Wavelength Selection in the Noisy Stabilized Kuramoto-Sivashinskii Equation

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INTRODUCTION:

Problem is to find stationary state of driven out of equilibrium system.

- System evolves via Langevin dynamics

$$\frac{\partial\psi(x,t)}{\partial t} = K[\psi] + \eta(x,t) \qquad \langle\eta(x,t)\eta(x',t')\rangle = 2\epsilon\delta(x-x')\delta(t-t')$$

Equivalent to Fokker-Planck equation:

$$\frac{\partial P(\psi(x),t)}{\partial t} = -\frac{\delta}{\delta\psi}\left(K[\psi]P - \epsilon\frac{\delta P}{\delta\psi}\right)$$

Stationary state:

$$\epsilon\frac{\delta P}{\delta\psi} = K[\psi]P \Rightarrow P_0(\psi_0 \to \psi_f) \propto \exp\left(\frac{1}{\epsilon}\int_{\psi_0}^{\psi_f}\mathcal{D}\psi K[\psi]\right)$$

- Assume that free energy potential exists and K[ψ] is derivative of the potential:

$$K[\psi] = -\frac{\delta \mathcal{F}[\psi]}{\delta \psi} \Rightarrow P_0[\psi_0 \rightarrow \psi] \propto \exp\left(-\frac{1}{\epsilon}(\mathcal{F}[\psi] - \mathcal{F}[\psi_0])\right)$$

- Boltzmann form for stationary distribution:

$$P[\psi] = \frac{1}{Z}\exp\left(-\frac{\mathcal{F}[\psi]}{\epsilon}\right) \qquad Z = \int \mathcal{D}\psi \exp\left(-\frac{\mathcal{F}[\psi]}{\epsilon}\right)$$

- Ingredients of K[ψ]: Set of stationary state solutions to K[ψ]=0.
  K[ψ] must be non-linear (insoluble) and not derivative of potential.

Hypothesis: Additive stochastic noise distinguishes between stationary states and selects one or a band whose width vanishes in the thermodynamic limit.

Methods: Numerical simulations and stability analysis of stationary states – most stable state is selected??

Simplest model we could think of is stabilized Kuramoto-Sivashinsky equation in 1D with additive stochastic noise:

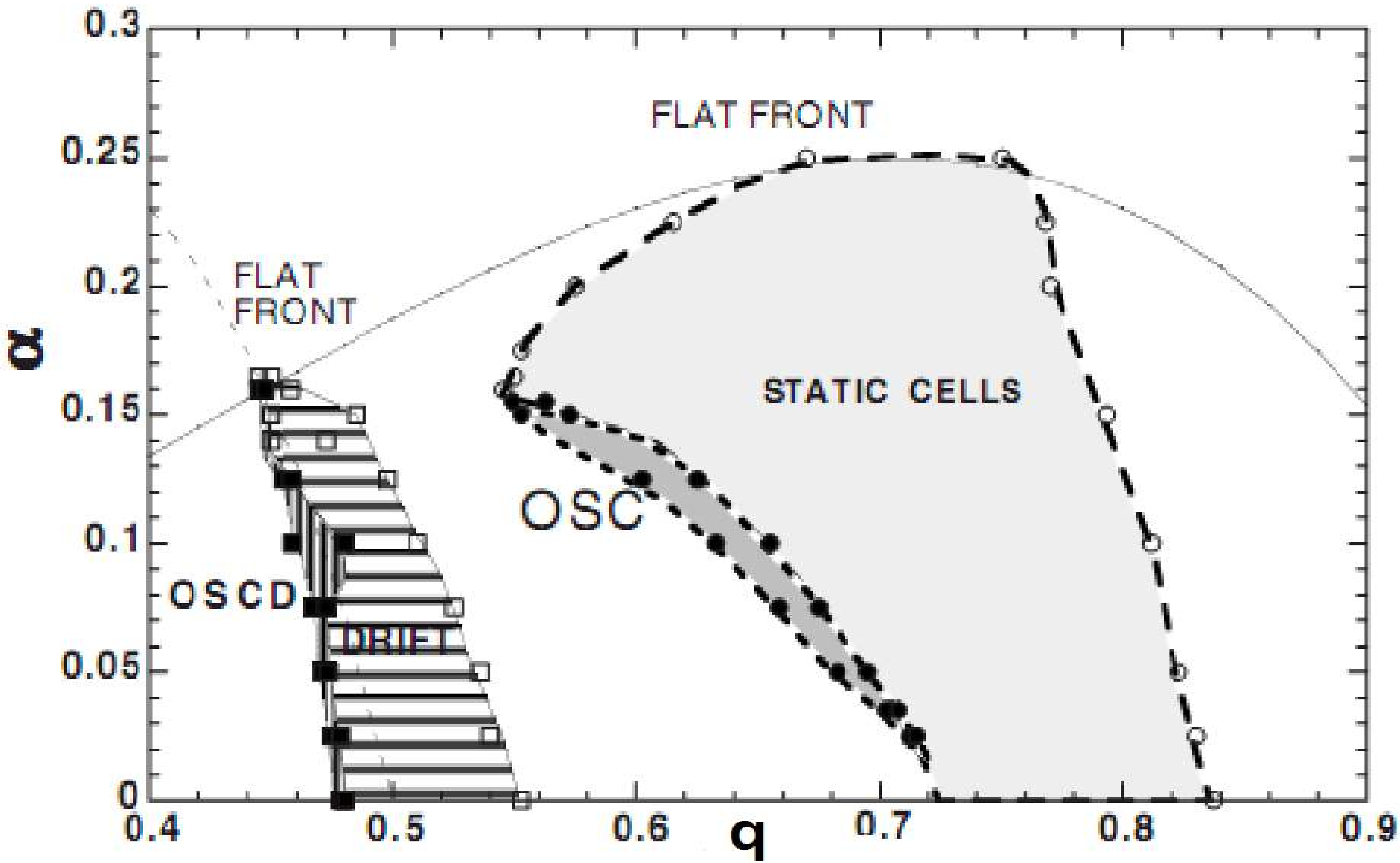
$$\frac{\partial h(x,t)}{\partial t} = -\left(\alpha + \frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right) h(x,t) + \left(\frac{\partial h}{\partial x}\right)^2 + \eta(x,t)$$

