First passage time for random walks on complex networks

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Non-equilibrium statistical physics of complex systems
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I. Random walks in homogeneous systems

Rammal and Toulouse (1983)

J. Physique — LETTRES **44** (1983) L-13 - L-22 1er JANVIER **1983,** PAGE L-13

Classification
Physics Abstracts
05.40 — 72.90 — 64.70

Random walks on fractal structures and percolation clusters

R. Rammal (*) and G. Toulouse

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(*Reçu le 28 octobre 1982, accepté le 10 novembre 1982*)

Rammal and Toulouse (1983)

$$p_{is}(t) = \sum_{j \in nn(i)} \frac{1}{z_j} p_{js}(t-1)$$

Occupation probability

$$\left\langle r^2(t) \right\rangle \sim t^{2/d_w}$$

$$\Sigma(t) \sim r^{d_f} \sim t^{d_f/d_w} \sim t^{d_s/2}$$

The number of accessible sites in random walks of t steps

$$\rho(\omega) \sim \omega^{d_s - 1}$$

Using the scaling relation of $\rho(\omega)$

$$\frac{d_s}{2} = \frac{d_f}{d_w}$$

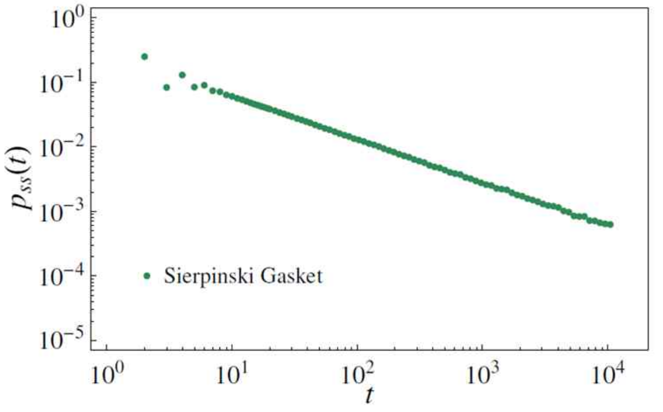
The probability of return to the origin after *t* steps

$$P_o(t) = \frac{1}{N}\sum_{s=1} p_{ss}(t)$$

$$\approx \frac{1}{N}\sum_{i=0}^{N-1}(1-\mu_i)^t$$

$$\approx \int_0^\infty d\mu \rho_\Delta(\mu) e^{-\mu t} \approx \int_0^\infty d\mu \mu^{\frac{d_s}{2}-1} e^{-\mu t}$$

$$\sim t^{-\frac{d_s}{2}} \qquad (t \to \infty)$$



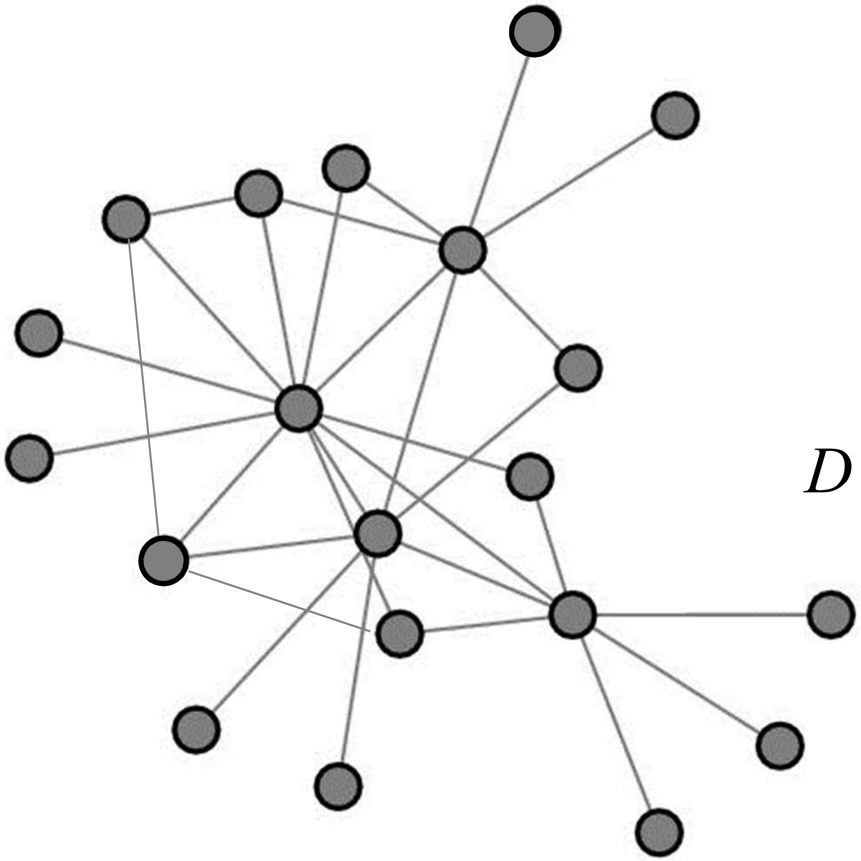
$$P_o(t) \sim \frac{1}{\Sigma(t)} \sim \frac{1}{t^{d_s/2}}$$

The number of distinct sites visited

$$S(t) \sim \begin{cases} t^{d_s/2} & d_s < 2 \\ t & d_s > 2 \end{cases}$$

II. Random walks on a heterogeneous network

Scale-free networks $P_d(k) \sim k^{-\gamma}$



$$D = \langle d_{ij} \rangle \sim \begin{cases} \ln \ln N & \text{for } 2 < \gamma < 3 \\ \ln N & \text{for } \gamma > 3 \end{cases}$$

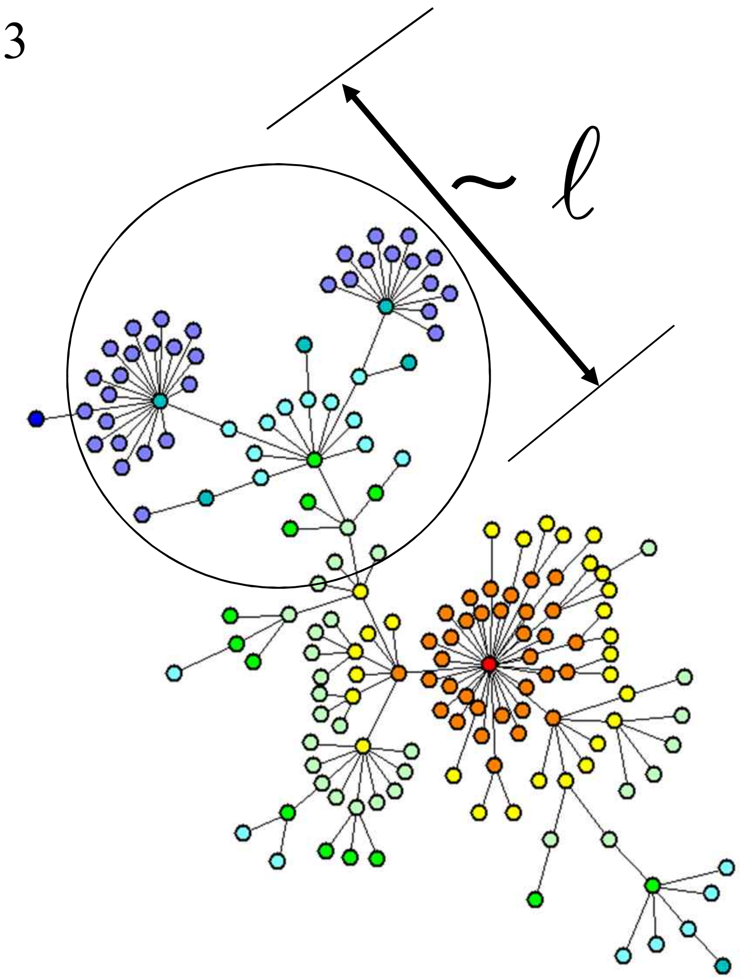
Critical branching tree

$$b_0 = 1 - \sum_{m=1} b_m \qquad b_m = \frac{\langle m \rangle_b}{\varsigma(\gamma - 1)} m^{-\gamma} \quad (m \ge 1)$$

$$\langle M \rangle \sim \ell^{d_B} \qquad d_B = \begin{cases} 2 & \gamma > 3 \\ \dfrac{\gamma - 1}{\gamma - 2} & 2 < \gamma < 3 \end{cases}$$

M is the mass within the circle

Goh PRL (2003), Burda PRE (2001)

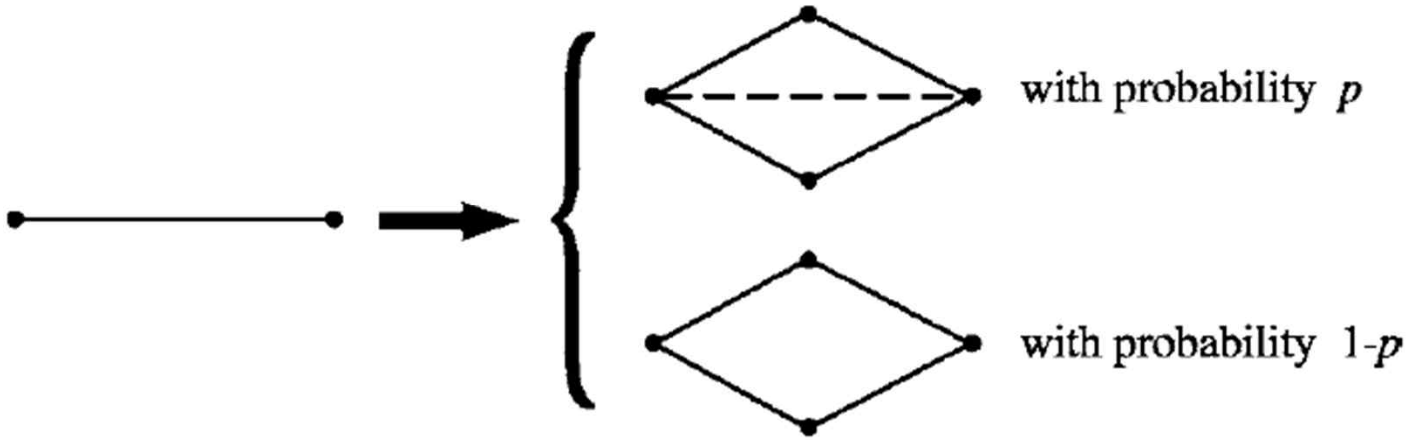


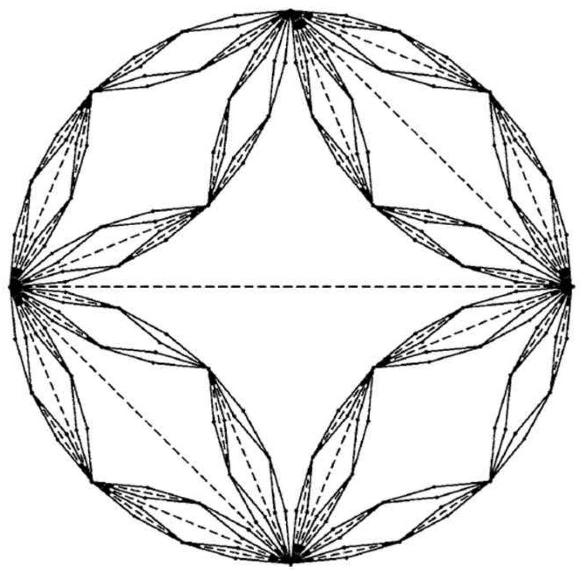
$$d_s = \begin{cases} \dfrac{2(\gamma - 1)}{2\gamma - 3}, & \text{for } 2 < \gamma < 3, \\ \dfrac{4}{3}, & \text{for } \gamma > 3. \end{cases}$$

