n}{\pi \sqrt{x(1-x)}} \int_0^\infty u e^{-u^2} [\operatorname{erf}(u\sqrt{x})]^{n-1} du$$

Marginals of M and τ^* for arbitrary n

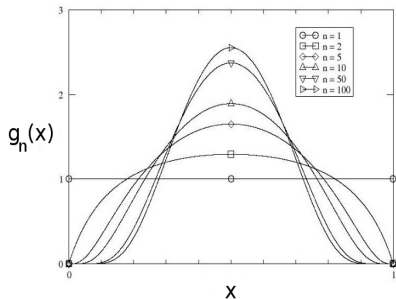
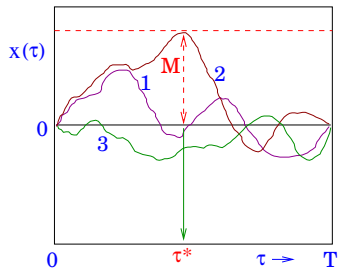
- Marginals: $\sigma_n(M|T) = \sqrt{\frac{2}{\pi T}} n e^{-M^2/2T} \left[\operatorname{erf} \left(\frac{M}{\sqrt{2T}} \right) \right]^{n-1}$

$$\rho_n(\tau^*|T) = \frac{1}{T} f_n(\tau^*/T)$$

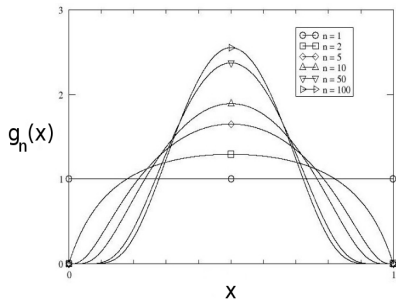
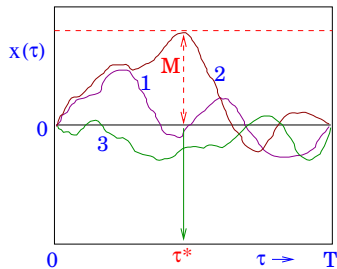
$$f_n(x) = \frac{2n}{\pi \sqrt{x(1-x)}} \int_0^\infty u e^{-u^2} [\operatorname{erf}(u\sqrt{x})]^{n-1} du$$



Marginals for n Independent Brownian Bridges

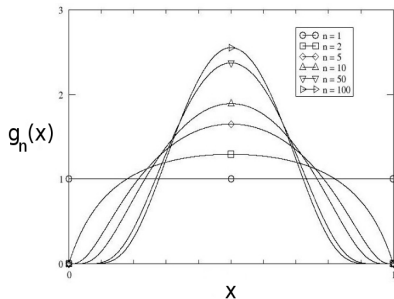
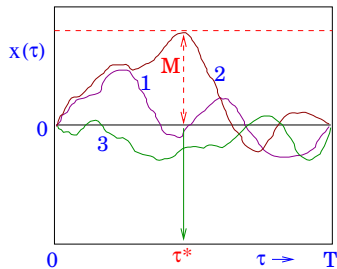


Marginals for n Independent Brownian Bridges



- Marginals: $\sigma_n(M|T) = \frac{4n}{T} M \left(1 - e^{-2M^2/T}\right)^{n-1}$

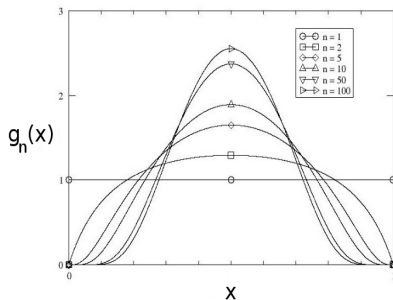
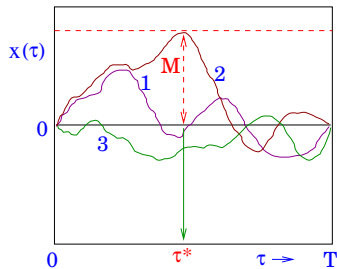
Marginals for n Independent Brownian Bridges



• Marginals: $\sigma_n(M|T) = \frac{4n}{T} M \left(1 - e^{-2M^2/T}\right)^{n-1}$

$$\rho_n(\tau^*|T) = \frac{1}{T} g_n(\tau^*/T)$$

Marginals for n Independent Brownian Bridges

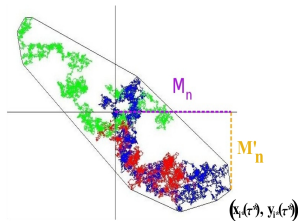


- Marginals: $\sigma_n(M|T) = \frac{4n}{T} M \left(1 - e^{-2M^2/T}\right)^{n-1}$

$$\rho_n(\tau^*|T) = \frac{1}{T} g_n(\tau^*/T)$$

$$g_n(x) = n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{[1 + 4kx(1-x)]^{\frac{3}{2}}}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



