Multi-dimensional quantum paths to optimization problems

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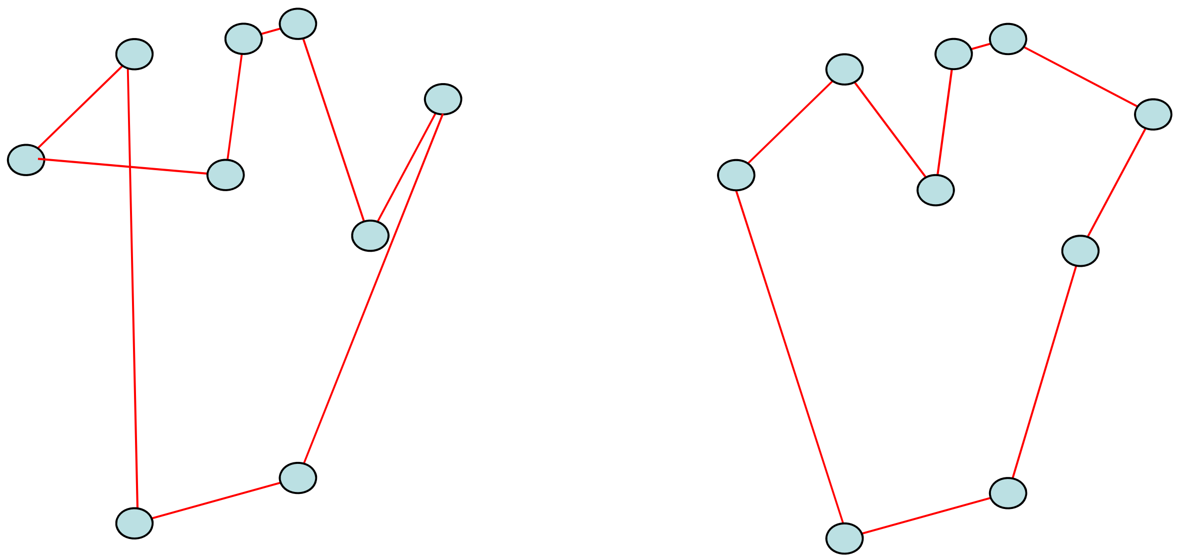
Quantum annealing

Combinatorial optimization

- Travelling Salesman Problem (TSP)



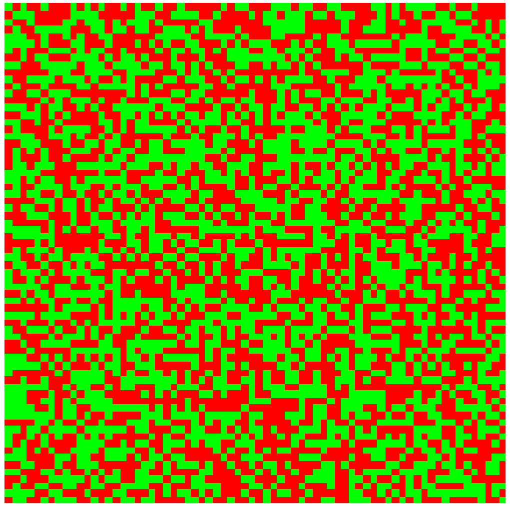
Configuration 1 Configuration 2

Minimize the <u>cost function</u> (=*tour length*)