Hierarchical model

(u,v) flower model

(2,2) flower model



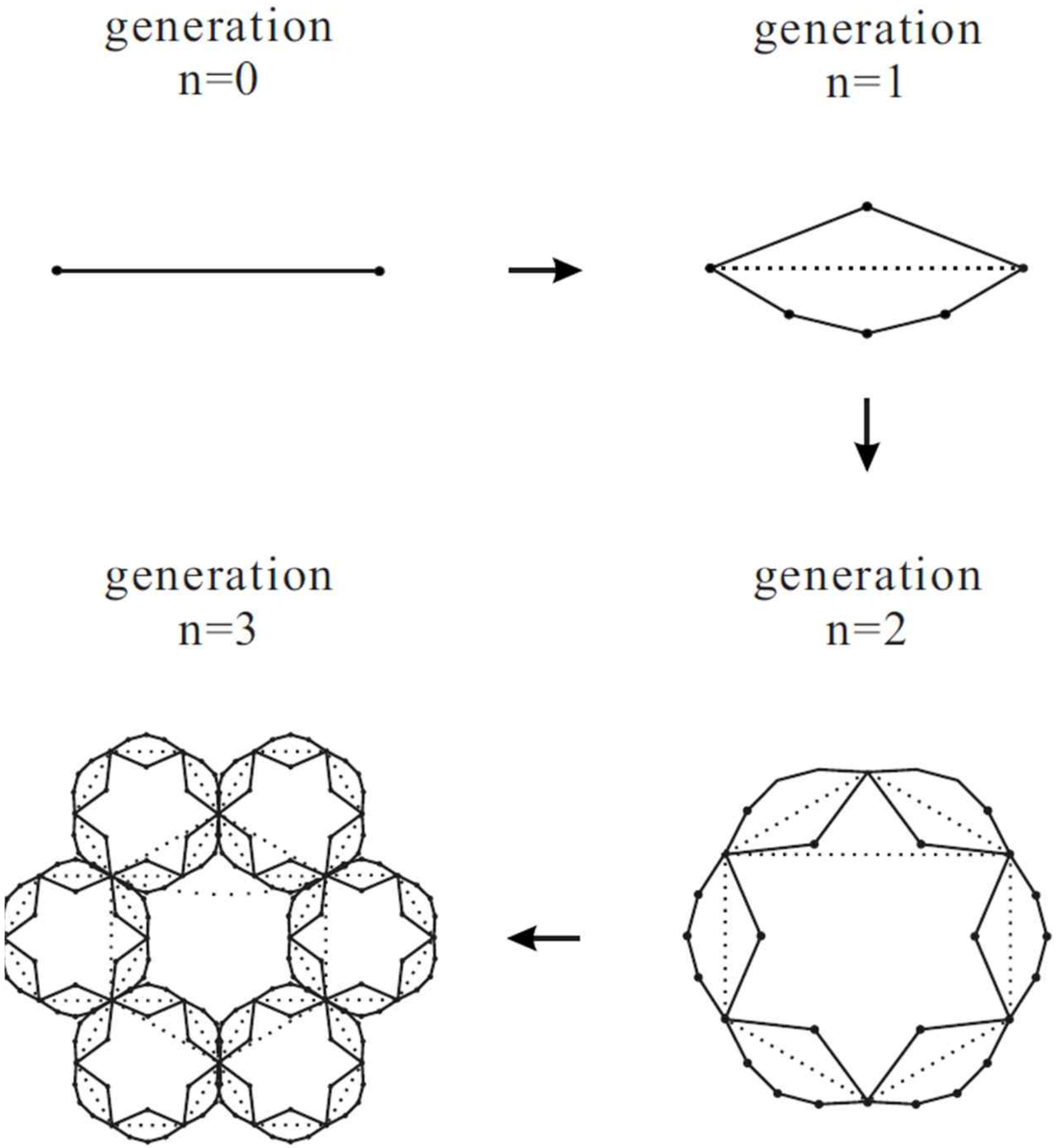


$n=4$, $p=0.6$

Berker and Ostlund, (1979)

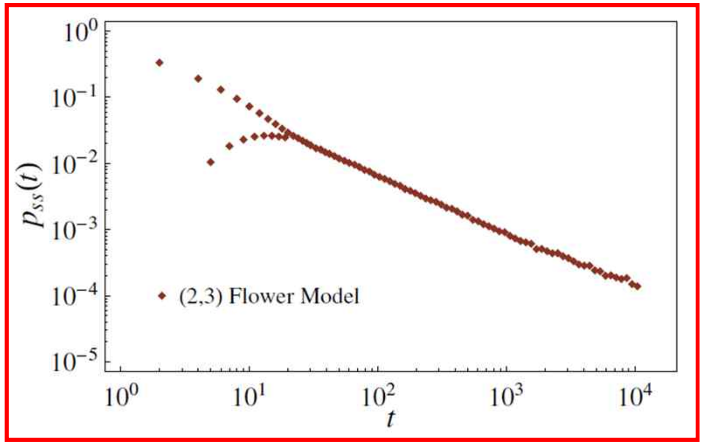
Hinczewski and Berker, (2006)

(2,4)-flower model



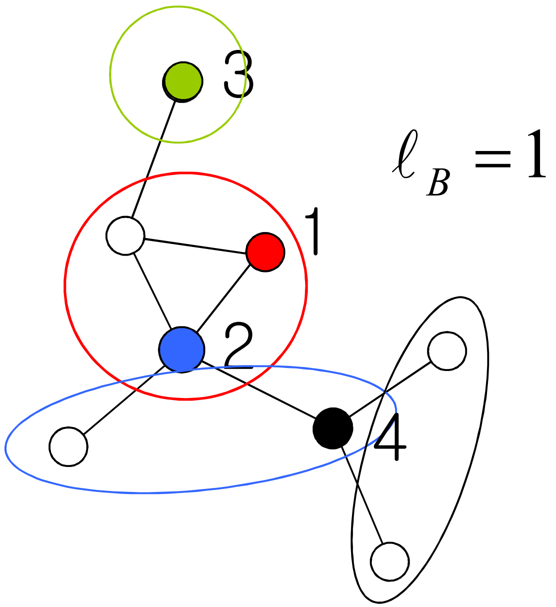
$$\langle d_{ij} \rangle \sim \begin{cases} N^{1/d_f} & \text{for } p = 0 \\ \ln N & \text{for } p \neq 0 \end{cases}$$

$$d_s \sim \begin{cases} \dfrac{2\ln(u+v)}{\ln(uv)} & \text{for } p = 0 \\ 2 & \text{for } p \neq 0 \end{cases}$$



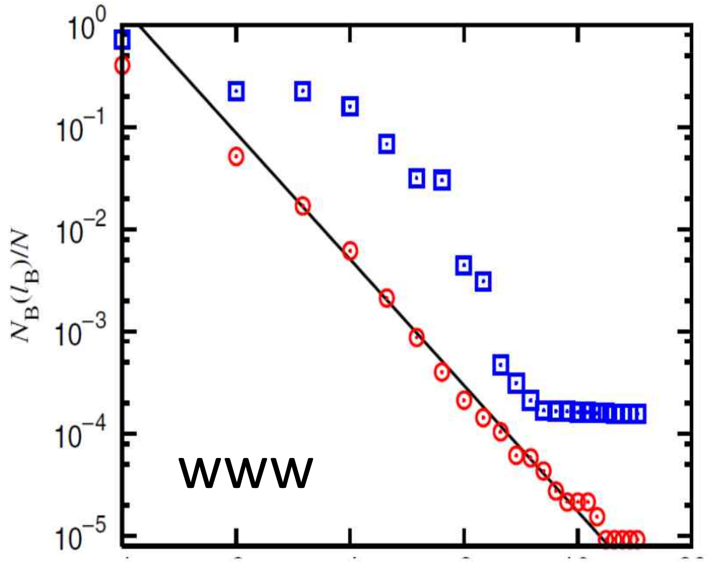
Hwang et al. PRE (2009)

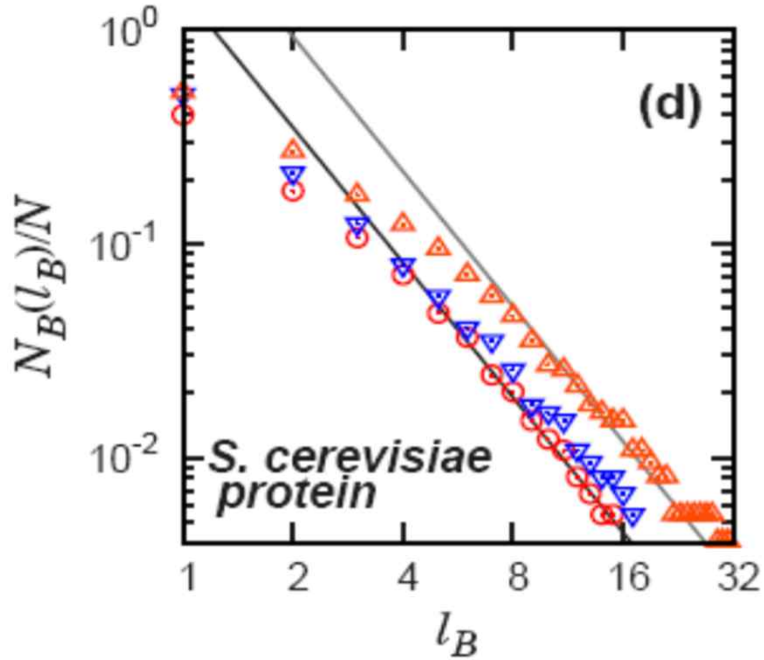
Random sequential packing:

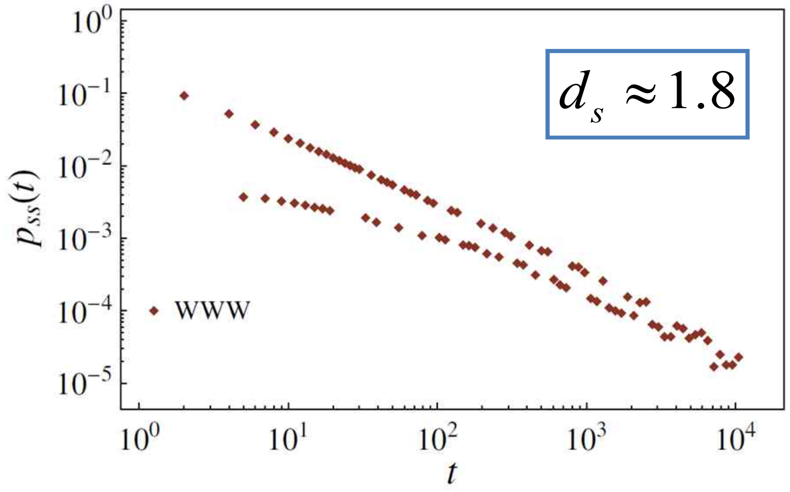


Goh, et al. PRL (2006)

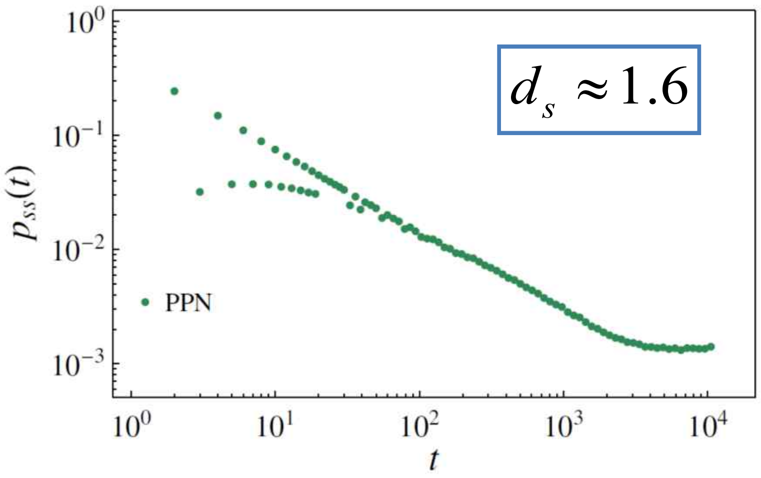
$$d_s \sim \begin{cases} 1.8 & \text{for www} \\ 1.6 & \text{for yeast} \end{cases}$$



