The current distribution of the pushing asymmetric simple exclusion process

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Introduction and Motivation

- Central Limit Theorem states the the limiting distribution for the scaled fluctuation is Gaussian. ($1/2$ power law of exponent)
- KPZ(Kadar-Parisi-Zhang) predicted scale exponents $1/3$ for fluctuation (and $2/3$ for spatial correlation) for a large class of models.
- What are limiting distributions for KPZ universality class ?

Tracy-Widom distribution

GUE ensemble

The Gaussian Unitary Ensemble (GUE) is defined as a sequence $\mathbb{P}_N$ of Gaussian probability measure on $N \times N$ Hermitian matrices of the form

$$d\mathbb{P}_N(M) = \frac{1}{Z_N} e^{tr(M^2)} dM$$

where Z_N is a normalizing constant.

Tracy-Widom distribution;1994

Distribution of the largest eigenvalue in a Gaussian unitary random matrix ensemble

$$\lim_{N\to\infty} \mathbb{P}_N\left(\frac{\sqrt{2N}\lambda_1(M) - 2N}{N^{1/3}} \leq s\right) := TW_2(s)$$

Fluctuation of the length of the longest increasing subsequence of a random permutation (Baik, Deift, Johansson; 1999)

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \le s\Big) = TW_2(s)$$

Fluctuation of the current of the TASEP on $\mathbb{Z}$ with step initial condition (the corner growth model) (Johansson;2000)

$F_0(t)$: the total number of particles that have crossed from 0 and 1 during the time $[0, 1]$

$$\lim_{t\to\infty} \mathbb{P}\Big(\frac{F_0(t) - Jt}{Vt^{1/3}} \le s\Big) = 1 - TW_2(-s)$$

where J and V are some constants

Techniques to obtain the Tracy-Widom distribution

- Combinatorics : Robinson-Schensted-Knuth (RSK) type bijection
  - Longest increasing subsequence of a random permutation
  - Discrete TASEP
- (Coordinate) Bethe ansatz for continuous-time ASEP on $\mathbb{Z}$

Bethe ansatz for ASEP on $\mathbb{Z}$

- (Schütz;1997) Transition probability $P_Y(X;t)$ of N-particle (continuous-time) TASEP
- It is a determinant of which entries are hypergeometric functions.
- Entries have *special* properties.
- $\mathbb{P}_Y(x_m(t)=x)$ is obtained by summing $P_Y(X;t)$ over all possible configurations
- Asymptotic study of $\mathbb{P}_Y(x_m(t)=x)$ for step initial condition rediscovers Johannsson'result (Nagao, Sasamoto;2004, Rákos, Schütz;2005)
- How about ASEP ?

Tracy and Widom's works on ASEP;2008,2009

- Exact expression for the transition probability of N-particle system : the sum of $N!$ N-dimensional contour integrals
- $\mathbb{P}_Y(x_m(t) \leq x)$ to use some special identities
- Obtained a closed form of $\mathbb{P}_Y(x_m(t) \leq x)$ when Y is step initial condition and expressed it as a Fredholm determinant
- Asymptotic analysis of the Fredholm determinant $\rightarrow 1/3$ law

$$\lim_{t\to\infty} \mathbb{P}\Big(\frac{\mathcal{T}([-vt],t) - a_1 t}{a_2 t^{1/3}}\Big) = 1 - TW_2(-s)$$

where $\mathcal{T}([-vt],t)$ is the number of particles of which positions are less than or equal to $[-vt]$ at t, and $0 \leq v < 1$, a_1, a_2 are some constants.

More results on particle models

- TASEP with flat initial condition : TW_1 (Borodin, Ferrari, Sasamoto; 2007)
- (one-sided) pushing asymmetric simple exclusion process (PushASEP) : TW_1, TW_2 (Borodin, Ferrari;2008)
- Depending on the geometry of initial conditions
- Based on determinantal form of transition probabilities
- Distribution of the Hopt-Cole solution of the KPZ equation with narrow edge initial condition based on Tracy-Widom's ASEP results

Dynamics of N-particle two-sided PushASEP

- Each site is occupied by at most one particle or empty.
- Each particle is equipped with 2N Poisson clocks with rates pr_n and ql_n with $p + q = 1$ and $n = 1, \cdots, N$ and all clocks are independent.
- If (1) the clock with rate $pr_n(ql_n)$ of the particle at x rings, (2) $x + 1, \cdots x + n - 1$ $(x - 1, \cdots, x - (n - 1))$ are occupied and (3) $x + n$ $(x - n)$ is empty, then the particle at x jump to $x + n$ $(x - n)$.
- Otherwise, nothing happens and the clock resumes.