Problem in physics

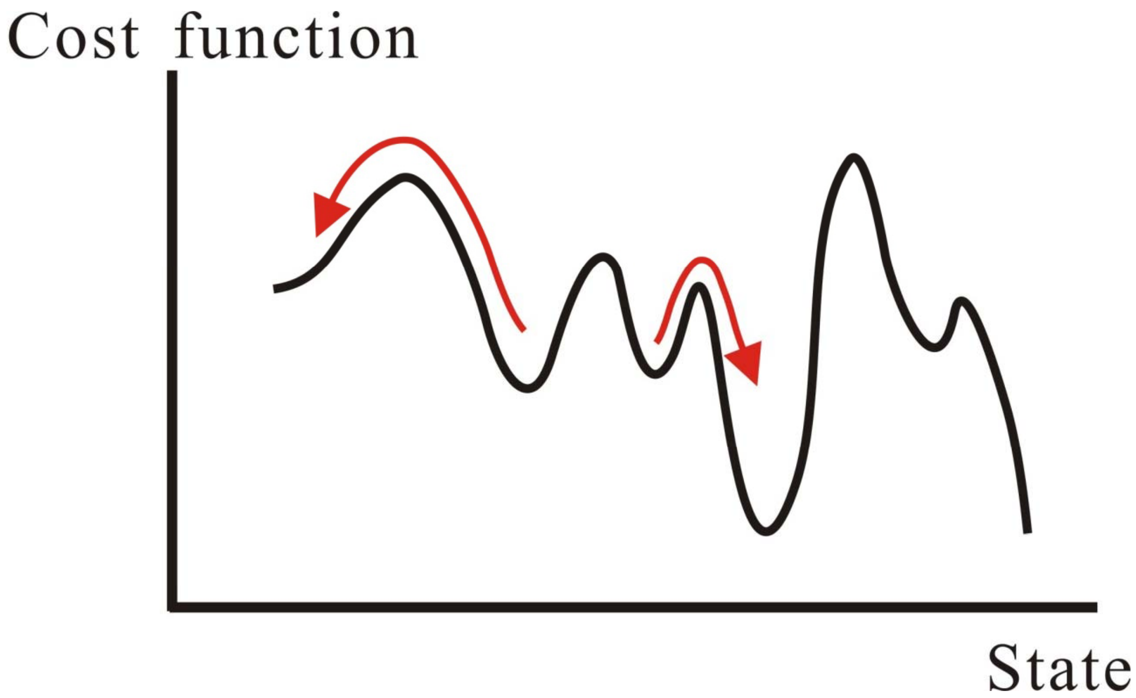
Ground state of Ising SG

$$H = -\sum J_{ij} S_i S_j$$



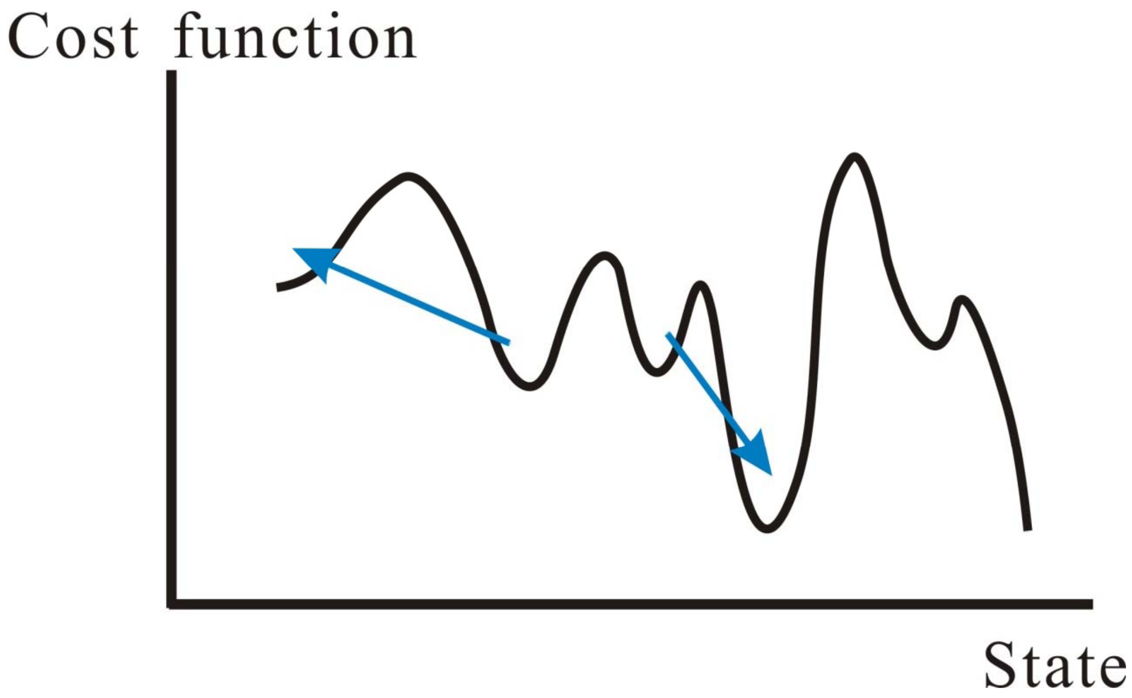
Simulated Annealing (SA)