$$\dot{h}(q,t) = -\left(\alpha - \frac{1}{4} + \left(q^2 - \frac{1}{2}\right)^2\right) h(q,t) - \int \frac{dk}{2\pi} k(q-k)h(k)h(q-k) + \eta(q,t)$$

Modes are linearly stable if $\alpha > 1/4$ and linearly unstable if $\alpha < 1/4$.

$$1/2 + \sqrt{1/4 - \alpha} \geq q^2 \geq 1/2 - \sqrt{1/4 - \alpha}$$

- Stability diagram of deterministic SKS equation.
- P. Brunet, Phys. Rev. E 76, 017204 (2007)



Numerical simulations:

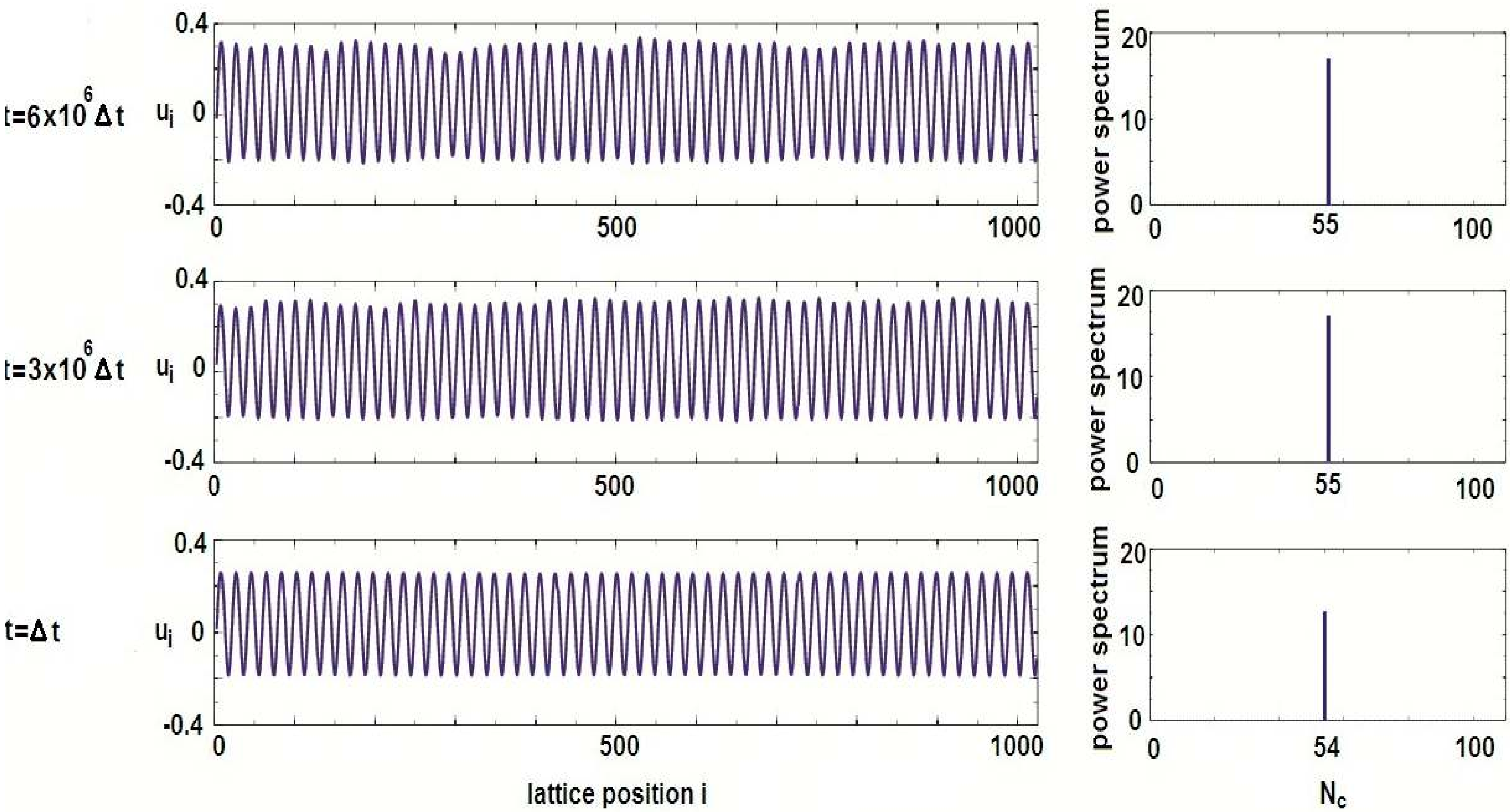
$$h_i^n = h(x_i, t_n) \qquad t_n = n\Delta t \quad x_i = i\Delta x \quad 1 \le i \le N \quad h_{i+N}^n = h_i^n$$

$$h_i^{n+1} = h_i^n + \Delta t K_i^n[h] + \sqrt{\frac{2\epsilon\Delta t}{\Delta x}}\eta_i^n \qquad \langle \eta_i^n \rangle = 0 \quad \langle \eta_i^n \eta_j^m \rangle = \delta_{ij}\delta_{nm}$$