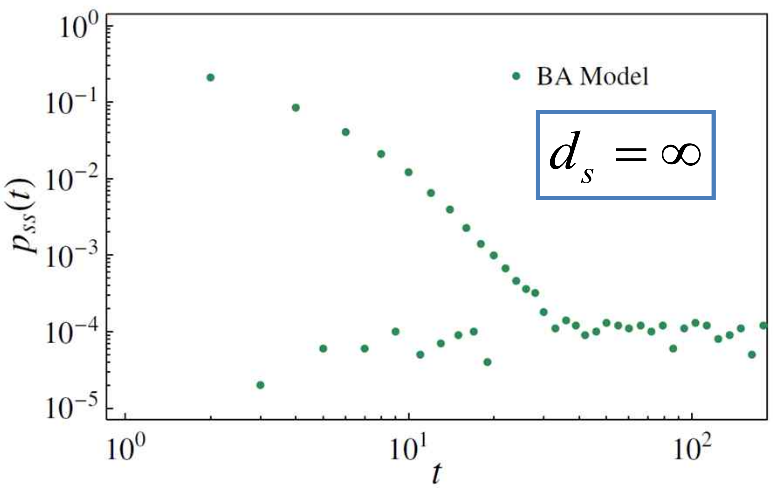




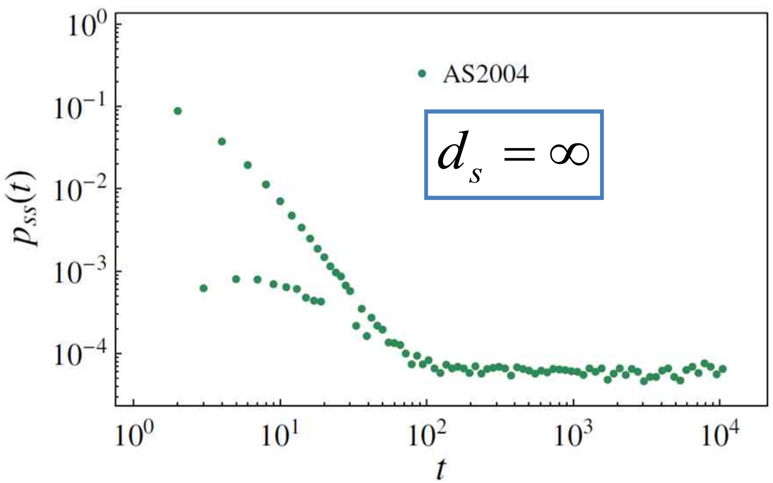
$d_s \approx 1.8$
WWW
$p_{ss}(t)$
t
10^0 10^{-1} 10^{-2} 10^{-3} 10^{-4} 10^{-5}
10^0 10^1 10^2 10^3 10^4



$d_s \approx 1.6$
PPN
$p_{ss}(t)$
t
10^0 10^{-1} 10^{-2} 10^{-3}
10^0 10^1 10^2 10^3 10^4



BA Model
$d_s = \infty$
$p_{ss}(t)$
t
10^0 10^{-1} 10^{-2} 10^{-3} 10^{-4} 10^{-5}
10^0 10^1 10^2



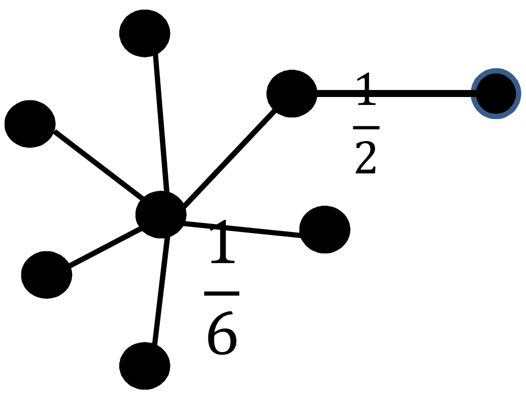
AS2004
$d_s = \infty$
$p_{ss}(t)$
t
10^0 10^{-1} 10^{-2} 10^{-3} 10^{-4}
10^0 10^1 10^2 10^3 10^4

Random walks on a heterogeneous network

$$p_{is}(t \to \infty) = \frac{k_i}{2L}$$

Noh and Rieger, PRL (2004)

$$P_o(t) = \frac{1}{N}\sum_{s=1} p_{ss}(t)$$



$$p_{ss}(t) = ?$$

Purposes:

1) Probability to return to the origin $p_{ss}(t)$

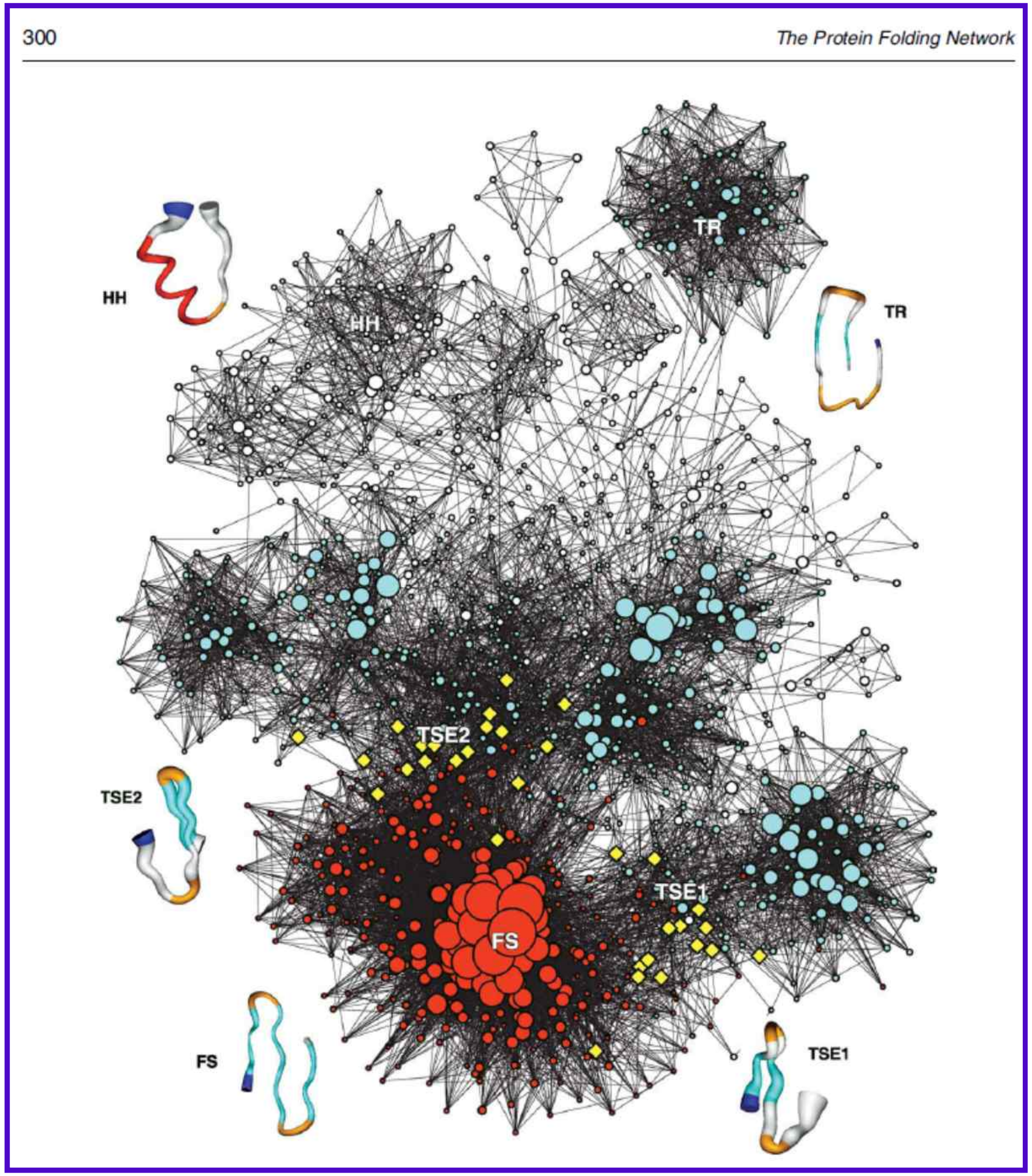
2) Global first passage time:

- GFPT distribution $$F_m(t) = \sum_{i=1}^{N} \frac{k_i}{2L} F_{mi}(t)$$

- Mean GFPT $$T_m = \sum_t t F_m(t)$$

as a function of d_s and γ.

- Many studies on these have been performed on deterministic SF nets,
- but not on undeterministic networks, or
- asymptotic behaviors for some limited cases



The protein folding network

- F. Rao and A. Caflisch

JMB (2004)

Probability to return to the origin *s*

$$p_{is}(t \to \infty) = \frac{k_i}{2L} \quad \Rightarrow \quad p_{is}(t) = \frac{\hat{k}_i(t)}{2\hat{L}(t)}$$

$$\widehat{k}_i(t) = \sum_{j \in nn(i)} \widehat{L}_{ij}(t)$$

Sum of the link accessibility from node j to i

$$\widehat{L}(t) = \sum_{i=1}^{N} \widehat{k}_i(t)/2$$

Number of accessed links

$$\widehat{L}(t) \simeq \frac{<k>}{2P_o(t-2)} \sim \begin{cases} t^{d_s/2} & \text{for } t \ll t_x \\ L & \text{for } t \gg t_x \end{cases}$$

cf. $\Sigma(t) \sim t^{d_s/2}$

Number of accessible sites

$$t_x \sim L^{2/d_s}$$

For the hub

$$\hat{k}_h(t) \sim \hat{L}(t)^{1/(\gamma-1)}$$ Similar to natural cutoff relation

$$\sim \; t^{d_s/2(\gamma-1)}$$ assumed $\{\hat{k}_i(t)\}$ follow $P_d(\hat{k})$

$$\hat{k}_h \sim \begin{cases} t^{d_s/2(\gamma-1)} & \text{for } \; t \ll t_x \\ k_h & \text{for } \; t \gg t_x \end{cases}$$

$$t_x \sim k_h^{\;2(\gamma-1)/d_s} \sim L^{2/d_s}$$

$$p_{hh}(t) = \frac{\hat{k}_h(t)}{2\hat{L}(t)} \sim \begin{cases} t^{-d_s^{(\mathrm{hub})}/2} & \text{for } t \ll t_{\mathrm{x}}, \\ \frac{k_h}{2L} & \text{for } t \gg t_{\mathrm{x}}, \end{cases}$$

$$d_s^{(\mathrm{hub})} = d_s \frac{\gamma - 2}{\gamma - 1}$$

For a node *m*

$$\hat{k}_m \sim \begin{cases} t^{d_s/2(\gamma-1)} & \text{for } \; t \ll t_c(k_m) \\ k_m & \text{for } \; t \gg t_c(k_m) \end{cases}$$

$$t_c(k_m) \sim k_m^{\;2(\gamma-1)/d_s}$$

$$\hat{k}_m \sim \begin{cases} t^{d_s/2(\gamma-1)} & \text{for } t \ll t_c(k_m) \\ k_m & \text{for } t \gg t_c(k_m) \end{cases} \qquad t_c(k_m) \sim k_m^{2(\gamma-1)/d_s}$$

