Ageing and dynamical symmetries in non-equilibrium systems without detailed balance

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Overview:

1. Ageing phenomena
2. Interface growth (KPZ universality class)
3. Critical contact process (DP universality class)
4. Form of the scaling functions
5. Logarithmic conformal & ageing invariance
6. Numerical experiments (KPZ and DP classes in 1D)
7. Conclusions
1. Ageing phenomena

known & practically used since prehistoric times (metals, glasses)
systematically studied in physics since the 1970s
occur in widely different systems
(structural glasses, spin glasses, polymers, simple magnets, . . .)

Struik ’78

Three defining properties of ageing:

1. slow relaxation (non-exponential!)
2. no time-translation-invariance (TTI)
3. dynamical scaling without fine-tuning of parameters

Most existing studies on ‘magnets’: relaxation towards equilibrium

Question: what can be learned about intrisically irreversible systems by studying their ageing behaviour?
1. observe slow relaxation after quenching PVC from melt to low $T$
2. creep curves depend on waiting time $t_e$ and creep time $t$
3. find master curve for all $(t, t_e)$ → dynamical scaling
   → three defining properties of physical ageing
master curves of distinct materials are identical
independent of ‘details’
→ Universality!
good for theorists . . .
hint for hidden symmetry?

conceptual confirmation in phase-ordering: Allen-Cahn equation

Struik 78
\begin{align*}
t & = t_1 \\
t & = t_2 > t_1
\end{align*}

\text{magnet } T < T_c \quad \rightarrow \quad \text{ordered cluster}

\text{magnet } T = T_c \quad \rightarrow \quad \text{correlated cluster}

critical contact process

\rightarrow \quad \text{cluster dilution}

\begin{align*}
\text{voter model, contact process, \ldots}

L(t) & \sim t^{1/z} \\
\end{align*}

\text{common feature: growing length scale}

z : \text{dynamical exponent}
Two-time observables: analogy with ‘magnets’

time-dependent order-parameter \( \phi(t, r) \)

two-time **correlator**
\[
C(t, s) := \langle \phi(t, r) \phi(s, r) \rangle - \langle \phi(t, r) \rangle \langle \phi(s, r) \rangle
\]

two-time **response**
\[
R(t, s) := \frac{\delta \langle \phi(t, r) \rangle}{\delta h(s, r)} \bigg|_{h=0} = \langle \phi(t, r) \tilde{\phi}(s, r) \rangle
\]

\( t \): observation time, \( s \): waiting time

**a)** system at equilibrium: **fluctuation-dissipation theorem**

\[
R(t - s) = \frac{1}{T} \frac{\partial C(t - s)}{\partial s}, \quad T : \text{temperature}
\]

**b)** far from equilibrium: \( C \) and \( R \) independent!

The **fluctuation-dissipation ratio** (FDR) (Cugliandolo, Kurchan, Parisi ’94)

\[
X(t, s) := \frac{TR(t, s)}{\partial C(t, s)/\partial s}
\]

measures the distance with respect to equilibrium:

\[
X_{eq} = X(t - s) = 1
\]
Scaling regime: \[ t, s \gg \tau_{\text{micro}} \text{ and } t - s \gg \tau_{\text{micro}} \]

\[ C(t, s) = s^{-b} f_C \left( \frac{t}{s} \right), \quad R(t, s) = s^{-1-a} f_R \left( \frac{t}{s} \right) \]

asymptotics: \[ f_{C,R}(y) \sim y^{-\lambda_{C,R}/z} \text{ for } y \gg 1 \]

\( \lambda_C \): autocorrelation exponent, \( \lambda_R \): autoresponse exponent, 
\( z \): dynamical exponent, \( a, b \): ageing exponents

\textbf{ex.:} critical particle-reaction model (contact process), 
initial particle density > 0

\[ \lambda_C = \lambda_R = d + z + \frac{\beta}{\nu_{\perp}}, \quad b = \frac{2\beta'}{\nu_{\parallel}} \]

\[ \rightarrow \text{stationary-state critical exponents } \beta, \beta', \nu_{\perp}, \nu_{\parallel} = z\nu_{\perp} \]
2. Interface growth

deposition (evaporation) of particles on a substrate → height profile \( h(t, r) \)

generic situation: RSOS (restricted solid-on-solid) model Kim & Kosterlitz 89

\[ h = \text{deposition prob.} \]
\[ 1 - p = \text{evap. prob.} \]

here \( p = 0.98 \)

some universality classes:

(a) KPZ \[ \partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta \]
(b) EW \[ \partial_t h = \nu \nabla^2 h + \eta \]
(c) MH \[ \partial_t h = -\nu \nabla^4 h + \eta \]

\( \eta \) is a gaussian white noise with \( \langle \eta(t, r)\eta(t', r') \rangle = 2\nu T \delta(t - t')\delta(r - r') \)

\( T \) is a temperature.

Kardar, Parisi, Zhang 86
Edwards, Wilkinson 82
Mullins, Herring 63; Wolf, Villain 80
Family-Viscek scaling on a spatial lattice of extent $L^d$: $\overline{h}(t) = L^{-d} \sum_j h_j(t)$

\[ w^2(t; L) = \frac{1}{L^d} \sum_{j=1}^{L^d} \left\langle \left( h_j(t) - \overline{h}(t) \right)^2 \right\rangle = L^{2\zeta} f(tL^{-z}) \sim \begin{cases} L^{2\zeta} & \text{if } tL^{-z} \gg 1 \\ t^{2\beta} & \text{if } tL^{-z} \ll 1 \end{cases} \]

$\beta$: growth exponent, $\zeta$: roughness exponent, $\zeta = \beta z$

two-time correlator:

\[ C(t, s; r) = \left\langle h(t, r)h(s, 0) \right\rangle - \left\langle \overline{h}(t) \right\rangle \left\langle \overline{h}(s) \right\rangle = s^{-b} F_C \left( \frac{t}{s}, \frac{r}{s^{1/z}} \right) \]

with ageing exponent: $b = -2\beta$

two-time integrated response:

* sample A with deposition rates $p_i = p \pm \epsilon_i$, up to time $s$,
* sample B with $p_i = p$ up to time $s$;
  then switch to common dynamics $p_i = p$ for all times $t > s$

\[ \chi(t, s; r) = \int_0^s du R(t, u; r) = \frac{1}{L} \sum_{j=1}^{L} \left\langle \frac{h_{j+r}^{(A)}(t; s) - h_{j+r}^{(B)}(t)}{\epsilon_j} \right\rangle = s^{-a} F_\chi \left( \frac{t}{s}, \frac{|r|^z}{s} \right) \]

Kallabis & Krug 96
Effective action of the KPZ equation:

\[ \mathcal{J}[\phi, \tilde{\phi}] = \int dt dr \left[ \tilde{\phi} \left( \partial_t \phi - \nu \nabla^2 \phi - \frac{\mu}{2} (\nabla \phi)^2 \right) - \nu T \tilde{\phi}^2 \right] \]

\[ \Rightarrow \text{Very special properties of KPZ in } d = 1 \text{ spatial dimension!} \]

Exact critical exponents: \[ \beta = 1/3, \ \zeta = 1/2, \ z = 3/2, \ \lambda_C = 1 \]

related to precise symmetry properties:

A) tilt-invariance (Galilei-invariance)

kept under renormalisation!

B) time-reversal invariance

Forster, Nelson, Stephen 77

Medina, Hwa, Kardar, Zhang 89

Lvov, Lebedev, Paton, Procaccia 93

Frey, Täuber, Hwa 96
A) **tilt-invariance** (holds for any dimension $d$)

$$
t \mapsto t \quad , \quad r \mapsto r - \epsilon t \quad , \quad h(t, r) \mapsto h(t, r - \epsilon t) - \frac{1}{\mu} \epsilon \cdot r + \frac{t}{2\mu} \epsilon^2
$$

$$
\eta(t, r) \mapsto \eta(t, r - \epsilon t)
$$

combination with dynamical scaling gives the exponent relation

$$
\zeta + z = 2
$$

this is preserved in the loop expansion !