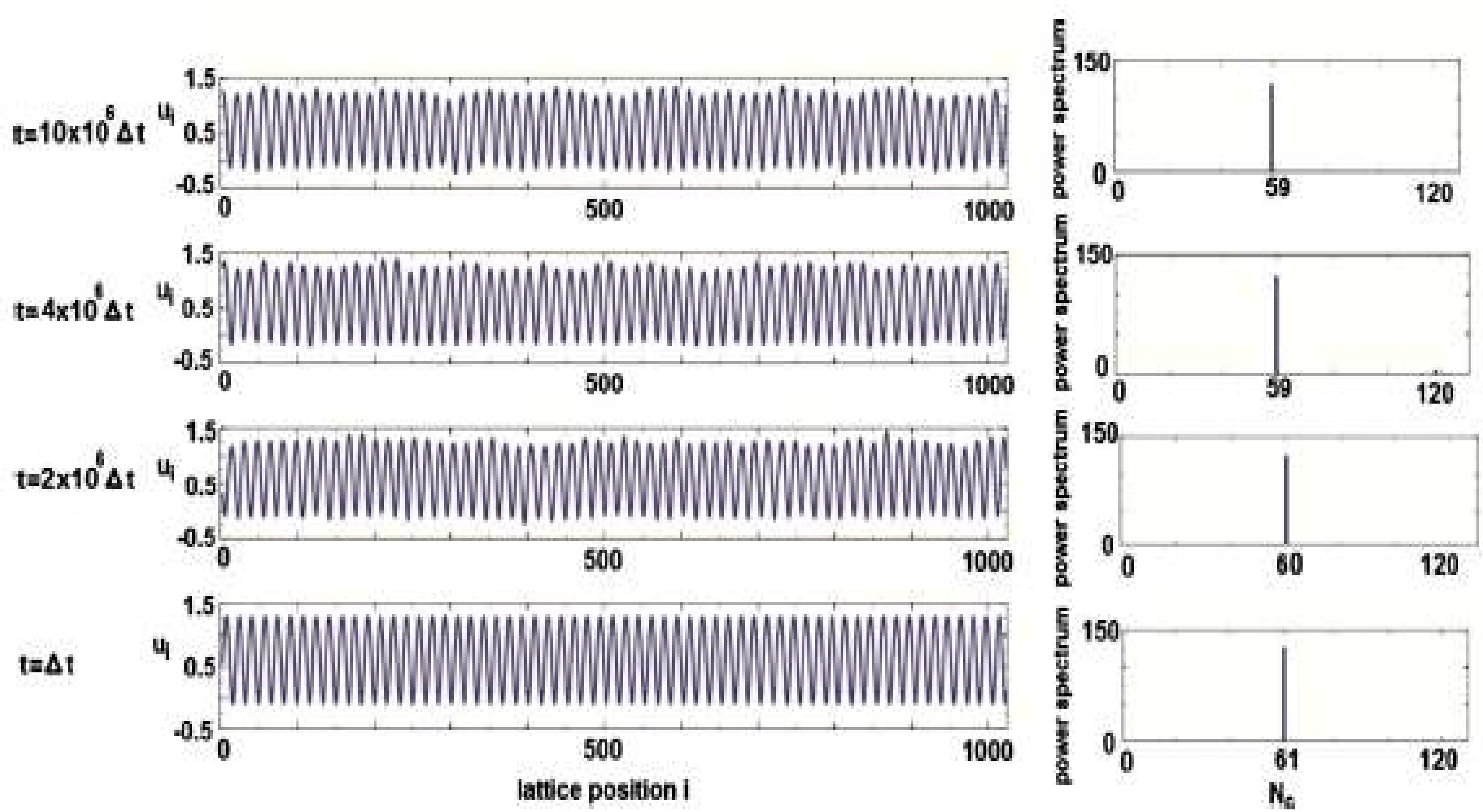
$$K_i^n[h] = -\alpha h_i^n - \frac{1}{(\Delta x)^2}\left(h_{i+1}^n - 2h_i^n + h_{i-1}^n\right) - \frac{1}{(\Delta x)^4}\left(h_{i+2}^n - 4h_{i+1}^n + 6h_i^n - 4h_{i-1}^n + h_{i-2}^n\right) + \frac{1}{4(\Delta x)^2}\left(h_{i+1}^n - h_{i-1}^n\right)^2$$

$$q = \frac{2\pi N_c}{N\Delta x} \quad N = 1024 \quad \Delta x = \frac{1}{2} \quad \Delta t \le C(\Delta x)^4 \quad \Delta t = 0.006$$