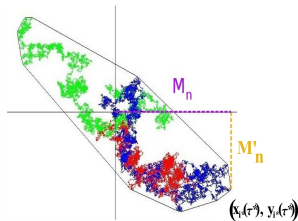
$x_i(\tau), y_i(\tau) \rightarrow 2n$
independent
one-dimensional Brownian
paths over $0 \leq \tau \leq T$

Mean Perimeter (open paths)

$$\langle L_n \rangle = 2\pi \langle M_n \rangle$$

$$\text{with } M_n = \max_{\tau, i} \{x_i(\tau)\} \equiv x_{i^*}(\tau^*)$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



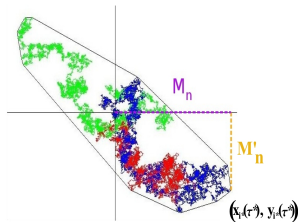
$x_i(\tau), y_i(\tau) \rightarrow 2n$
 independent
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 paths over $0 \leq \tau \leq T$

Mean Perimeter (open paths)

$$\langle L_n \rangle = \alpha_n \sqrt{T}$$

$$\alpha_n = 4n\sqrt{2\pi} \int_0^\infty du \, u \, e^{-u^2} [\operatorname{erf}(u)]^{n-1}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



$x_i(\tau), y_i(\tau) \rightarrow 2n$
 independent
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Mean Perimeter (open paths)

$$\langle L_n \rangle = \alpha_n \sqrt{T}$$

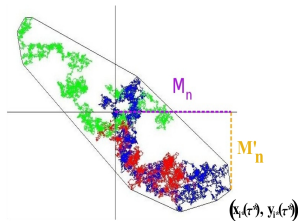
$$\alpha_n = 4n\sqrt{2\pi} \int_0^\infty du \, u \, e^{-u^2} [\operatorname{erf}(u)]^{n-1}$$

$$\alpha_1 = \sqrt{8\pi} = 5,013..$$

$$\alpha_2 = 4\sqrt{\pi} = 7,089..$$

$$\alpha_3 = 24 \frac{\tan^{-1}(1/\sqrt{2})}{\sqrt{\pi}} = 8,333..$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



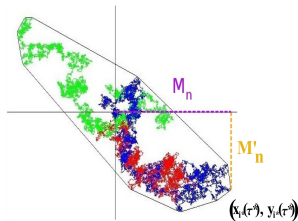
$x_i(\tau), y_i(\tau) \rightarrow 2n$
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Mean Area (open paths)

$$\langle A_n \rangle = \pi \left[\langle M_n^2 \rangle - \langle [M'_n]^2 \rangle \right]$$

with $M'_n = y_{i^*}(\tau^*)$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



$x_i(\tau), y_i(\tau) \rightarrow 2n$
 independent
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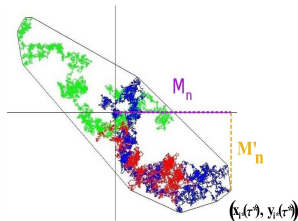
Mean Area (open paths)

$$\langle A_n \rangle = \beta_n T$$

$$\beta_n = 4n \sqrt{\pi} \int_0^\infty du \, u [\operatorname{erf}(u)]^{n-1} (ue^{-u^2} - h(u))$$

$$h(u) = \frac{1}{2\sqrt{\pi}} \int_0^1 \frac{e^{-u^2/t} dt}{\sqrt{t(1-t)}}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



$x_i(\tau), y_i(\tau) \rightarrow 2n$
independent
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Mean Area (open paths)

$$\langle A_n \rangle = \beta_n T$$

$$\beta_n = 4n\sqrt{\pi} \int_0^\infty du \, u [\operatorname{erf}(u)]^{n-1} (ue^{-u^2} - h(u))$$

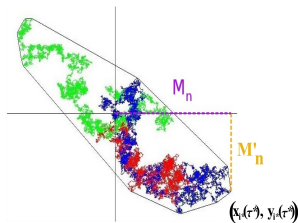
$$h(u) = \frac{1}{2\sqrt{\pi}} \int_0^1 \frac{e^{-u^2/t} dt}{\sqrt{t(1-t)}}$$

$$\beta_1 = \frac{\pi}{2} = 1,570..$$

$$\beta_2 = \pi = 3,141..$$

$$\beta_3 = \pi + 3 - \sqrt{3} = 4,409..$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