**N.B.** in 1D, gaussian stationary distribution fixes $\zeta = \frac{1}{2}$.  

B) Special KPZ symmetry in 1D: let \( v = \frac{\partial \phi}{\partial r}, \tilde{\phi} = \frac{\partial}{\partial r} (\tilde{p} + \frac{v}{2T}) \)

\[
\mathcal{J} = \int dt dr \left[ \tilde{p} \partial_t v - \frac{v}{4T} (\partial_r v)^2 - \frac{\mu}{2} v^2 \partial_r \tilde{p} + \nu T (\partial_r \tilde{p})^2 \right]
\]
is invariant under time-reversal

\[
t \mapsto -t \text{ , } v(t, r) \mapsto -v(-t, r) \text{ , } \tilde{p} \mapsto +\tilde{p}(-t, r)
\]
\[\Rightarrow \text{ fluctuation-dissipation relation for } t \gg s\]

\[
TR(t, s; r) = -\partial_r^2 C(t, s; r)
\]
distinct from the equilibrium FDT \( TR(t - s) = \partial_s C(t - s) \)

Combination with ageing scaling, gives the ageing exponents:

\[
\lambda_R = \lambda_C = 1 \text{ and } 1 + a = b + \frac{2}{z}
\]
1D relaxation dynamics, starting from an initially flat interface

observe all 3 properties of ageing:
- slow dynamics
- no TTI
- dynamical scaling

confirm expected exponents $b = -2/3, \lambda_C/z = 2/3$

N.B. : this confirmation is out of the stationary state

Kallabis & Krug 96; Krech 97; Bustingorry et al. 07-10; Chou & Pleimling 10; D’Aquila & Täuber 11
observe all 3 properties of **ageing** : slow dynamics, no TTI, dynamical scaling

exponents $a = -1/3$, $\lambda_R/z = 2/3$, as expected from FDR

**N.B.** : numerical tests for 2 models in KPZ class
Simple ageing is also seen in space-time observables

\[
C(t, s; r) = s^{2/3} F_C \left( \frac{t}{s}, \frac{r^{3/2}}{s} \right)
\]

integrated response \( \chi(t, s; r) = s^{1/3} F_\chi \left( \frac{t}{s}, \frac{r^{3/2}}{s} \right) \) \( z = 3/2 \)
Values of some growth and ageing exponents in 1D

<table>
<thead>
<tr>
<th>model</th>
<th>$z$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\lambda_R = \lambda_C$</th>
<th>$\beta$</th>
<th>$\zeta$</th>
</tr>
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<tbody>
<tr>
<td>KPZ</td>
<td>3/2</td>
<td>$-1/3$</td>
<td>$-2/3$</td>
<td>1</td>
<td>1/3</td>
<td>1/2</td>
</tr>
<tr>
<td>exp</td>
<td></td>
<td></td>
<td>$\approx -2/3$</td>
<td>$\approx 1\dagger$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EW</td>
<td>2</td>
<td>$-1/2$</td>
<td>$-1/2$</td>
<td>1</td>
<td>1/4</td>
<td>1/2</td>
</tr>
<tr>
<td>MH</td>
<td>4</td>
<td>$-3/4$</td>
<td>$-3/4$</td>
<td>1</td>
<td>3/8</td>
<td>3/2</td>
</tr>
</tbody>
</table>

$\dagger$ scaling holds only for flat interface

Two-time space-time responses and correlators consistent with simple ageing for 1D KPZ

Similar results known for EW and MH universality classes

Roethlein, Baumann, Pleimling 06
3. Critical contact process

\[ A \overset{p}{\rightarrow} 2A, \quad A \overset{1}{\rightarrow} \emptyset \]

ageing and scaling for \( C(t, s) \): critical contact process

main figures: 1D, insets: 2D

observe all 3 properties of ageing:

- slow dynamics
- no TTI
- dynamical scaling

contrast to critical magnets: \( a \neq b \implies \text{no finite FDR!} \)