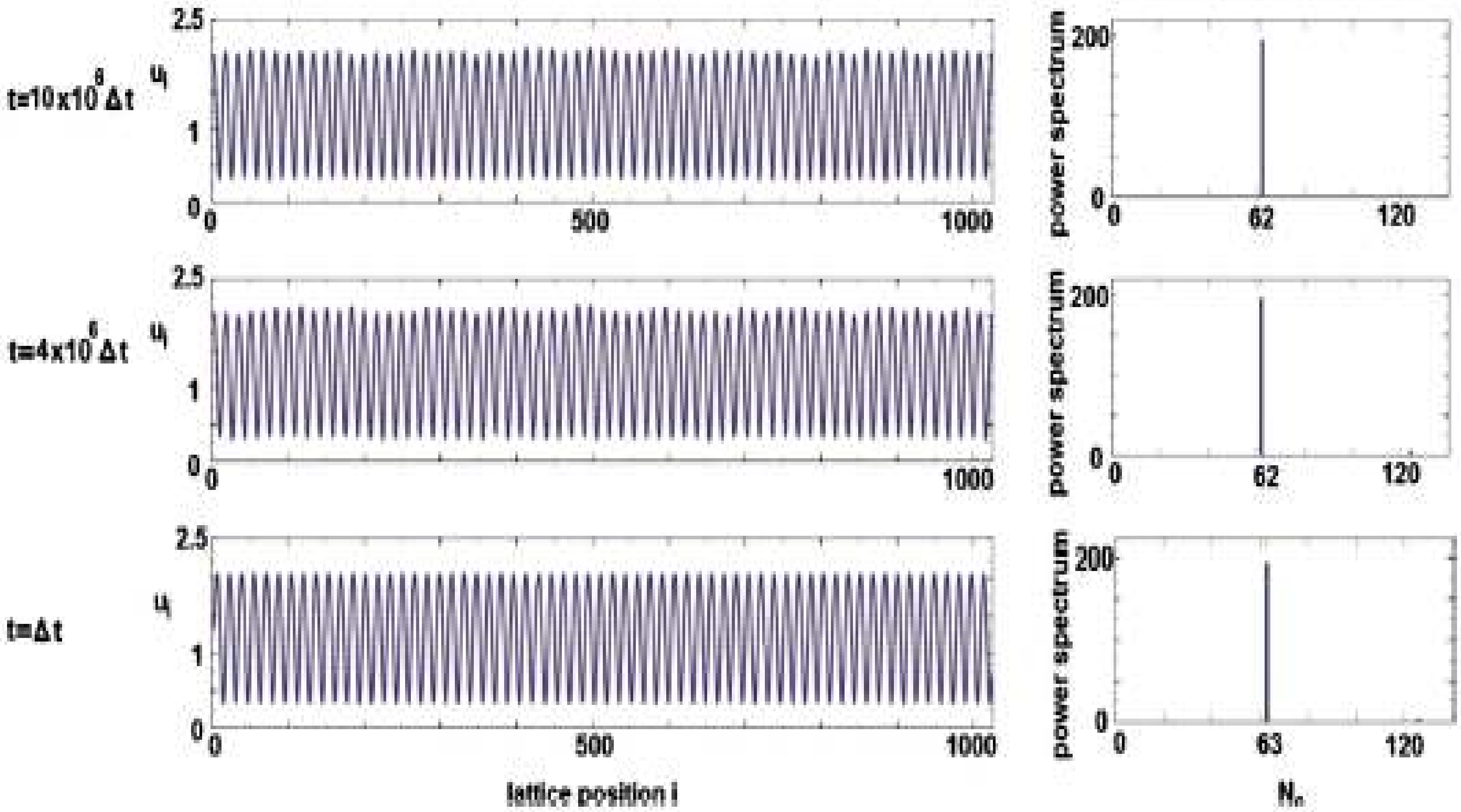
- α=0.24 Eckhaus stable band without noise 53<N_c<61. Simulation done with noise $\varepsilon=0.00001$. Initial state by evolving deterministic equation.