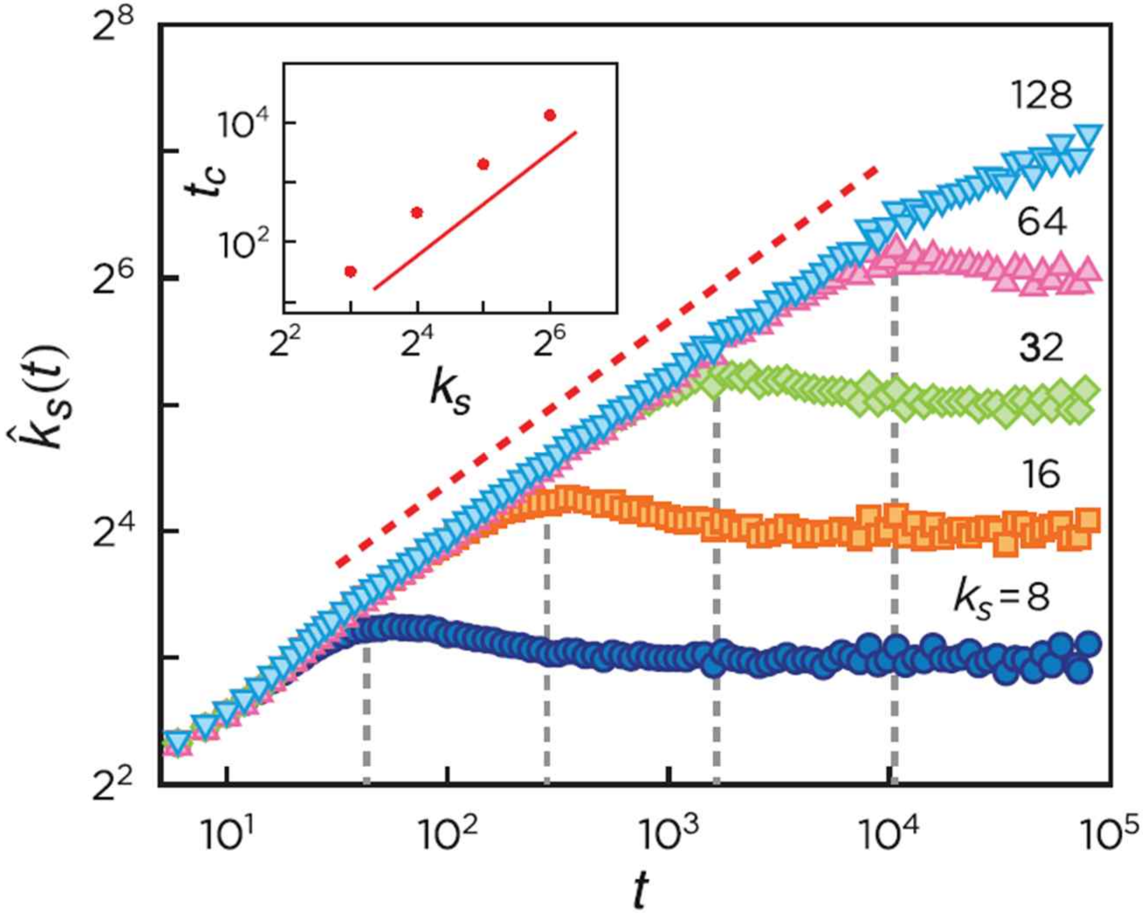
$$p_{mm}(t) = \frac{\hat{k}_m(t)}{2\hat{L}(t)} \sim \begin{cases} t^{-d_s^{(h)}/2} & \text{for} \quad t \ll t_c(k_m) \\ k_m t^{-d_s/2} & \text{for} \quad t_c(k_m) \ll t \ll t_x \\ \dfrac{k_m}{2L} & \text{for} \quad t \gg t_x \end{cases}$$

$$d_s^{(\text{hub})} = d_s \frac{\gamma - 2}{\gamma - 1}$$

when $\gamma \to 2$, $d_s^{(\text{hub})} \to 0$, and $p_{ss}(t) \to$ const. during $t_c(k_m)$.

Random walks are trapped at local hubs,
Minotaur's labyrinth.

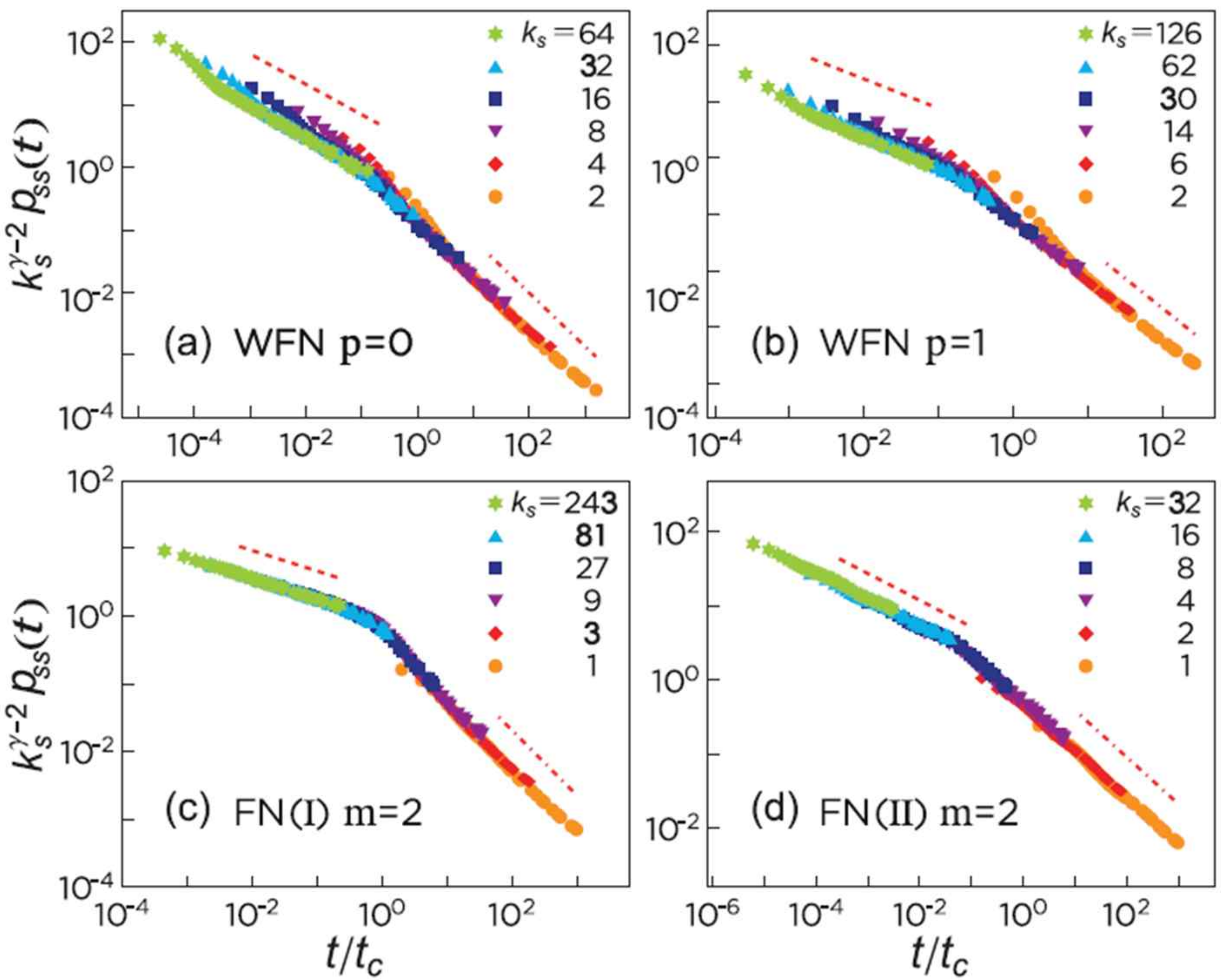
Effective degree of starting node vs time



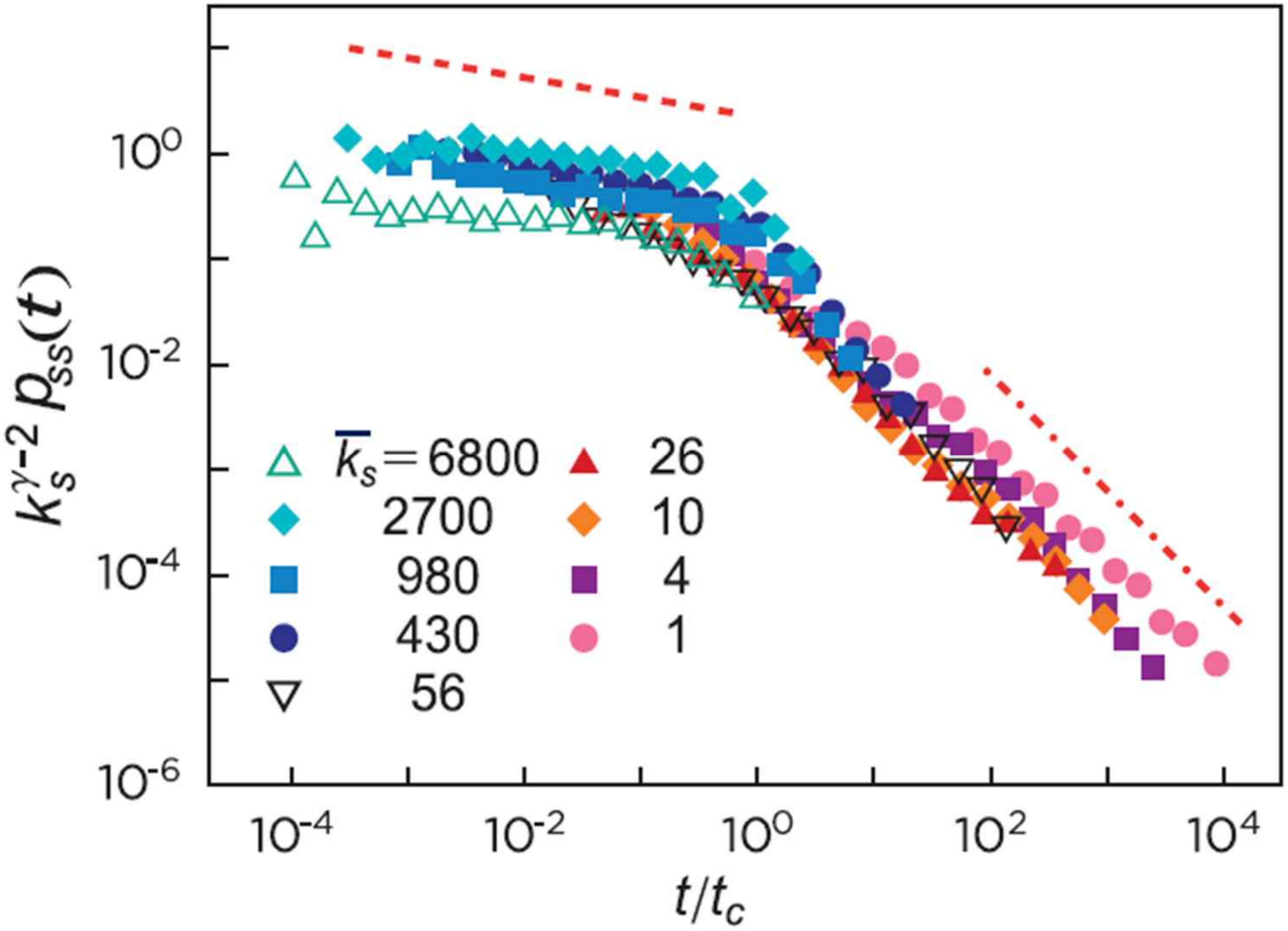
$$\hat{k}_m \sim \begin{cases} t^{d_s/2(\gamma-1)} & \text{for } t \ll t_c(k_m) \\ k_m & \text{for } t \gg t_c(k_m) \end{cases}$$

$$t_c(k_m) \sim k_m^{\,2(\gamma-1)/d_s}$$

Probability to return to the origin on model nets



Probability to return to the origin on the WWW



First passage time distribution for RWs

$$F_m(t) = \sum_{s=1}^{N} \frac{k_s}{2L} F_{m\,s}(t)$$

FPT probability for RWs starting from s to m

Using the renewal equation,

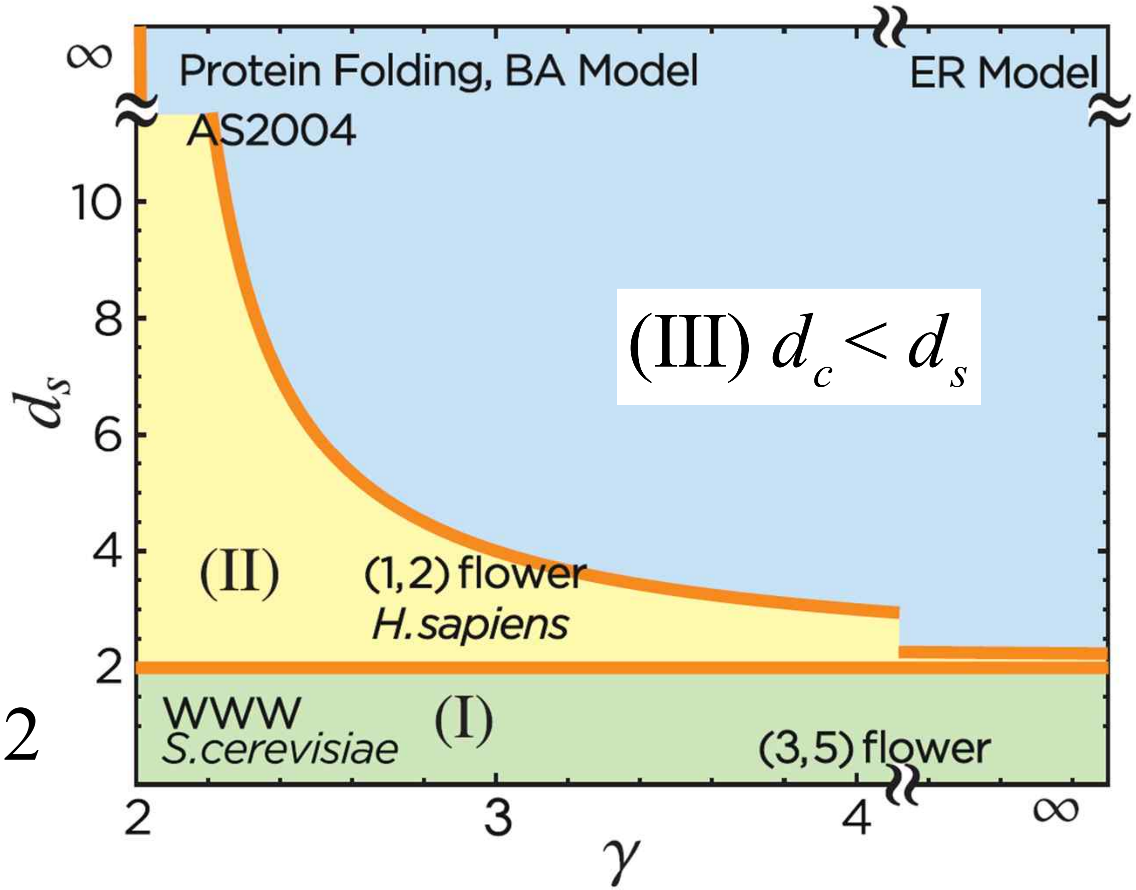
$$p_{m\,s}(t) = \delta_{m\,s}\delta_{t0} + \sum_{t'=0}^{t} F_{m\,s}(t')p_{m\,m}(t-t')$$