Bethe ansatz applicability (Sasamoto, Wadati;1998)

Constraint on r_n and l_n;

$$\frac{1-\left(\frac{\lambda}{\mu}\right)^n}{1-\frac{\lambda}{\mu}} = \frac{1}{r_n}$$
$$\frac{1-\left(\frac{\mu}{\lambda}\right)^n}{1-\frac{\mu}{\lambda}} = \frac{1}{l_n}$$

with $\lambda+\mu=1$.
$\lambda\rightarrow 0$; pushing dynamics on the right with constant rate p and TASEP dynamics on the left with rate q, that is, one-sided PushASEP.

Transition probability of the two-sided PushASEP (L;2012)

Let $\lambda + \mu = 1$ $(1/2 < \mu \leq 1)$ and $\mathcal{C}_{R_i}$ $(i = 1, \cdots, N)$ be a circle oriented counterclockwise, centered at 0 with radius R_i. Assume that $1 < R_1 < \cdots < R_N < c$ where

$$c = \begin{cases} \frac{\mu}{\lambda} & \text{if} \quad \lambda \neq 0 \\ \infty & \text{if} \quad \lambda = 0. \end{cases}$$

The transition probability of the two-sided PushASEP is

$$P_Y(X;t) = \sum_{\sigma \in \mathbb{S}_N} \left(\frac{1}{2\pi i}\right)^N \int_{\mathcal{C}_{R_N}} \cdots \int_{\mathcal{C}_{R_1}} A_\sigma^\dagger \prod_i^N \left(\xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\varepsilon(\xi_i)t}\right) d\xi_1 \cdots \xi_N$$

where

$$A_\sigma^\dagger = \prod_{\substack{i<j, \\ \sigma(i)>\sigma(j)}} \frac{\xi_{\sigma(i)}}{\xi_{\sigma(j)}} \cdot S_{\sigma(i)\sigma(j)}.$$

and

$$S_{\beta\alpha} := -\frac{\mu + \lambda\xi_\alpha\xi_\beta - \xi_\alpha}{\mu + \lambda\xi_\alpha\xi_\beta - \xi_\beta} \quad (\mu + \lambda = 1)$$

Difference between S-matrices of the ASEP and the two-sided PushASEP

For the Bethe ansatz solution

$$\sum_{\sigma\in\mathbb{S}_N} A_\sigma \prod_i \xi_{\sigma(i)}^{x_i}$$

- ASEP

  $$S_{\beta\alpha} := -\frac{p+q\xi_\alpha\xi_\beta-\xi_\beta}{p+q\xi_\alpha\xi_\beta-\xi_\alpha}$$

  where $\alpha<\beta$.
- Two-sided PushASEP

  $$S_{\beta\alpha} := -\frac{\xi_\beta}{\xi_\alpha}\cdot\frac{\mu+\lambda\xi_\alpha\xi_\beta-\xi_\alpha}{\mu+\lambda\xi_\alpha\xi_\beta-\xi_\beta}$$

  where $\alpha<\beta$.
- This difference requires new proof.

Contours of the ASEP and the two-sided PushASEP

- ASEP: $\mathcal{C}_r$ sufficiently small $r < 1$ and same for all variables
- Two-sided PushASEP: $\mathcal{C}_{R_i}$ with $1 < R_1 < \cdots < R_N < \frac{\mu}{\lambda}$ for variable ξ_i

Main part of the proof

Show the initial condition; $P_Y(X;0) = \delta_Y(X)$. In the sum $\sum_{\sigma \in \mathbb{S}_N}$

- the term for $\sigma = identity$ contributes to $\delta_Y(X)$
- all other terms for $\sigma \neq identity$ give 0.
- mathematical induction