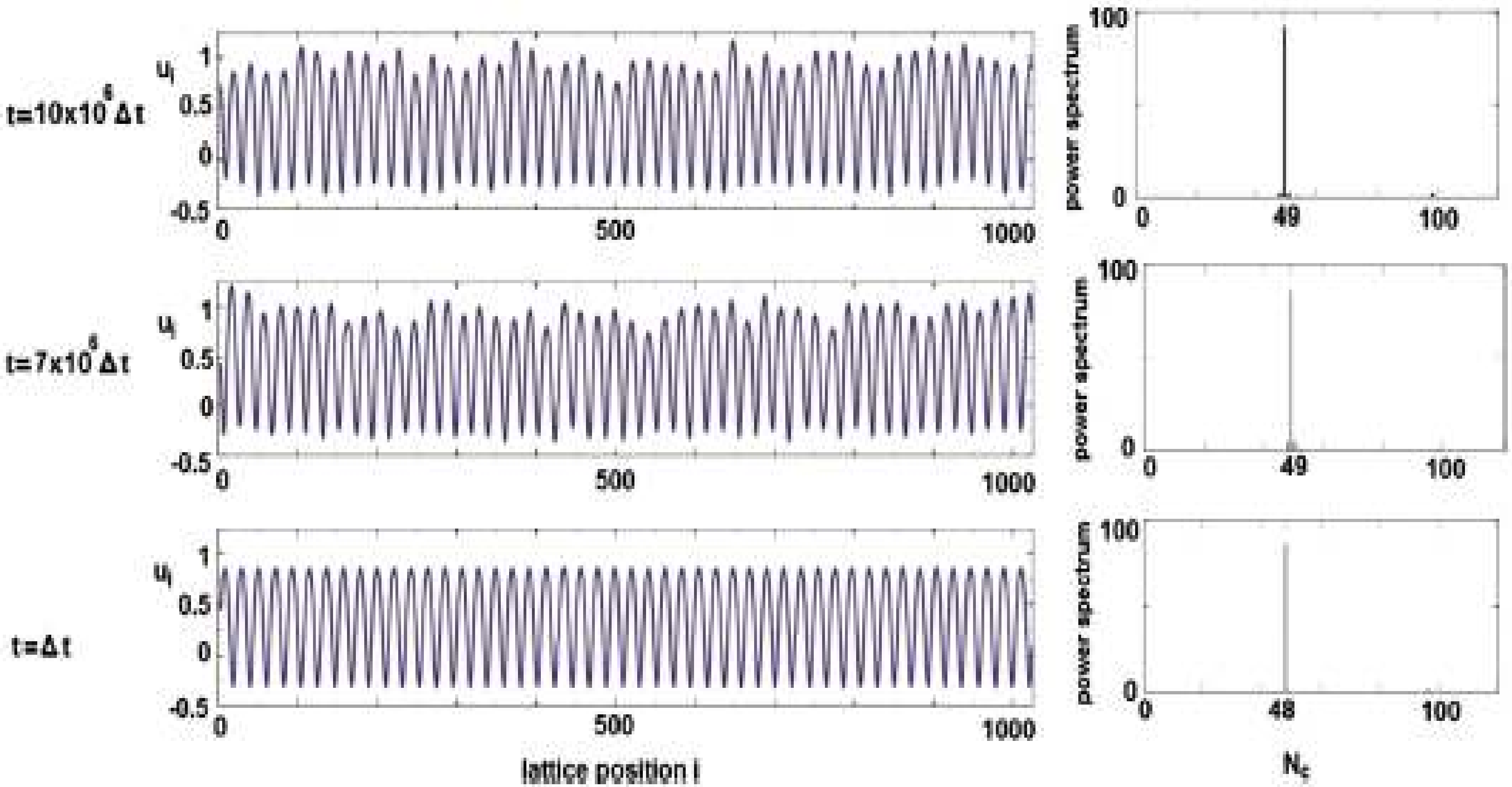
Time evolution of an N_c=61 state, $\alpha=0.20$, noise strength $\varepsilon=0.0005$



Evolution of N_c =63 state with noise ε=0.0005, $\alpha=0.17$



Evolution of N_c =48 state, $\alpha=0.17$, $\varepsilon=0.0005$



Phase diffusion stability: need eigenvalues for perturbations about periodic steady state H(x):

$$h(x,t) = H(x) + v(x,t) \qquad H(x+L) = H(x) \qquad v(x+L) = v(x)$$

$$-\left(\alpha + \partial_x^2 + \partial_x^4\right) H + (\partial_x H)^2 = 0 \qquad v_t = [-\alpha - \partial_x^2 - \partial_x^4 + 2(\partial_x H)\partial_x]v \equiv \mathcal{L}v$$

Eigenvalue equation: $$[\alpha + (\partial_x + \nu)^2 + (\partial_x + \nu)^4 - 2(\partial_x H)(\partial_x + \nu) + \lambda]v = 0$$

$$\lambda^0(\nu) = \lambda^0(0) + \nu\lambda^0_\nu(0) + \frac{1}{2}\nu^2\lambda^0_{\nu\nu}(0)\cdots$$

We find eigenvalues with ``Auto'' software package for system of first order ODE.

$$H \equiv (h, h_x, h_{xx}, h_{xxx}) \qquad H_x = LF(H,c)H \quad \text{where} \quad H(x+1) = H(x)$$

$$F(H,c) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha & (h_x - c) & -1 & 0 \end{pmatrix}$$

Linearize about steady state H(x) and expand to O(ν^2) and solve resulting eigenvalue problems with Auto:

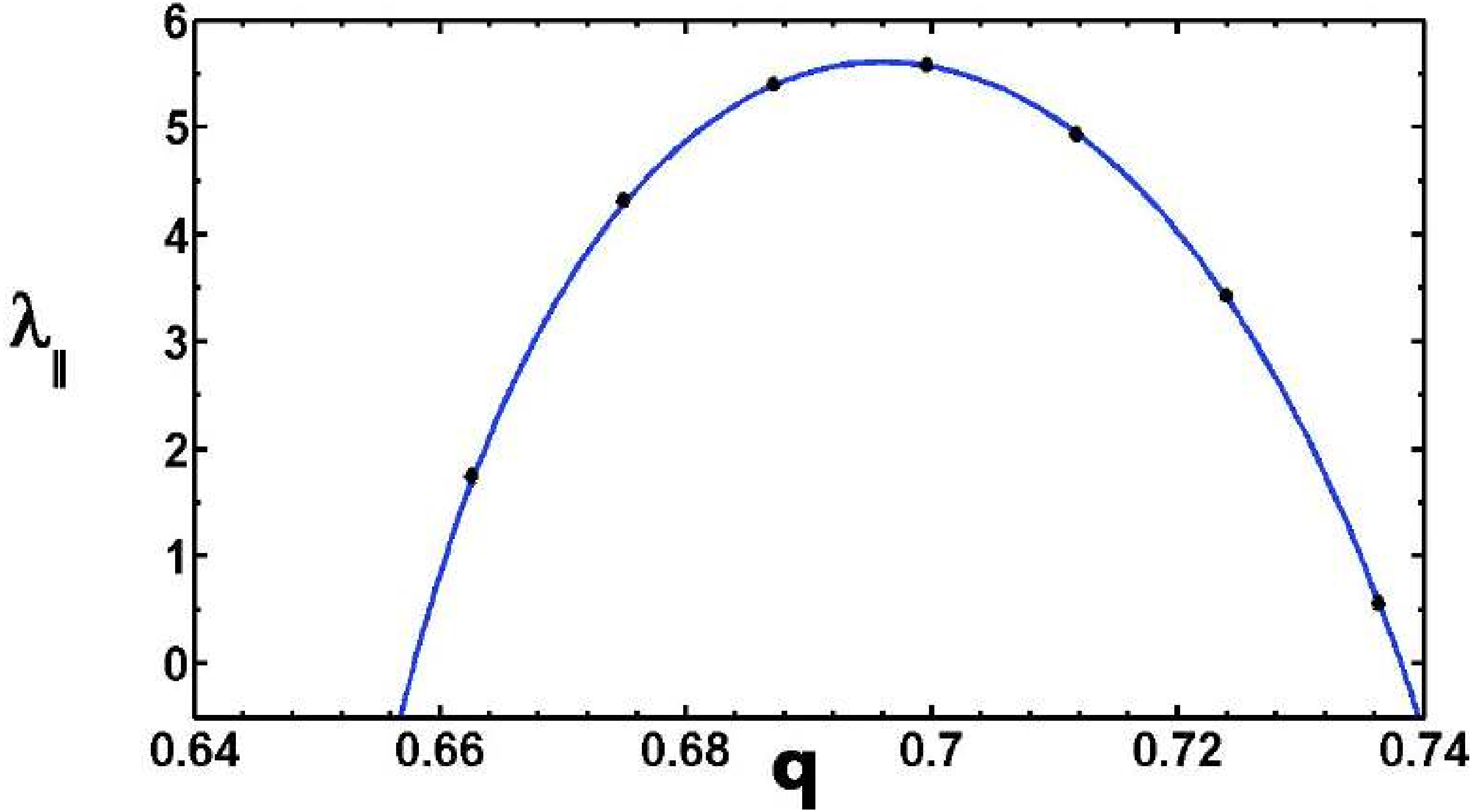
$$H \to H + V \qquad V = (v, v_x, v_{xx}, v_{xxx}) \qquad V_x = L[A + \lambda B - \nu]V$$

$$V_|' = L[AV_| + (\lambda_| B - 1)V] \qquad \lambda_| \equiv \frac{d\lambda^0}{d\nu}|_{\nu=0}$$