$$\mathcal{F}_m(z) = \frac{k_m\,z}{2L\,(1-z)} \frac{1}{R_m(z)}$$

$R_m(z)$ is a generating function of $p_{m\,m}(t)$

Phase diagram in (d_s, γ) space

(II) $2 < d_s < d_c = \dfrac{2(\gamma - 1)}{(\gamma - 2)}$



(I) $d_s < 2$

(I) $d_s < 2$

$1 \ll t \ll t_c(k_m)$

$$F_m(t) \sim \frac{k_m}{2L} t^{-(1-d_s^{(h)}/2)}$$

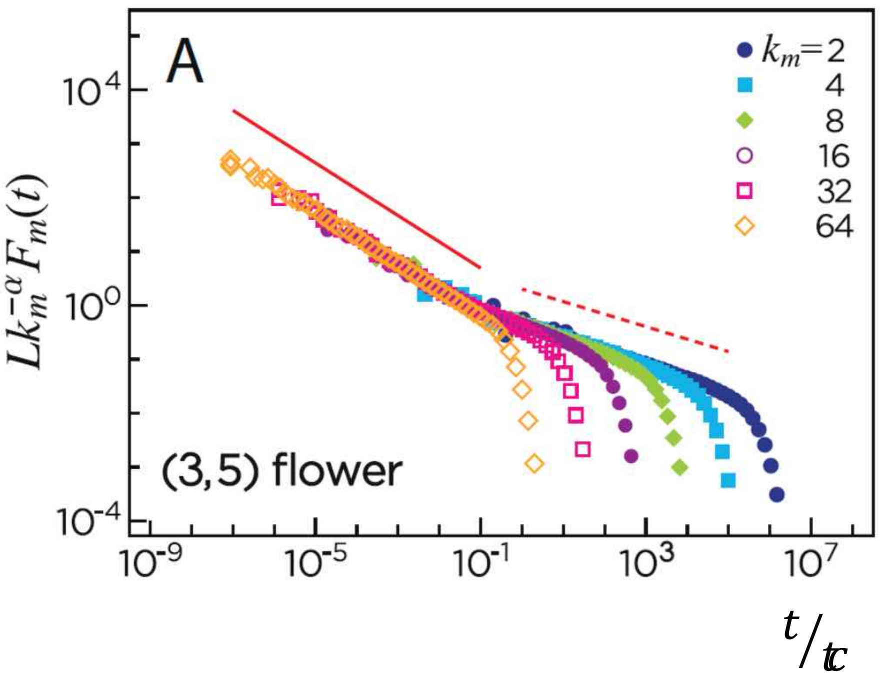
$$d_s^{(h)} = d_s \frac{\gamma - 2}{\gamma - 1}$$

$t_c(k_m) \ll t \ll t_x$

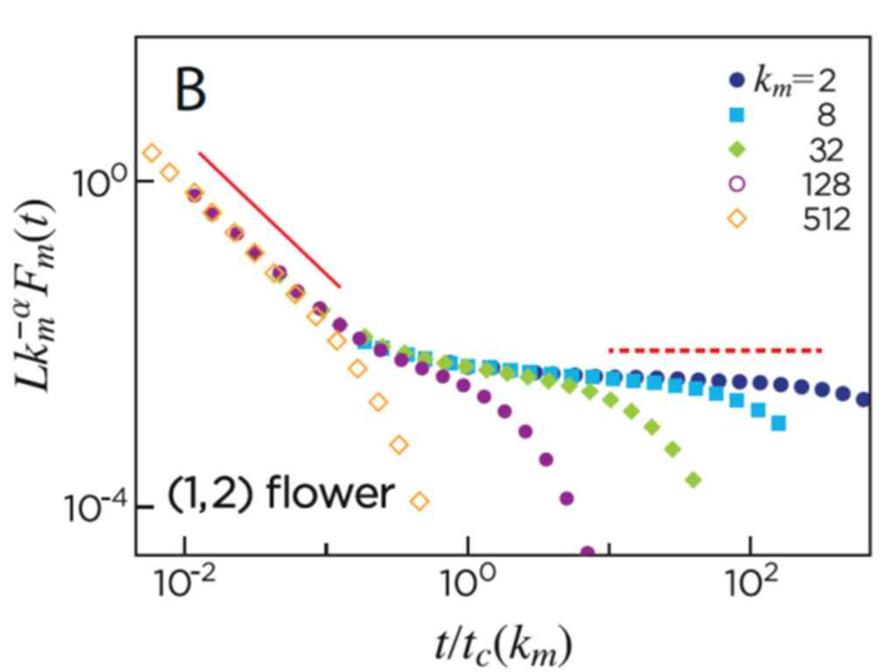
$$F_m(t) \sim \frac{1}{2L} t^{-(1-d_s/2)}$$

$t \gg t_x$

$$F_m(t) \sim N^{-d_s/2} e^{-t/N^{d_s/2}}$$



(II) $2 < d_s < d_c = \dfrac{2(\gamma-1)}{(\gamma-2)}$



$1 \ll t \ll t_c(k_m)$

$$F_m(t) \sim \frac{k_m}{2L} t^{-(1-d_s^{(h)}/2)}$$

$t_c(k_m) \ll t$

$$F_m(t) \sim N^{-1} k_m^{\alpha} e^{-t/Nk_m^{-\alpha}}$$

$$\alpha = (1-\frac{2}{d_s})(\gamma-1)$$

(III) $d_c < d_s$

$$F_m(t) \sim N^{-1} k_m e^{-t/Nk_m^{-1}} \qquad \text{for any } t$$

Mean First Passage Time

$$\mathcal{F}_m(z) = \frac{k_m z}{2L(1-z)} \frac{1}{\mathcal{R}_m(z)}$$

$$T_m = \frac{\partial}{\partial z}\mathcal{F}_m(z)\bigg|_{z=1} \approx \frac{2L}{k_m}\mathcal{R}_m^*(1) + 1$$

$$= \frac{2L}{k_m}\sum_{t=0}^{\infty}(R_m(t) - R_m(\infty)) + 1.$$

$$T_m \approx \frac{2L}{k_m}\int_1^{t_x}[R_m(t) - R_m(\infty)]dt$$

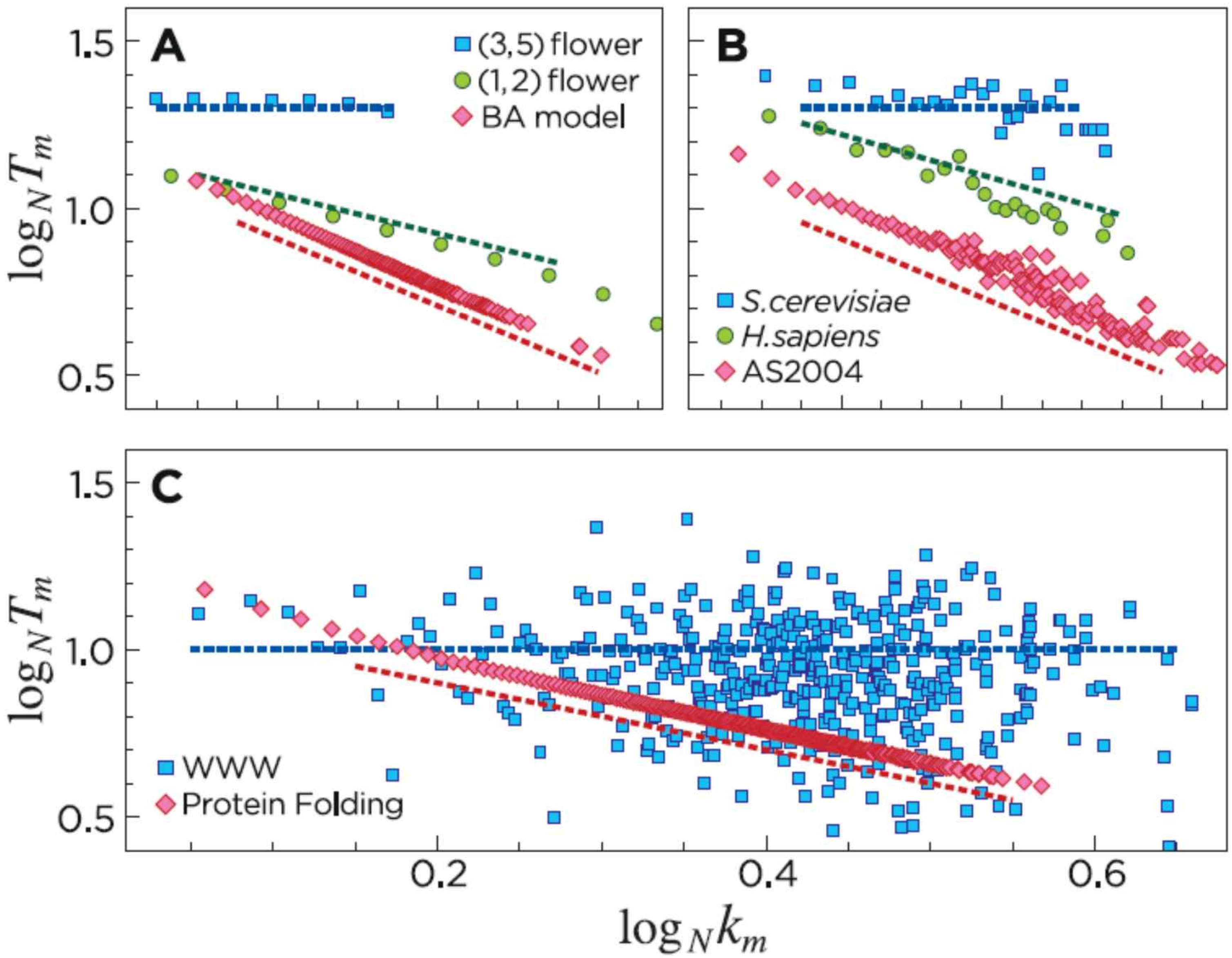
$$\sim \begin{cases} N^{2/d_s} & \text{(I)} \quad d_s < 2, \\ N k_m^{-\alpha} & \text{(II)} \quad 2 < d_s < d_c, \\ N k_m^{-1} & \text{(III)} \quad d_s > d_c, \end{cases}$$

1) When the target is a hub with degree $k_h \sim N^{1/(\gamma-1)}$ for $2 < \gamma < 3$,

$$T_h \sim \begin{cases} N^{2/d_s} & \text{(I and II)}, \\ N^{(\gamma-2)/(\gamma-1)} & \text{(III)}. \end{cases}$$

2) When the target is a hub with degree $k_h \sim N^{1/(5-\gamma)}$ for $2 < \gamma < 3$

$$T_h \sim \begin{cases} N^{2/d_s} & \text{(I)}, \\ N^{[2(3-\gamma)+2(\gamma-1)/d_s]/(5-\gamma)} & \text{(II)}, \\ N^{(4-\gamma)/(5-\gamma)} & \text{(III)}. \end{cases}$$



A
(3,5) flower
(1,2) flower
BA model
B
S.cerevisiae
H.sapiens
AS2004
C
WWW
Protein Folding
1.5
1.0
0.5
0.2
0.4
0.6
$\log_N T_m$
$\log_N k_m$

Conclusions

1. Probability to return to the origin has been studied in diverse scale-free networks

2. First passage time problems have been studied in diverse scale-free networks

Complete analytic formulae for those quantities are derived in terms of $d_s, \gamma, k_m,$ and N.