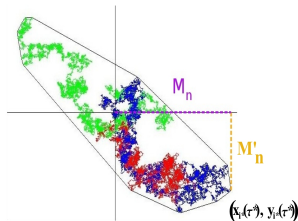
$x_i(\tau), y_i(\tau) \rightarrow 2n$
 independent
 one-dimensional Brownian
 bridges over $0 \leq \tau \leq T$

Mean Perimeter (Closed Paths)

$$\langle L_n^c \rangle = \alpha_n^c \sqrt{T}$$

$$\alpha_n^c = \frac{\pi^{3/2}}{\sqrt{2}} \sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k+1}}{\sqrt{k}}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



$x_i(\tau), y_i(\tau) \rightarrow 2n$
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Mean Perimeter (Closed Paths)

$$\langle L_n^c \rangle = \alpha_n^c \sqrt{T}$$

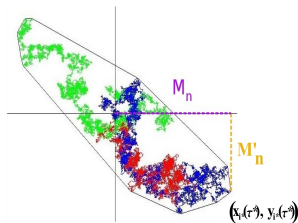
$$\alpha_n^c = \frac{\pi^{3/2}}{\sqrt{2}} \sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k+1}}{\sqrt{k}}$$

$$\alpha_1^c = \sqrt{\pi^3/2} = 3,937.$$

$$\alpha_2^c = \sqrt{\pi^3}(\sqrt{2} - 1/2) = 5,090..$$

$$\alpha_3^c = \sqrt{\pi^3} \left(\frac{3}{\sqrt{2}} - \frac{3}{2} + \frac{1}{\sqrt{6}} \right) = 5,732..$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



$x_i(\tau), y_i(\tau) \rightarrow 2n$
 independent
 one-dimensional Brownian
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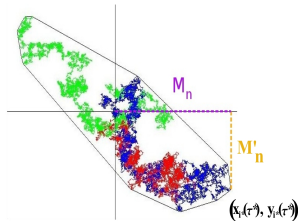
Mean Area (Closed Paths)

$$\langle A_n^c \rangle = \beta_n^c T$$

$$\beta_n^c = \frac{\pi}{2} \left[\sum_{k=1}^n \frac{1}{k} - \frac{n}{3} + \frac{1}{2} \sum_{k=2}^n (-1)^k w(k) \right]$$

$$w(k) = \binom{n}{k} (k-1)^{-3/2} (k \tan^{-1}(\sqrt{k-1}) - \sqrt{k-1})$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



$x_i(\tau), y_i(\tau) \rightarrow 2n$
 independent
 one-dimensional Brownian
 bridges over $0 \leq \tau \leq T$

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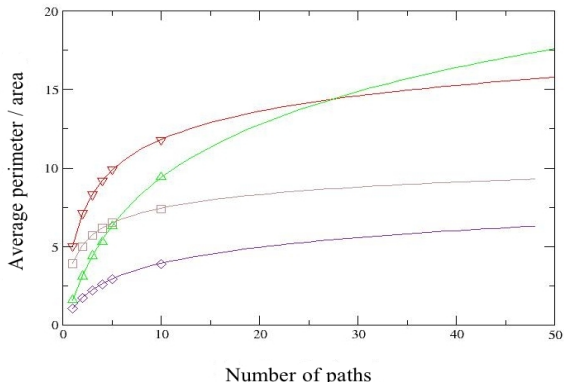
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$$\beta_1^c = \frac{\pi}{3} = 1,047..$$

$$\beta_2^c = \frac{\pi(4+3\pi)}{24} = 1,757..$$

$$\beta_3^c = 2,250..$$

Numerical Check



The coefficients α_n (mean perimeter) (lower triangle), β_n (mean area) (upper triangle) of n open paths and similarly α_n^c (square) and β_n^c (diamond) for n closed paths, plotted against n . The symbols denote numerical simulations (up to $n = 10$, with 10^3 realisations for each point)

Asymptotics for large n

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- Very slow growth with $n \implies$ good news for conservation

Summary and Conclusion

- Unified approach adapting Cauchy's formulae

⇒ Mean Perimeter and Area of Random Convex Hull

both for Independent and Correlated points

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⇒ Ecological Implication: Home Range Estimate

Very slow (logarithmic) growth of Home Range with population size n

Open Questions

- For n planar Brownian paths each of duration T

Mean no. of Vertices $\langle V_n(T) \rangle \rightarrow ?$

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- 3 dimensions: Random Convex Polytopes ?