Ramasco, mh, Santos, da Silva Santos 04; Enss, mh, Schollwöck 04
Effective action at criticality

\[ \mathcal{J}[\tilde{\phi}, \phi] = \int dt dr \left[ \tilde{\phi} \left( D \partial_t \phi - \nabla^2 \phi \right) - \kappa \tilde{\phi} \left( \tilde{\phi} - \phi \right) \phi \right] \]

rapidity-reversal symmetry: \( \mathcal{J} \) is invariant under Grassberger 79, Janssen 81

\[ t \mapsto -t \ , \ \phi(t, r) \mapsto -\tilde{\phi}(-t, r) \ , \ \tilde{\phi}(t, r) \mapsto -\phi(-t, r) \]

\[ \Rightarrow \phi \text{ and } \tilde{\phi} \text{ must have equal scaling dimensions } \Rightarrow 1 + a = b \]

new form of FDR, \( \frac{1}{\Xi} \) measures distance from stationarity Enss et al. 04

Baumann & Gambassi 07

\[ \Xi(t, s) := \frac{R(t,s)}{C(t,s)} = \frac{f_R(t/s)}{f_C(t/s)} \ , \ \Xi_{\infty} := \lim_{s \to \infty} \left( \lim_{t \to \infty} \Xi(t, s) \right) \]

for \( d = 4 - \varepsilon \) (one-loop calculation) & 1D num. TMRG estimate B & G 07

Enss et al. 04

\[ \Xi_{\infty} = 2 \left[ 1 - \varepsilon \left( \frac{119}{480} - \frac{\pi^2}{120} \right) \right] + O(\varepsilon^2) \ , \ \Xi_{\infty} = 1.15(5) \]

NB: \( 1 + a = b \) invalid in other non-equilibrium universality classes
numerical values of some non-equilibrium exponents

contact process (CP) $A \rightarrow 2A, A \rightarrow \emptyset$, parity-conserved model (PC) $A \leftrightarrow 3A, 2A \rightarrow \emptyset$, diffusion-coagulation (DC) $2A \rightarrow A$

<table>
<thead>
<tr>
<th></th>
<th>$d$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\lambda_C/z$</th>
<th>$\lambda_R/z$</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>CP</td>
<td>1</td>
<td>$-0.68(5)$</td>
<td>0.32(5)</td>
<td>1.85(10)</td>
<td>1.85(10)</td>
<td>TMRG</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-0.57(10)$</td>
<td>0.3189</td>
<td>1.9(1)</td>
<td>1.9(1)</td>
<td>MC</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-0.6810$</td>
<td>0.3189</td>
<td>1.76(5)</td>
<td></td>
<td>MC</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-0.6810$</td>
<td>0.3189</td>
<td>1.7921</td>
<td>1.7921</td>
<td>scal</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.3(1)</td>
<td>0.901(2)</td>
<td>2.8(3)</td>
<td>2.75(10)</td>
<td>MC</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-0.198(2)$</td>
<td>0.901(2)</td>
<td>2.58(2)</td>
<td>2.58(2)</td>
<td>scal</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9(1)</td>
<td>2.5(1)</td>
<td></td>
<td></td>
<td>exp</td>
</tr>
<tr>
<td></td>
<td>$&gt;4$</td>
<td>$d/2 - 1$</td>
<td>$d/2$</td>
<td>$d/2 + 2$</td>
<td></td>
<td>MF</td>
</tr>
<tr>
<td>PC</td>
<td>1</td>
<td>$-0.430(4)$</td>
<td>0.570(4)</td>
<td>1.9(1)</td>
<td>1.9(2)</td>
<td>MC</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-0.430(4)$</td>
<td>0.570(4)</td>
<td>1.86(1)</td>
<td>1.86(1)</td>
<td>scal</td>
</tr>
<tr>
<td>DC</td>
<td>1</td>
<td>$-1/2$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>exact</td>
</tr>
</tbody>
</table>

4. Form of the scaling functions

Question: Are there model-independent results on the form of universal scaling functions?