References:

1) Hwang et al. PRE 82, 056110 (2010)
2) Hwang et al. PRE 85, 046110 (2012)
3) Hwang et al. Preprint (2012)

Suppression effect on explosive percolations

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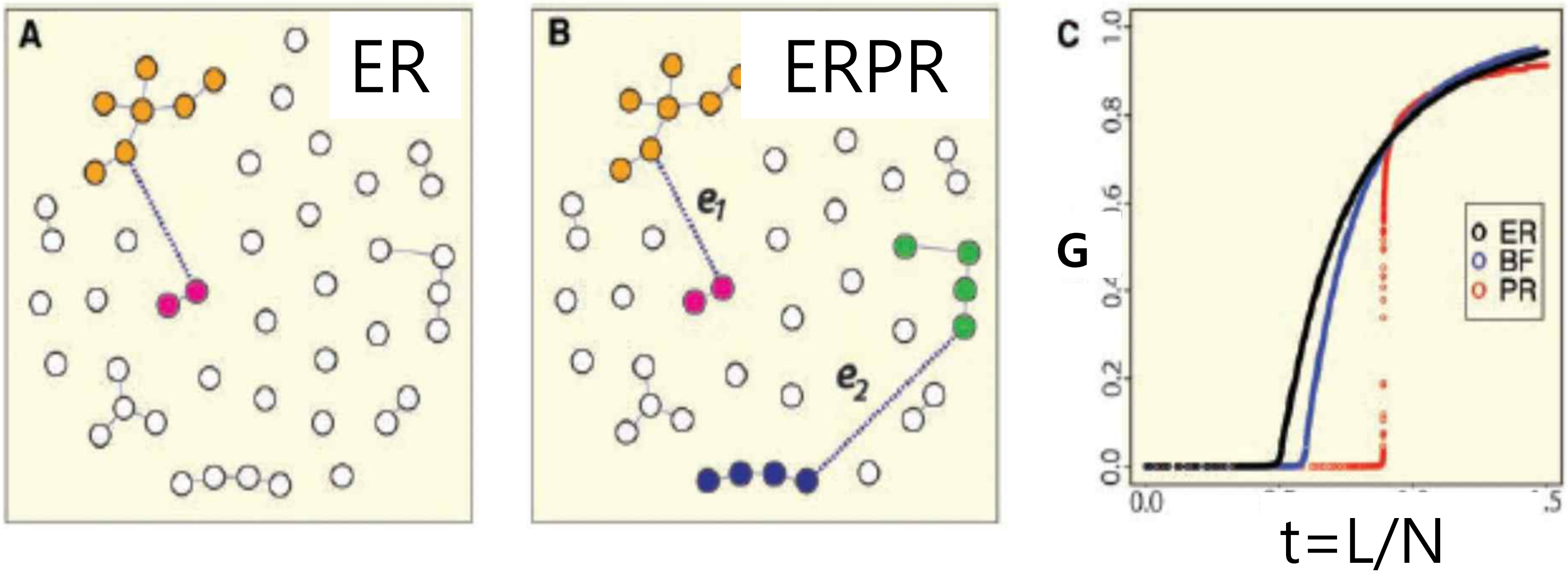
The 5th KIAS conference on statistical physics
Non-equilibrium statistical physics of complex systems
At KIAS, Seoul, July 3-6, 2012

1. Background



1) The number of nodes is fixed as *N*.

2) Edges are added one by one to the system between two nodes randomly chosen at each time step.

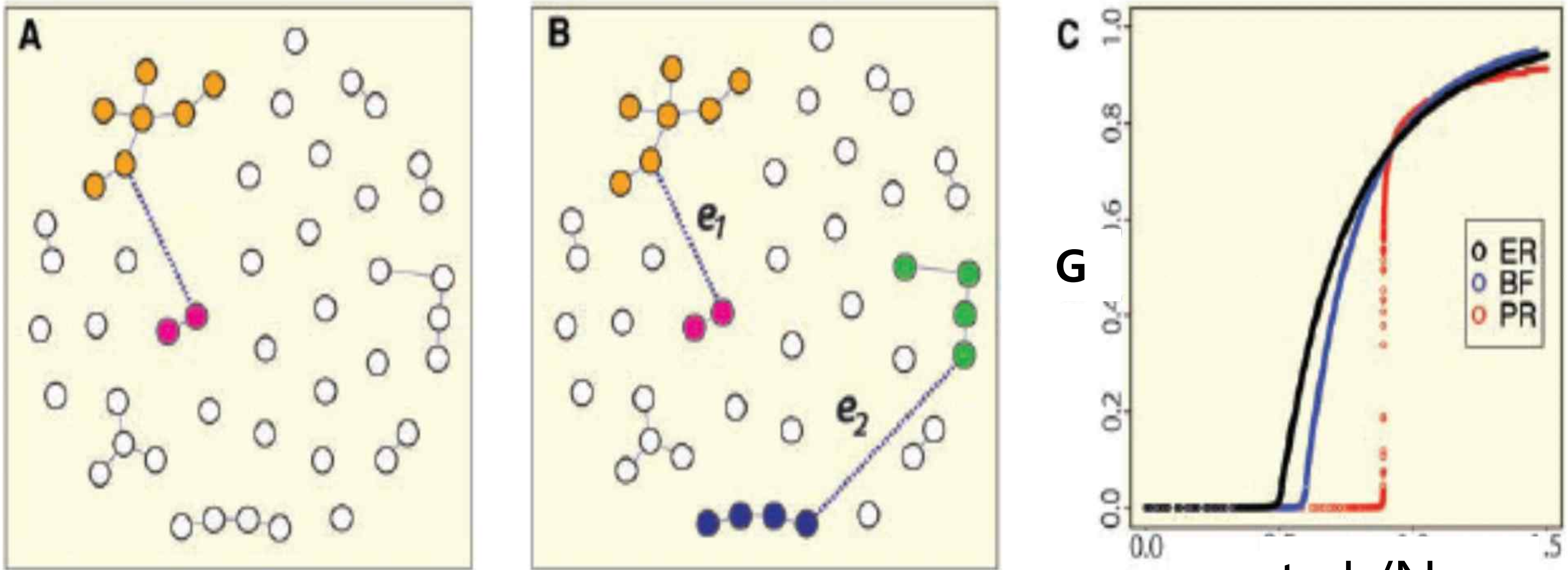
➔ Percolation transition at $t_c = L_c / N = 1/2$

➔ Continuous transition

Achlioptas process

Achlioptas et al, Science (2009,3)

ER　　　　ERPR



t=L/N

1. Pick up two edge candidates randomly.
2. Calculate the product of two-cluster sizes:
   By e_1, 7*2=14 vs. by e_2, 4*4=16 ➔ $e_1 < e_2$ (product rule)
3. Then, e_1 is attached, and e_2 is discarded.

➔ Growth of large clusters is suppressed.
➔ Percolation transition point is delayed.

ERPR

2. Goal

Is the explosive percolation transition continuous or discontinuous ?

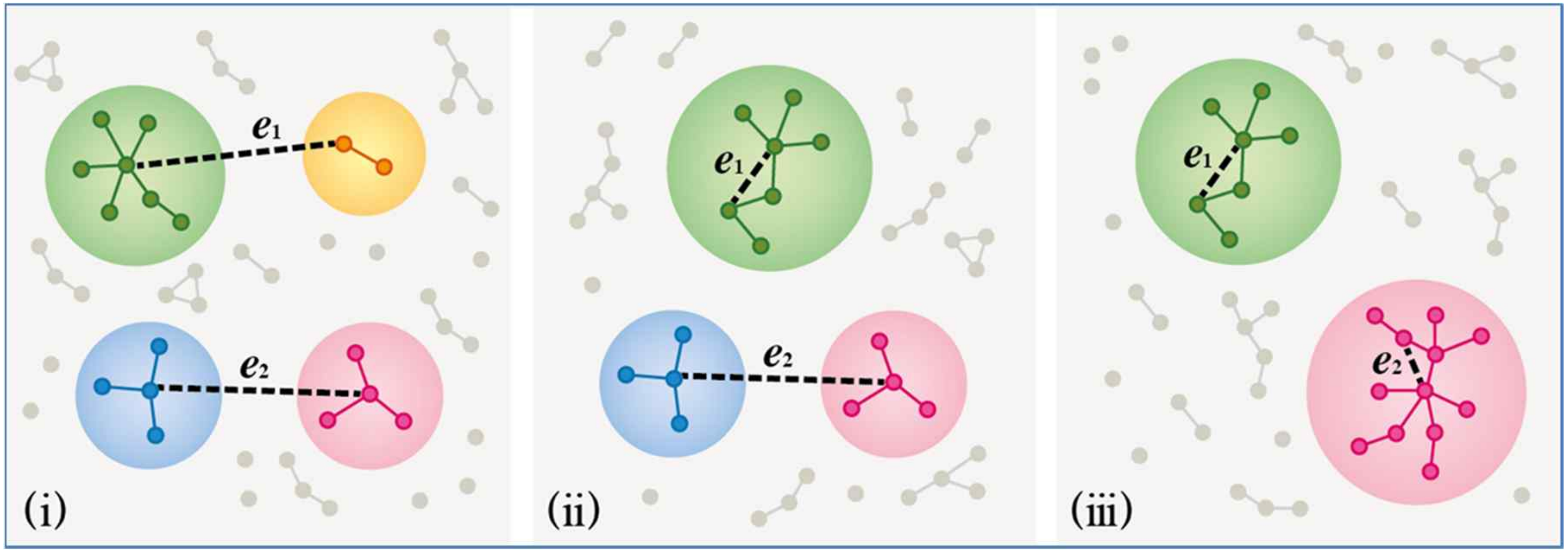
1) Achlioptas et al, **Explosive percolation transition**, Science (2009,3).
2) Many others.

1) R.A. da Costa, S.N. Dorogovtsev, A.V. Goltsev, J.F.F. Mendes **Explosive Percolation Transition is Actually Continuous,** PRL 105, 255701 (2010).
2) P. Grassberger, C. Christensen, G. Bizhani, S.-W. Son, M. Paczuski, **Explosive percolation is continuous, but with unusual finite size behavior,** PRL 106, 225701 (2011).
3) O. Riordan and L. Warnke, **Explosive percolation is continuous**, Science 333, 322 (2011).
4) H.K. Lee, B.J. Kim, and H. Park, **Continuity of the explosive PT**, PRE 84, 020101 (2011).

- ✓ The Achlioptas process (AP):
  the dynamics avoiding the formation
  of a given pattern in evolution of graph.