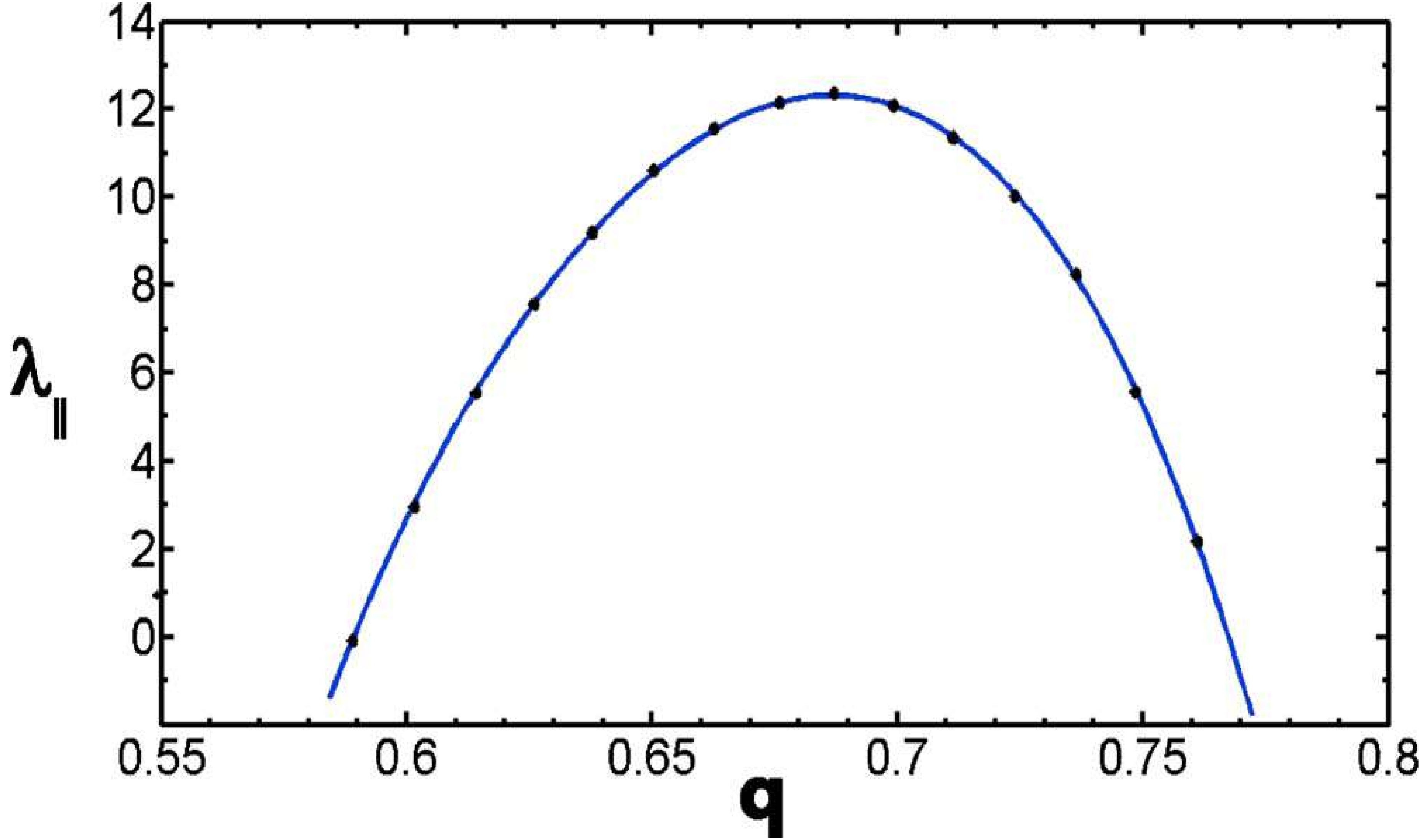
$$V_{||}' = L[AV_{||} + 2(\lambda_| B - 1)V_| + \lambda_{||} BV] \qquad \lambda_{||} \equiv \frac{d^2\lambda^0}{d\nu^2}|_{\nu=0}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha & (2H_x - c) & -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$\lambda_{||}(q)=2D(q)$ for $\alpha=0.24$. Dots are $q=2\pi N_c/N\Delta x$ of simulation with N=1024, Δx=0.50. $\lambda_{||}(max)=5.5$