- *Generic, approximate* algorithm
- Phase-space search by thermal fluctuations



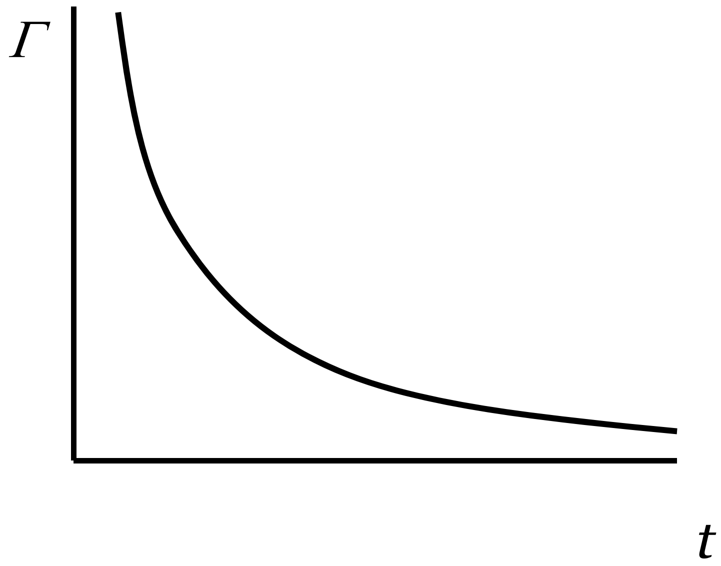
Quantum Annealing (QA)

- *Generic, approximate* algorithm
- Phase-space search by quantum fluctuations



Implementation

$$H(t) = H_{\text{classical}} + H_{\text{quantum}} = -\sum J_{ij}\sigma_i^z\sigma_j^z - \Gamma(t)\sum \sigma_i^x$$