- ✓ The percolation model following the AP:
  the target pattern is giant component.
  Thus, **the dynamics has to be proceeded to avoid the formation of a giant cluster.**

3. Classification of edge candidates

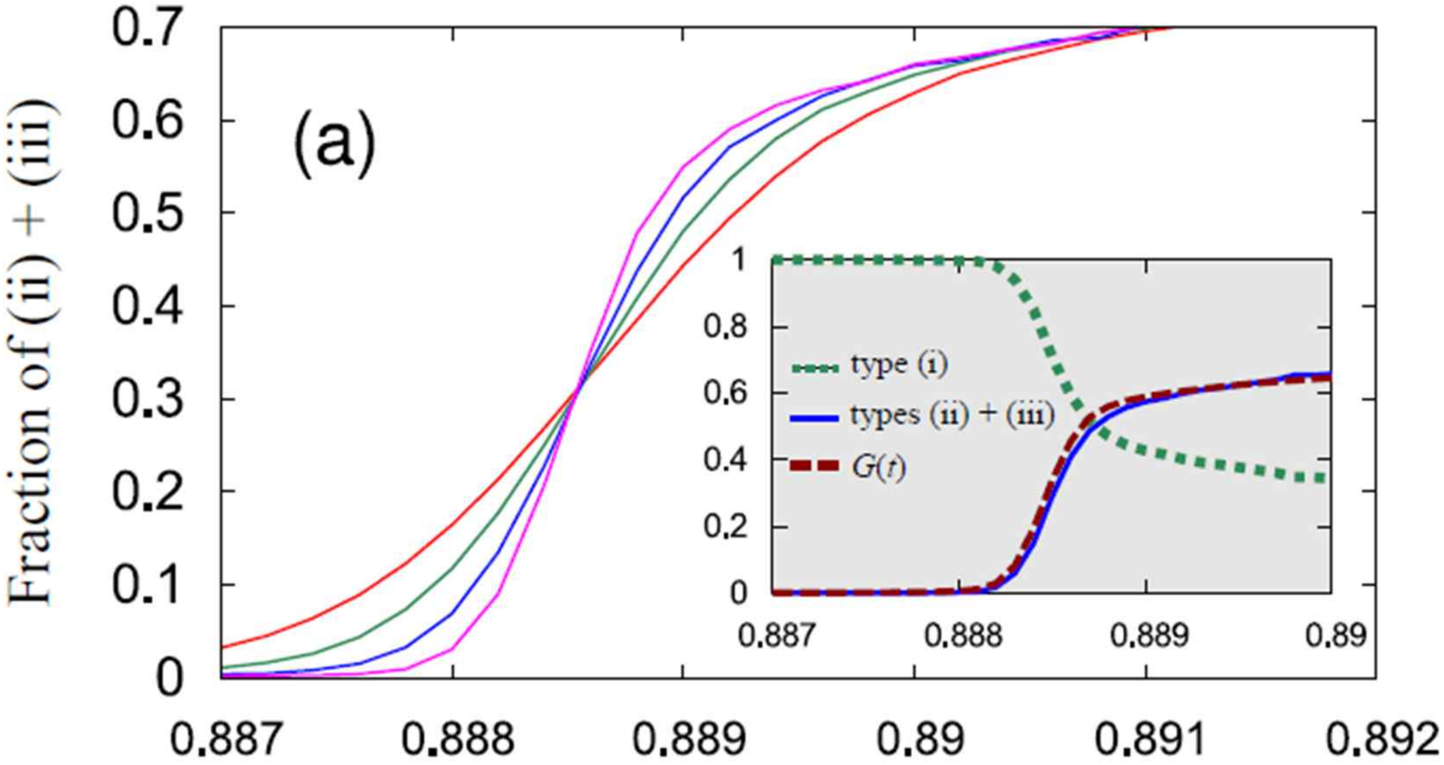


Inter-cluster edges

Inter-cluster edge
+
Intra-cluster edge

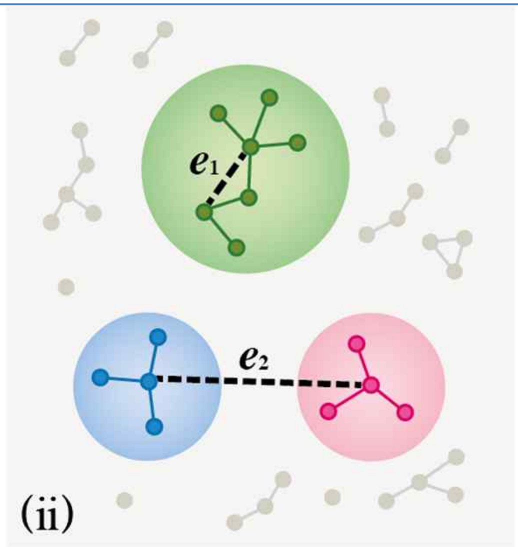
Intra-cluster edges

Fraction of type (ii) & (iii)



t=L/N

4. Model Variants (Product Rule)



For the case (ii)

ERPR-A (original rule)

$S_1^2=7^2$ vs. $S_{2a}*S_{2b}=4*4=16$

➔Take e_2

But e_1 is desirable

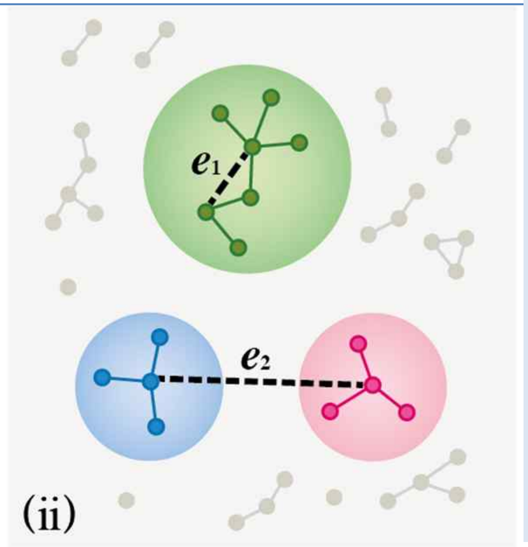
ERPR-B

➔Take e_1 (Absolutely)
Cluster size unchanged

ERPR-C

Case (ii) is excluded.

Model Variants (Sum Rule)



For the case (ii)

ERSR-A

$2S_1$=2*7 vs. $S_{2a}+S_{2b}$=4+4=8

➔Take e_2

But e_1 is desirable

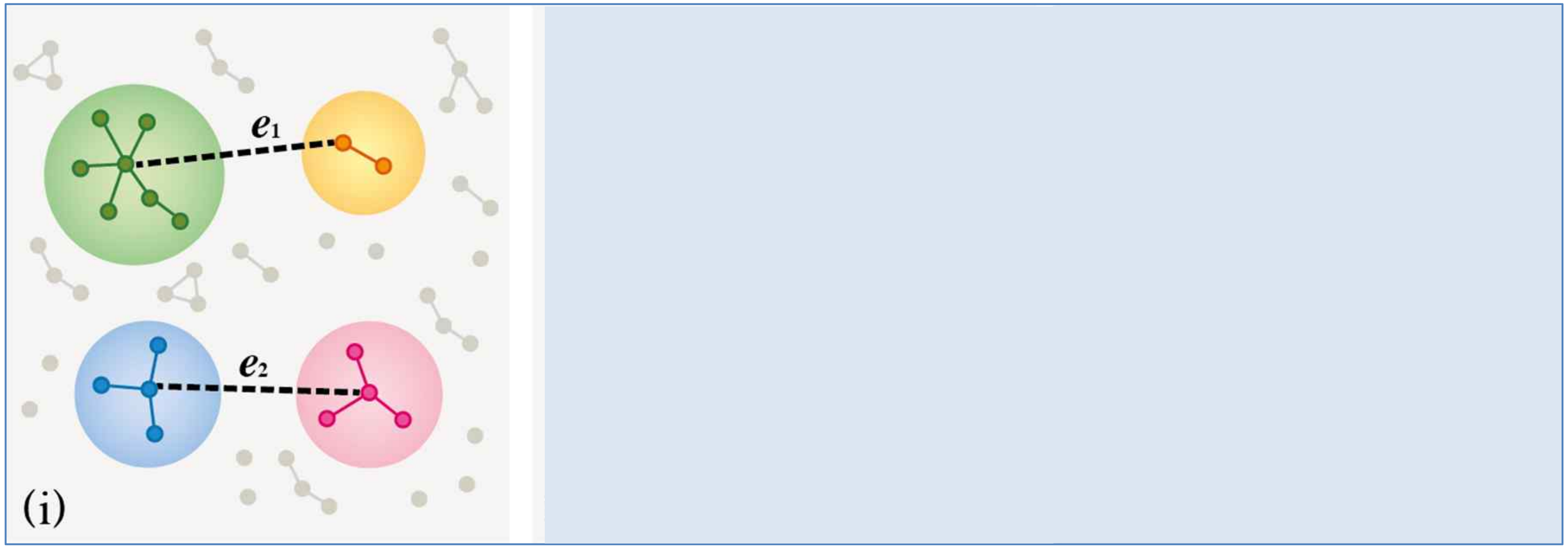
ERSR-B

➔Take e_1 (Absolutely)
Cluster size unchanged

ERSR-C

Case (ii) is excluded.

5. Intrinsic fault of product rule



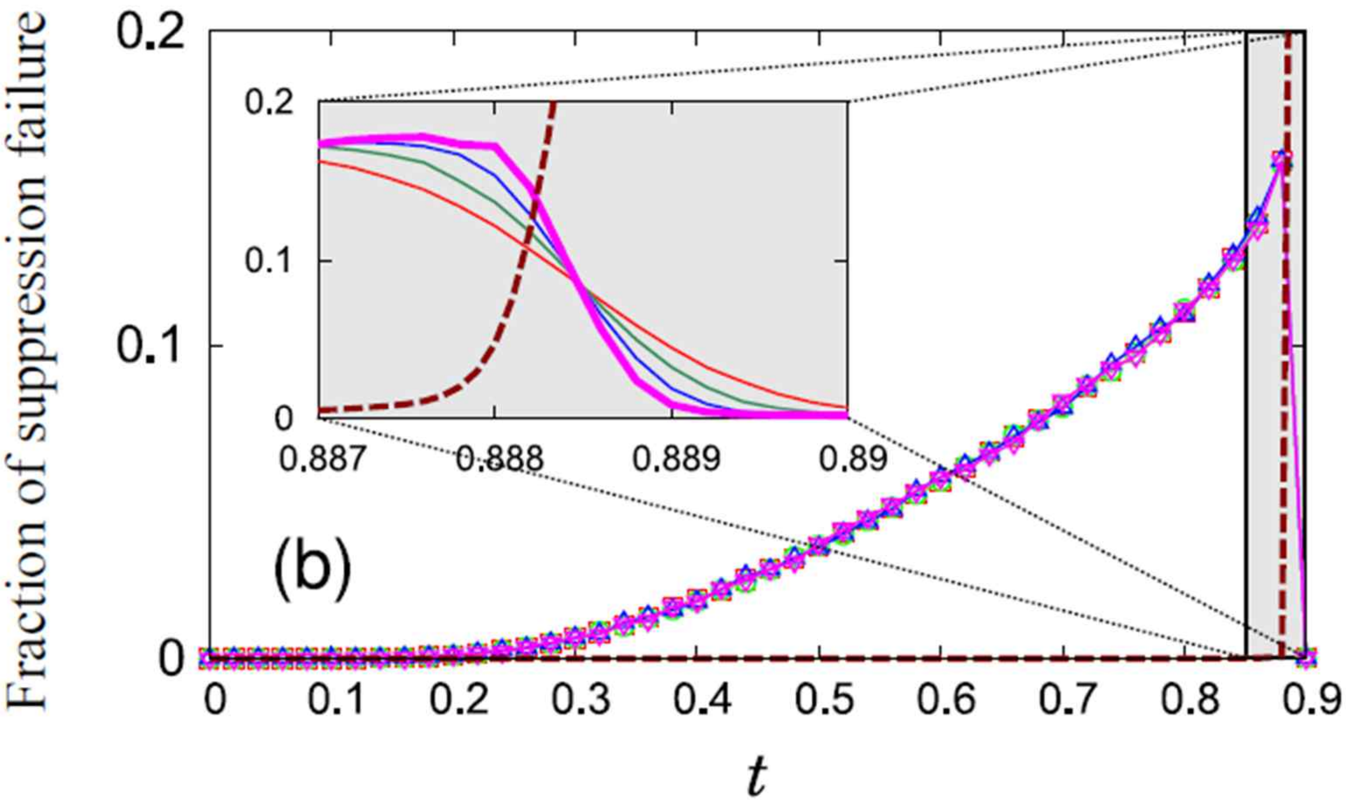
For the case (i)

$S_{1a}*S_{1b}=7*2=14$ vs.
$S_{2a}*S_{2b}=4*4=16$

e_1 was taken in PR.