‘Natural’ starting point: try to draw analogies with conformal invariance at equilibrium

* Equilibrium critical phenomena: scale-invariance
* For sufficiently local interactions: extend to conformal invariance space-dependent re-scaling (angles conserved) \( r \mapsto \frac{r}{b(r)} \)

In two dimensions: infinitely many conformal transformations \( w \mapsto \beta(w) \) complex analytic

\( \Rightarrow \) exact predictions for critical exponents, correlators, …

Bateman & Cunningham 1909/10, Polyakov 70

BPZ 84
Hidden assumptions:

1) extension scale-invariance $\rightarrow$ conformal invariance?  
   formally: energy-momentum tensor symmetric & traceless \cite{Callan, Coleman, Jackiw 70}  
   but counterexamples:  
   \begin{align*} 
   & \text{lattice animals} \\
   & \text{hydrodynamics} \\
   & \text{renormalised FT} 
   \end{align*} \cite{Miller & De Bell 93, Riva & Cardy 05, Fortin, Grinstein, Stergiou 12}

2) choice of so-called ‘primary’ scaling operators  
   not all physical models are unitary minimal CFTs $\rightarrow$ SLE

3) how do primary operators transform?  
   usual form  
   $$ \phi'(w) = \beta'(w)\Delta \phi(\beta(w)) $$  
   alternative: logarithmic partner $\psi$  
   $$ \psi'(w) = \beta'(w)\Delta \left[ \psi(\beta(w)) + \ln \beta'(w) \cdot \phi(\beta(w)) \right] $$ \cite{Gurarie 93, Khorrami et al. 97,..}

Logarithmic conformal invariance has been found in, e.g.  
- critical 2D percolation \cite{Cardy 92, Watts 96, Mathieu & Ridout 07/08}  
- disordered systems \cite{Caux et al. 96}  
- sand-pile models \cite{Ruelle et al. 08-10}
What about **time**-dependent critical phenomena?

Characterised by **dynamical exponent** $z : t \mapsto tb^{-z}$, $r \mapsto rb^{-1}$

Can one extend to **local** dynamical scaling, with $z \neq 1$?

If $z = 2$, the **Schrödinger group** is an example:

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta}, \quad r \mapsto \frac{Dr + vt + a}{\gamma t + \delta}; \quad \alpha \delta - \beta \gamma = 1$$

⇒ study **ageing** phenomena as paradigmatic example

**essential**: (i) **absence** of TTI & (ii) **Galilei**-invariance

Transformation $t \mapsto t'$ with $\beta(0) = 0$ and $\dot{\beta}(t') \geq 0$ and

$$t = \beta(t'), \quad \phi(t) = \left(\frac{d\beta(t')}{dt'}\right)^{-x/z} \left(\frac{d \ln \beta(t')}{dt'}\right)^{-2\xi/z} \phi'(t')$$

**out of equilibrium**, have 2 distinct scaling dimensions, $x$ and $\xi$.

**mean-field for magnets**: expect

\[
\begin{cases}
\xi = 0 \text{ in ordered phase } T < T_c \\
\xi \neq 0 \text{ at criticality } T = T_c
\end{cases}
\]

**NB**: if TTI (equilibrium criticality), then $\xi = 0$. 

Cardy 85
physical requirement:
co-variance of \textbf{response functions} under local scaling!

why: certain extended scaling symmetries predict \textbf{causality} for co-variant \(n\)-point functions!

\(\Rightarrow\) set of linear differential equations for \(R(t,s)\)

most simple case:

\[
R(t,s) = \left\langle \phi(t) \tilde{\phi}(s) \right\rangle = s^{-1-a} f_R \left( \frac{t}{s} \right)
\]

\[
f_R(y) = f_0 y^{1+a'-\lambda_R/z} (y-1)^{-1-a'} \Theta(y-1)
\]

causality

\[
a = \frac{1}{z} (x+\tilde{x}) - 1, \quad a' - a = 2 \left( \xi + \tilde{\xi} \right), \quad \frac{\lambda_R}{z} = x + \xi
\]

magnetic example: 1D Glauber-Ising model at \(T = T_c = 0\):

\[
a = 0, \quad a' - a = -\frac{1}{2}, \quad \lambda_R = 1, \quad z = 2
\]

Picone, mh 04
mh, Enss, Pleimling 06
Particle models: comparison of $R(t,s)$ with LSI-prediction:

**contact process (CP)**

$$a' - a \simeq 0.27$$

mh, Enss, Pleimling 06
Enss 06; Hinrichsen 06

**nonequil. kinetic Ising (PC)**

$$a' - a \simeq 0.00(1)$$

Ódor 06

**voter Potts-3 (VP3)**

$$a' - a \simeq -0.1$$

Chatelain, Tomé, de Oliveira 11

? is this good general agreement already conclusive?