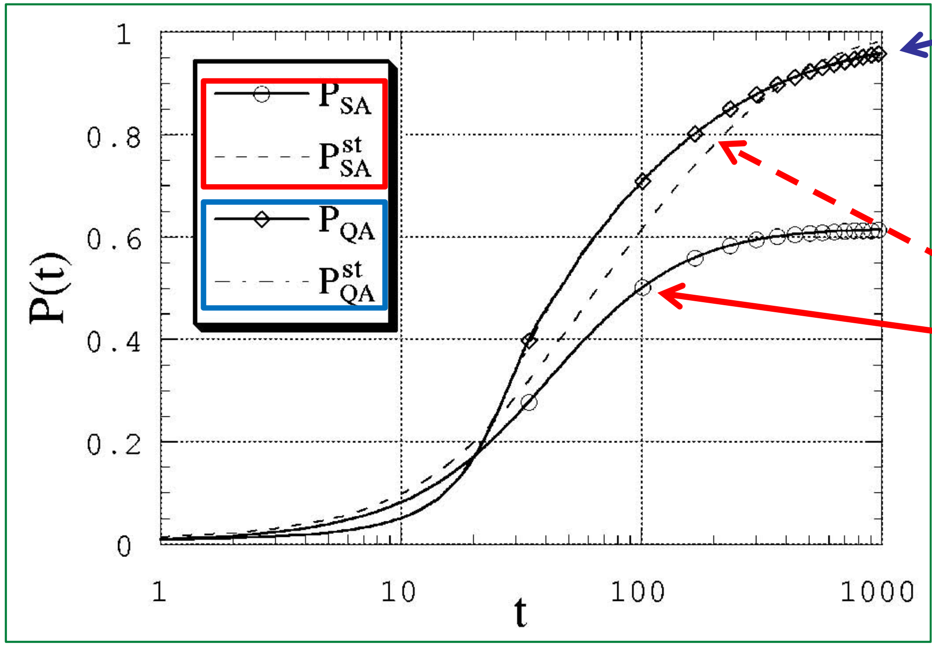


Numerical evidence

$$H = -\sum J_{ij}\sigma_i^z\sigma_j^z - \Gamma(t)\sum_i \sigma_i^x$$

Master eqn vs. Schrödinger eqn

Spin glass (SK model) with 8 spins



$$\Gamma(t) = \frac{3}{\sqrt{t}}$$

Schrödinger

$$T(t) = \frac{3}{\sqrt{t}}$$

Master /Equilibrium

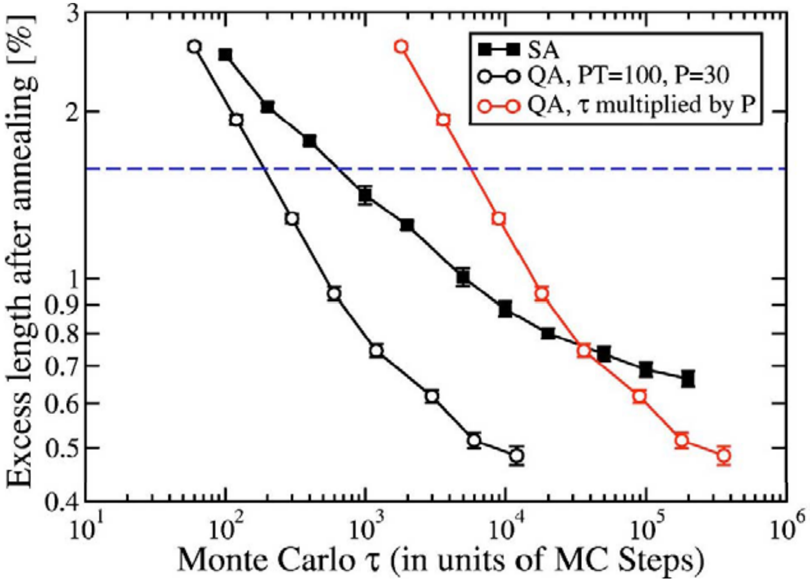
Monte Carlo for TSP (1002 cities)

$$H(t) = \frac{t}{\tau} H_{\text{classical}} + \left(1 - \frac{t}{\tau}\right) H_{\text{quantum}}$$

$$H(0) = H_{\text{quantum}} \Rightarrow H(\tau) = H_{\text{classical}}$$

cf: classical SA $T(0) = \text{large} \Rightarrow T(\tau) = 0$

Residual energy: $H(\tau) - E_{\text{true}}$



Martonak, Santoro &Tosatti (2004)

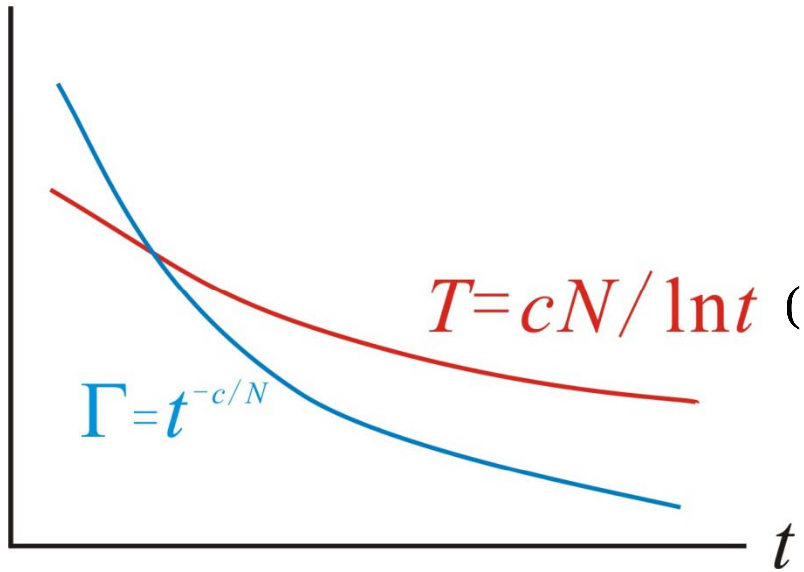
Theoretical background

$$H = -\sum J_{ij}\sigma_i^z\sigma_j^z - \Gamma\sum_i \sigma_i^x$$

Convergence theorem

$$H = H_{\text{classical}} + H_{\text{quantum}} = -\sum J_{ij}\sigma_i^z\sigma_j^z - \Gamma(t)\sum \sigma_i^x$$