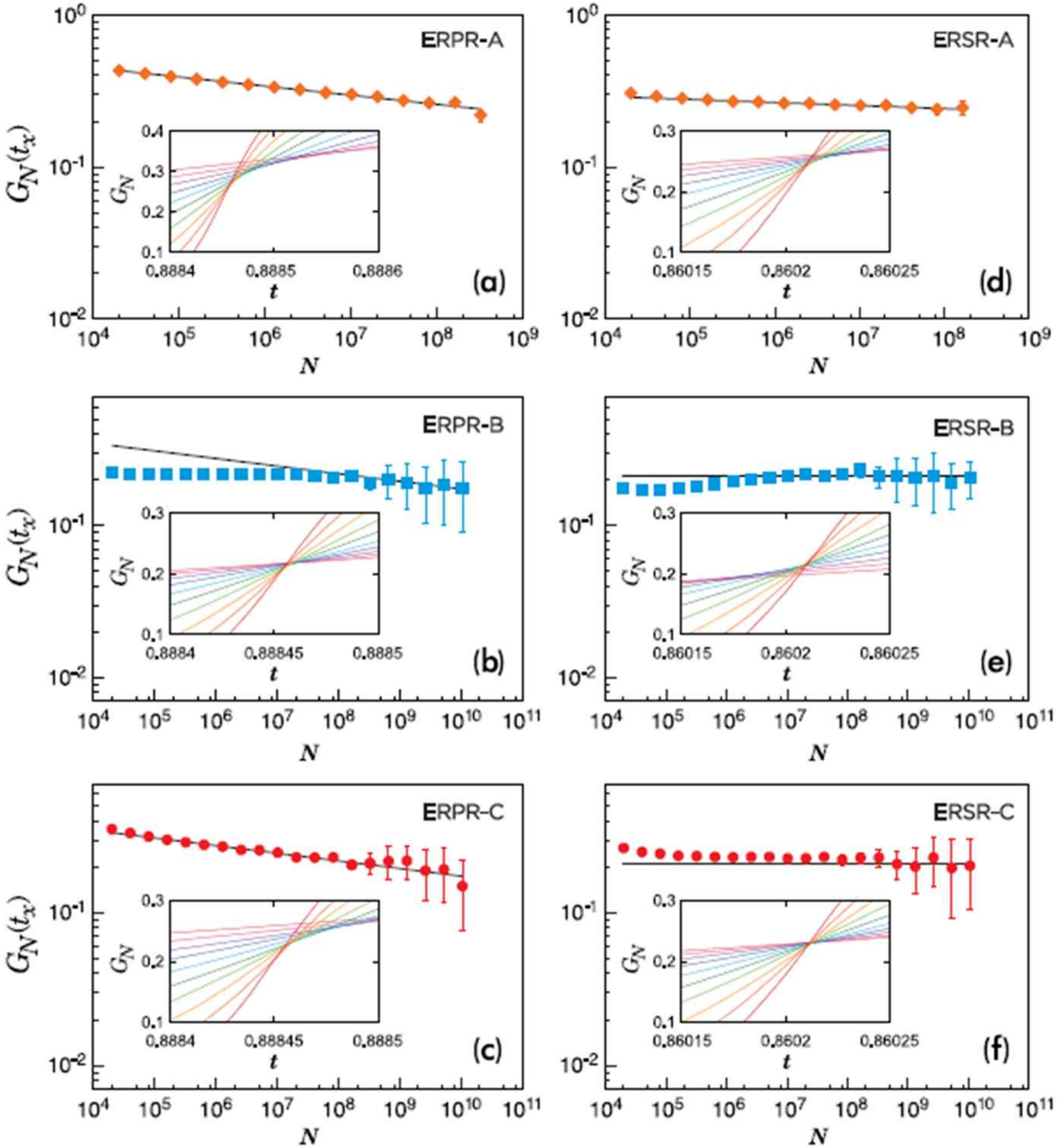
$S_{1a}+S_{1b}=7+2=9$ vs.
$S_{2a}+S_{2b}=4+4=8$

➔ e_2 has to be taken

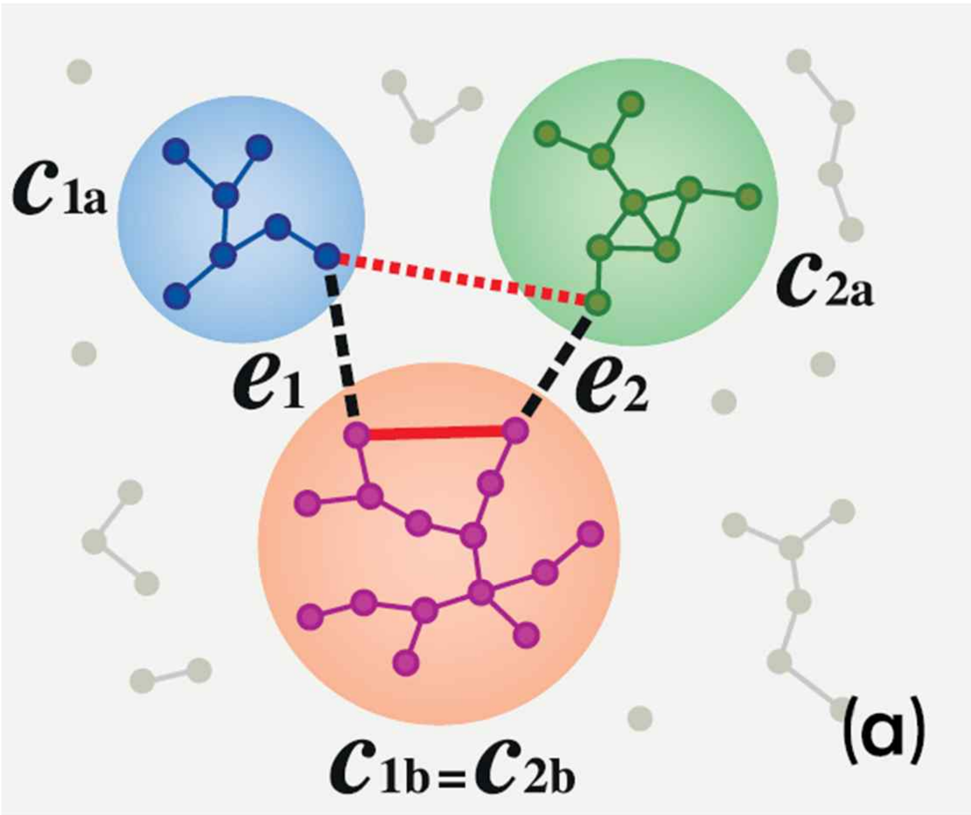


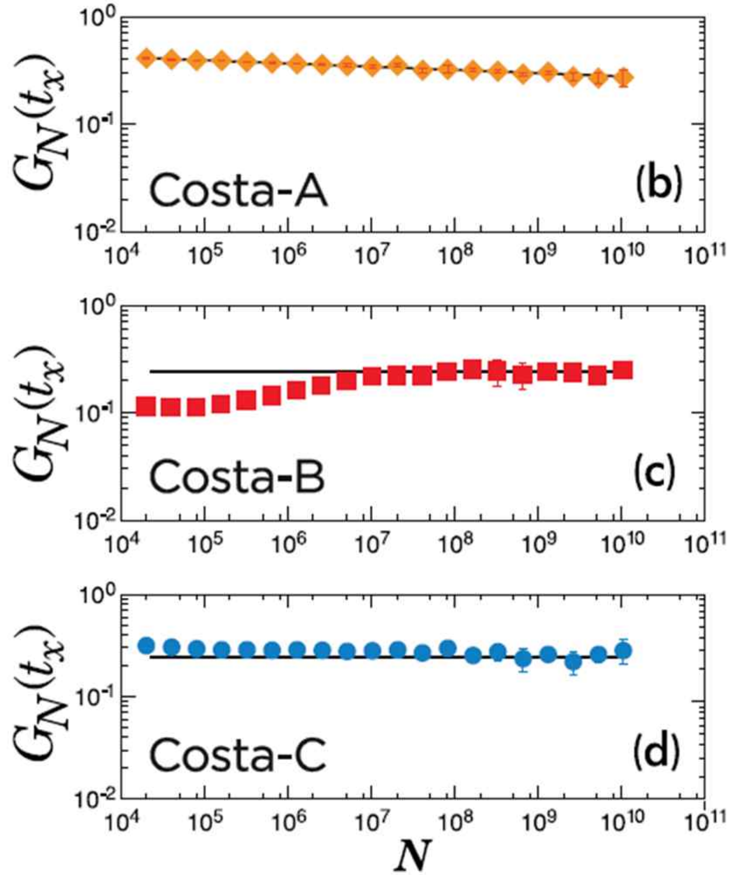
Fraction of suppression failure
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
t
(b)
0.2
0.1
0
0.887
0.888
0.889
0.89

6. Results

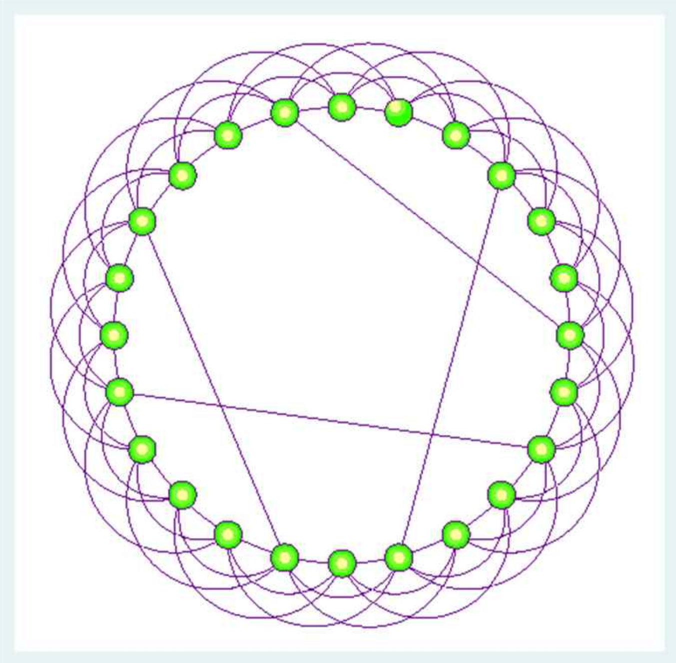


7. da Costa, Dorogovtsev, Goltsev, & Mendes model

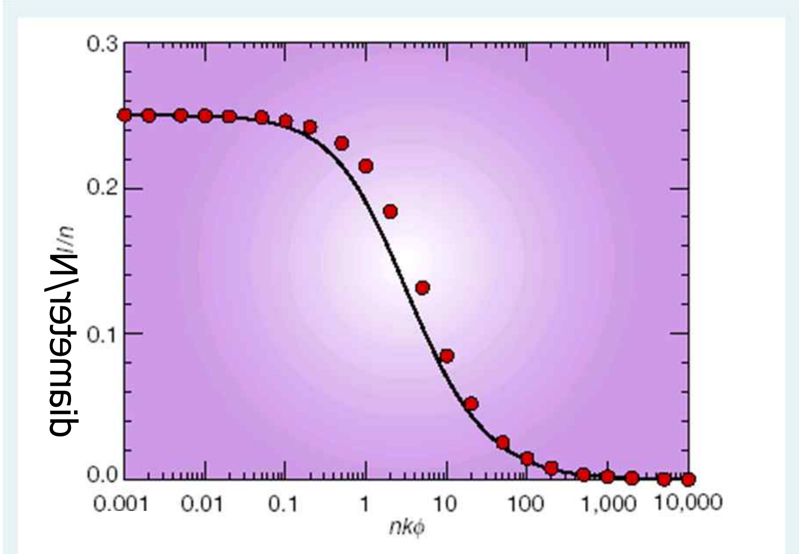




Small-world network model by Watts & Strogatz



[Strogatz 1998]



average number of shortcuts

MBER 15 PHYSICAL REVIEW LETTERS

Small-World Networks: Evidence for a Crossover Picture

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(Received 8 December 1998)

Addition or rewiring of p=1/N fraction of links changes to the SW network

Conclusions

1. Size-dependent behavior of the order parameter is sensitive to the dynamic rules.

2. This makes it hard to reach a conclusion (discontinuous or continuous transition) based on numerical data.

3. Comparison between randomness in choosing edge candidates and suppression strength should to be made analytically. The difference should be compared with the order of time delayed due to the addition of intra-cluster edges.

Y.S. Cho and BK, Phys. Rev. Lett. 107, 275703 (2011).