$\mathbb{P}_Y(x_m(t) = x)$

- Probability that the mth leftmost particle is at x at time t

$$\mathbb{P}_Y(x_m(t) = x) = \sum_{\text{all possible } X} P_Y(X;t)$$

- $x_{m+1} = x + z_1, x_{m+2} = x + z_1 + z_2, \cdots, x_N = x + z_1 + \cdots z_{N-m},\ z_i \in \mathbb{N}$
- $x_{m-1} = x - v_1, x_{m-2} = x - v_1 - v_2, \cdots, x_1 = x + v_1 + \cdots v_{m-1},\ v_i \in \mathbb{N}$

$$\mathbb{P}_Y(x_m(t) = x) = \sum_{\text{all } v_i, z_i = 1}^{\infty} P_Y(X;t)$$

- The sum is a multiple geometric series which can be shown to be convergent.

Main result $\mathbb{P}_Y(x_m(t) = x)$ (L;2012)

Let $S = \{s_1, \cdots, s_k\} \subset \{1, \cdots, N\}$ with $s_i < s_{i+1}$ and

$$I(\xi; s_1, \cdots, s_k) = \prod_{i<j} \frac{\xi_{s_i} - \xi_{s_j}}{\mu + \lambda \xi_{s_i}\xi_{s_j} - \xi_{s_j}} \cdot \frac{1}{\prod_{s\in S}(\xi_s - 1)} \cdot \Big(\prod_{s\in S} \xi_s - 1\Big)$$

Then

$$\begin{aligned}
&\mathbb{P}_Y(x_m(t) = x) \\
&= \sum_{|S|\geq m} c_S \Big(\frac{1}{2\pi i}\Big)^k \int_{\mathcal{C}_{R_{s_k}}} \cdots \int_{\mathcal{C}_{R_{s_1}}} I(\xi; s_1, \cdots, s_k) \\
&\qquad \prod_{s\in S} \xi_s^{x-(y_s - s)-1} e^{\varepsilon(\xi_s)t} \, d\xi_{s_1} \cdots d\xi_{s_k}.
\end{aligned}$$

where

$$\varepsilon(\xi_s) = \frac{p}{\xi_s} + q\xi_s - 1$$

Coefficient c_S

Notation

$$[N] = \frac{\mu^N - \lambda^N}{\mu - \lambda}$$

and

$$[N]! = [N][N-1]\cdots[1], \ \begin{bmatrix} N \\ m \end{bmatrix} = \frac{[N]!}{[m]![N-m]!}$$

with $[0]! = 1$

$$c_S = (-1)^{|S|+m}(\mu\lambda)^{m(m-1)/2} \begin{bmatrix} |S|-1 \\ |S|-m \end{bmatrix} \frac{\lambda^{\Sigma[S]-m|S|}}{\mu^{\Sigma[S]-|S|(|S|+1)/2}}$$

where $\Sigma[S]$ is the sum of all elements in S.
- Surprisingly, c_S for the two-sided PushASEP is in the same form as the c_S for the ASEP. (p, q instead of λ, μ)

Sketch of the proof

(1) Find $\mathbb{P}(x_N(t) = x)$; easy part using the same technique as the ASEP's $\mathbb{P}(x_1(t) = x)$

(2) Find $\mathbb{P}(x_1(t) = x)$; hard part

(3) Use $\mathbb{P}(x_N(t) = x)$ and $\mathbb{P}(x_1(t) = x)$ to find general $\mathbb{P}(x_m(t) = x)$

- Interestingly, no new identities required (ASEP's identities appeared in the PushASEP as well.)

More results to report and ongoing works

- Possible to obtain Fredholm determinant representation of $\mathbb{P}(x_m(t) = x)$ for step (Bernoulli) initial condition by the same technique as the Tracy-Widom's.(But slightly different operator so easy to apply the ASEP's technique.)
- Asymptotic analysis of the Fredholm determinant representation for the Tracy-Widom law (ongoing)
- Transition probability of the Bethe ansatz solvable zero range process on $\mathbb{Z}$ (We found new identities which is a good signal in this direction of the work.)
- Ultimately, we want to generalize the ASEP's result to the Bethe ansatz solvable AZRP (ongoing).

Thank you.