**Observation:** the hidden assumption $a = a'$, uncritically taken over from equilibrium, is often invalid out of equilibrium. Observables cannot always be identified with scaling operators.
5. Logarithmic conformal & ageing invariance
generalise conformal invariance $\rightarrow$ doublets $\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$

**generators** : $\ell_n = -w^{n+1}\partial_w - (n+1)w^n \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix}$

**two-point functions** : have $\Delta_1 = \Delta_2$  

\[
F = \langle \phi_1(w_1)\phi_2(w_2) \rangle = 0 \\
G = \langle \phi_1(w_1)\psi_2(w_2) \rangle = G_0|w|^{-2\Delta_1} \\
H = \langle \psi_1(w_1)\psi_2(w_2) \rangle = (H_0 - 2G_0 \ln |w|) |w|^{-2\Delta_1} \\
= w_2^{-2\Delta_1}(H_0 - 2G_0 \ln |y - 1| - 2G_0\ln |w_2|) |y - 1|^{-2\Delta_1}
\]

with $w = w_1 - w_2$ and $y = w_1/w_2$.

Simultaneous log corrections to scaling and modified scaling function

Logarithmic conformal invariance has been found in, e.g.
- critical 2D percolation Cardy 92, Watts 96, Mathieu & Ridout 07/08
- disordered systems Caux et al. 96
- sand-pile models Ruelle et al. 08-10
construct logarithmic ageing-invariance by the formal changes (generic case; \( x' = 0 \) or \( x' = 1 \)):

\[
\begin{align*}
\text{x} & \mapsto \hat{x} = \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}, \\
\xi & \mapsto \hat{\xi} = \begin{pmatrix} \xi & \xi' \\ 0 & \xi \end{pmatrix}
\end{align*}
\]

(must show: both dimension matrices \( \hat{x}, \hat{\xi} \) are simultaneously Jordan!)

we find the co-variant two-point functions (with \( y = t/s \)):

\[
\begin{align*}
\langle \phi(t) \tilde{\phi}(s) \rangle & = s^{-(x+\bar{x})/2} f(y) \\
\langle \phi(t) \tilde{\psi}(s) \rangle & = s^{-(x+\bar{x})/2} (g_{12}(y) + \ln s \cdot \gamma_{12}(y)) \\
\langle \psi(t) \tilde{\phi}(s) \rangle & = s^{-(x+\bar{x})/2} (g_{21}(y) + \ln s \cdot \gamma_{21}(y)) \\
\langle \psi(t) \tilde{\psi}(s) \rangle & = s^{-(x+\bar{x})/2} (h_0(y) + \ln s \cdot h_1(y) + \ln^2 s \cdot h_2(y))
\end{align*}
\]

all scaling functions explicitly known

**Question**: interesting models described by logarithmic LSI?
6. Numerical experiments

(A) Kardar-Parisi-Zhang (KPZ)
(B) directed percolation (DP)
(C) majority voter/Glauber models

**simple ageing** of the correlators and responses, especially

\[
C(t, s) = s^{-b} f_C \left( \frac{t}{s} \right), \quad R(t, s) = s^{-1-a} f_R \left( \frac{t}{s} \right)
\]

\[
f_C(y) \sim y^{-\lambda_C/z}, \quad f_R(y) \sim y^{-\lambda_R/z}, \quad y \gg 1
\]

values of the non-equilibrium exponents & scaling relations

**KPZ in 1D**: \( \lambda_C = \lambda_R = 1, \quad 1 + a = b + \frac{2}{z}, \quad b = -2\beta = -\frac{2}{3}, \quad z = \frac{3}{2} \)

**DP**: \( \lambda_C = \lambda_R = d + z + \frac{\beta}{\nu\perp}, \quad 1 + a = b = \frac{2\beta}{\nu\parallel} \)

what can be said on the form of the scaling function of the auto-response?