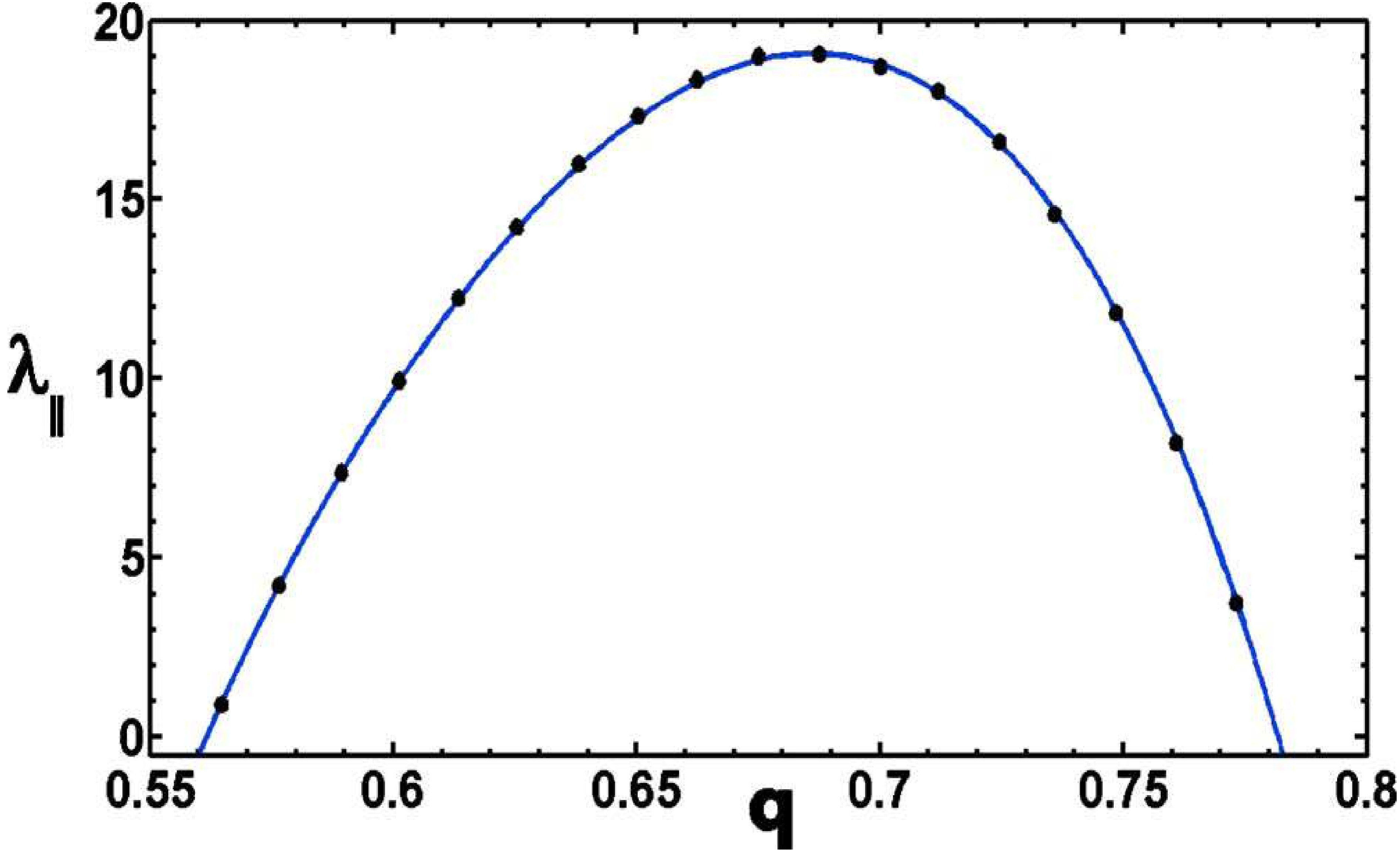


D(q) for $\alpha = 0.20$. N $\lambda_{||}$(max)=12.5

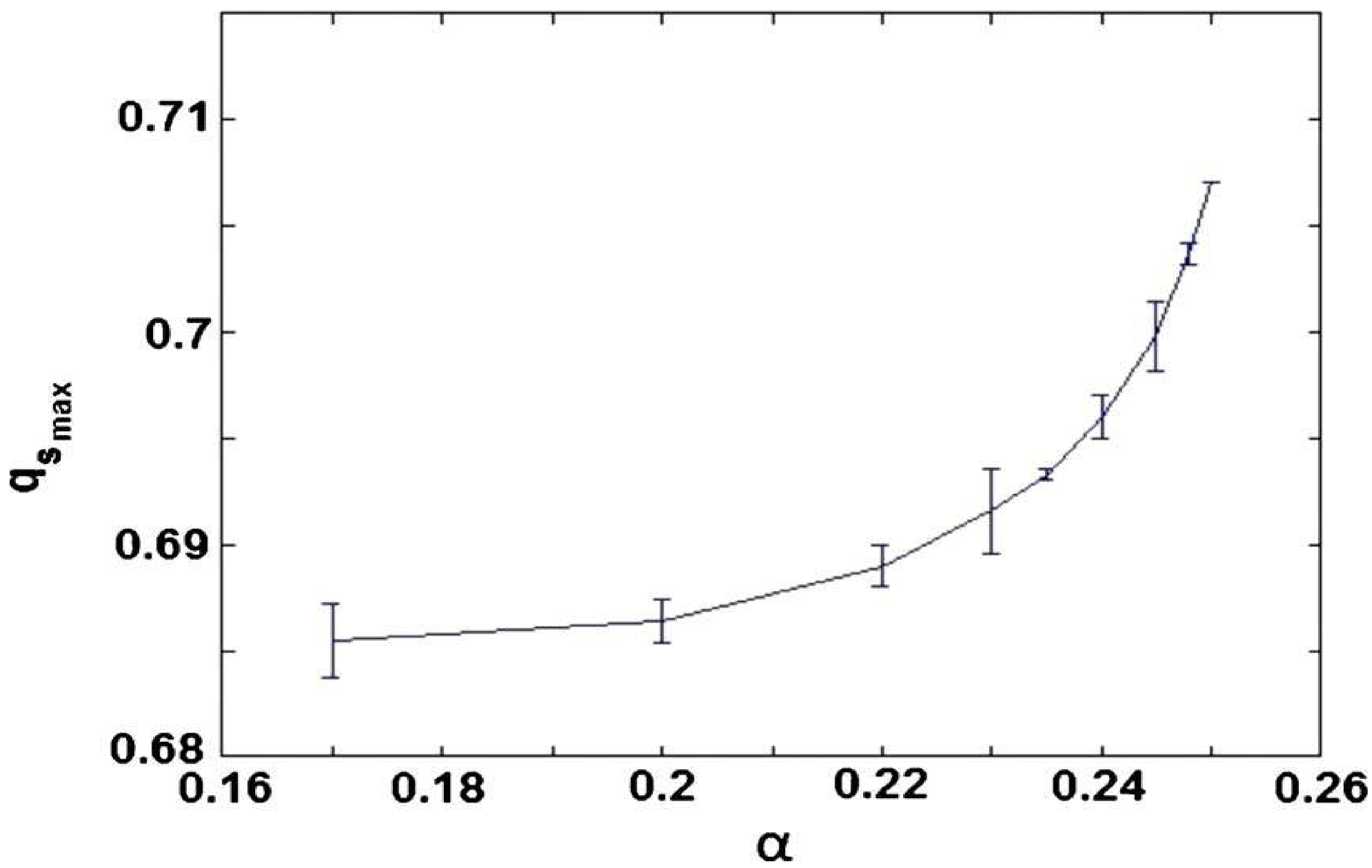


D(q) for $\alpha=0.17$. $\lambda_{\|}$(max)=19

D(q) increases as a decreases. Patterns more stable.



We assume that maximum of D(q) gives selected value since this is the most stable against perturbations. Agrees with simulation results for α close to 0.25.



Conclusions

- We have some evidence that a driven out of equilibrium system (eg. directional solidification, Rayleigh-Benard convection, etc) will eventually come to a unique stationary state.
- Evidence is mostly numerical on 1D stabilized KS equation.
- Analytical evidence is that the most stable state with the largest diffusion coefficient D(q) seems to be selected.
- Clearly, much more work required to really establish this.