**N.B.**: Galilei-invariance for KPZ is kept under renormalisation, unusual form
\((A)\) assumption : \(R(t, s) = \langle \psi(t)\overline{\psi}(s) \rangle\) 1D KPZ equation/RSOS model

good collapse \(\Rightarrow\) no logarithmic corrections \(\Rightarrow\) \[
\begin{align*}
\chi' &= \overline{\chi}' = 0
\end{align*}
\]

no logarithmic factors for \(y \gg 1\) \(\Rightarrow\) \[
\xi' = 0
\]

\(\Rightarrow\) only \(\overline{\xi}' = 1\) remains

\[
\begin{align*}
f_R(y) &= y^{-\lambda_R/z} \left( 1 - \frac{1}{y} \right)^{-1-a'} \left[ h_0 - g_0 \ln \left( 1 - \frac{1}{y} \right) - \frac{1}{2} f_0 \ln^2 \left( 1 - \frac{1}{y} \right) \right]
\end{align*}
\]

use specific values of 1D KPZ class \(\frac{\lambda_R}{z} - a = 1\)

find integrated autoresponse \(\chi(t, s) = \int_0^s du R(t, u) = s^{1/3} f_\chi(t/s)\)

\[
\begin{align*}
f_\chi(y) &= y^{1/3} \left\{ A_0 \left[ 1 - \left( 1 - \frac{1}{y} \right)^{-a'} \right] \\
&\quad + \left( 1 - \frac{1}{y} \right)^{-a'} \left[ A_1 \ln \left( 1 - \frac{1}{y} \right) + A_2 \ln^2 \left( 1 - \frac{1}{y} \right) \right] \right\}
\end{align*}
\]

with free parameters \(A_0, A_1, A_2\) and \(a'\)
non-log LSI with $a = a'$: deviations $\approx 20$

non-log LSI with $a \neq a'$: works up to $\approx 5$

log LSI: works better than $\approx 0.1$

### Table

<table>
<thead>
<tr>
<th>$R$</th>
<th>$a'$</th>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \phi \bar{\phi} \rangle$ – LSI</td>
<td>$-0.500$</td>
<td>0.662</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\langle \phi \psi \rangle$ – $L^1$LSI</td>
<td>$-0.500$</td>
<td>0.663</td>
<td>$-6 \cdot 10^{-4}$</td>
<td>0</td>
</tr>
<tr>
<td>$\langle \psi \bar{\psi} \rangle$ – $L^2$LSI</td>
<td>$-0.8206$</td>
<td>0.7187</td>
<td>0.2424</td>
<td>$-0.09087$</td>
</tr>
</tbody>
</table>

logarithmic LSI fits data at least down to $y \approx 1.01$, with $a' - a \approx -0.4873$ (can we make a conjecture?)
**Assumption:** \( R(t, s) = \langle \psi(t) \tilde{\psi}(s) \rangle \)  

1D critical contact process

Good collapse \( \Rightarrow \) no logarithmic corrections \( \Rightarrow x' = \tilde{x}' = 0 \)

\[
R(y) = \left(1 - \frac{1}{y}\right)^{a - a'} \left[ h_0 - g_{12,0} \xi' \ln(1 - 1/y) - g_{21,0} \xi' \ln(y - 1) \right.
\]
\[
- \frac{1}{2} f_0 \xi'^2 \ln^2(1 - 1/y) + \frac{1}{2} f_0 \xi'^2 \ln^2(y - 1) \right]
\]

Find empirically:  
very small amplitude of \( \ln^2 \)-terms

\( \Rightarrow f_0 = 0 \)

Require both \( \xi \neq 0, \tilde{\xi}' \neq 0 \)

**BUT:** logarithmic factor for \( y \gg 1 \)?

Logar. LSI fit data, at least down to \( y \approx 1.002 \); with \( a' - a \approx -0.002 \).
(C) assumption: \( R(t, s) = \left\langle \psi(t) \tilde{\psi}(s) \right\rangle \) 2D majority voter/Glauber model (triangular lattice)

good collapse \( \Rightarrow \) no logarithmic corrections \( \Rightarrow \) \( x' = \tilde{x}' = 0 \)

\[
h_R(y) = \left( 1 - \frac{1}{y} \right)^{a-a'} \left[ h_0 - g_{12,0} \ln(1 - 1/y) - \frac{1}{2} f_0 \ln^2(1 - 1/y) \right]
\]

no logarithmic terms for \( y \gg 1 \)
\( \Rightarrow \) \( \xi' = 0 \)

can normalise \( \tilde{\xi}' = 1 \)

F. Sastre (2012)

logar. LSI fit data, at least down to \( y \approx 1.005 \).
7. Conclusions

- Physical ageing occurs naturally in many irreversible systems relaxing towards non-equilibrium stationary states considered here: absorbing phase transitions & surface growth.
- Scaling phenomenology analogous to simple magnets.
- But finer differences in relationships between non-equilibrium exponents.
- A major difference w/ equilibrium: intrinsic absence of time-translation-invariance $\Rightarrow 2$ scaling dimensions.
- Shape of scaling functions: logarithmic local scale-invariance?
- Performed numerical experiments on auto-response function:
  1. $1D$ KPZ equation
  2. $1D$ critical directed percolation
  3. $2D$ majority voter/Glauber models
- Major open problem: Galilei-invariance!

Studies of the ageing properties, via two-time observables, might become a new tool, also for the analysis of complex systems!
This book is Volume I of a two-volume set describing the two main classes of non-equilibrium phase transitions. It covers the statics and dynamics of transitions into an absorbing state. Volume 2 will cover dynamical scaling in far-from-equilibrium relaxation behavior and ageing.

The first volume begins with an introductory chapter which recalls the main concepts of phase-transitions, set for the convenience of the reader in an equilibrium context. The extension to non-equilibrium systems is made by using directed percolation as the main paradigm of absorbing phase transitions and, in view of the richness of the known results, an entire chapter is devoted to it, including a discussion of recent experimental results. Scaling theories and a large set of both numerical and analytical methods for the study of non-equilibrium phase transitions are thoroughly discussed.

The techniques used for directed percolation are then extended to other universality classes and many important results of model parameters are provided for easy reference.
study more closely the limit $t, s \to \infty$, $y = t/s$ fixed; let $y \to 1$

$$R(t, s) = s^{-1-a} f_R \left( \frac{t}{s} \right), \quad h_R(y) := f_R(y) y^{\lambda_R/z} (1 - 1/y)^{1+a}$$

observe good collapse of data, when $y = t/s$ large enough

LSI with $a = a'$ predicts: $h_R(y) = f_0 = \text{cste}$.  
⇒ reproduces TMRG data for $y \gtrsim 3 - 4$
$$h_R(y) := f_R(y)y^{\lambda_R/z}(1 - 1/y)^{1+a} \overset{\text{LSI}}{=} f_0(1 - 1/y)^{a-a'}$$

with the choice $a' - a = 0.26$, LSI works well for $y \gtrsim 1.1$ but systematic deviations, still inside the ageing scaling region, for smaller values of $y = t/s$ (down to $y \simeq 1.001$)!

**Question**: improve the prediction of local scale-invariance (LSI)?
(C) assumption: \( R(t, s) = \langle \psi(t)\tilde{\psi}(s) \rangle \) 2D majority voter/Glauber model (triangular lattice)

good collapse \( \Rightarrow \) **no** logarithmic corrections \( \Rightarrow \) \( x' = \tilde{x}' = 0 \)

\[
h_R(y) = \left(1 - \frac{1}{y}\right)^{a-a'} \left[h_0 - g_{12,0} \ln(1 - 1/y) - \frac{1}{2} f_0 \ln^2(1 - 1/y)\right]
\]

no logarithmic terms for \( y \gg 1 \)
\( \Rightarrow \) \( \xi' = 0 \)

can normalise \( \tilde{\xi}' = 1 \)

F. Sastre (2012)

logar. LSI fit data, at least down to \( y \approx 1.005 \).