Convergence condition $\Gamma(t) = t^{-c/N}$ *Morita & Nishimori*

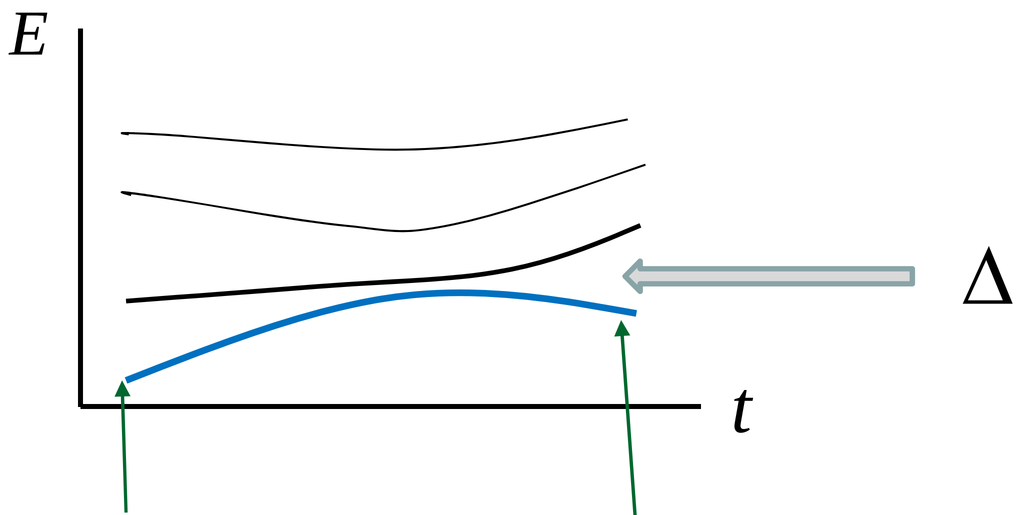


(*Geman-Geman* for SA)

p-spin ferromagnet

- Recent results -

Adiabatic evolution



Trivial initial state

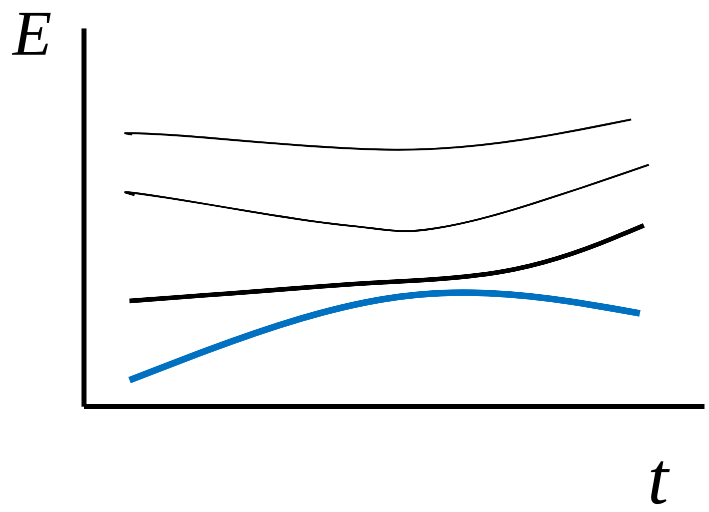
Non-trivial final state

$$H(t) = -\left(1 - \frac{t}{\tau}\right)\sum \sigma_i^x - \frac{t}{\tau}\sum J_{ij}\sigma_i^z\sigma_j^z$$

Computational complexity

Finite-size analysis

Adiabatic theorem $\tau \propto \Delta^{-2}$

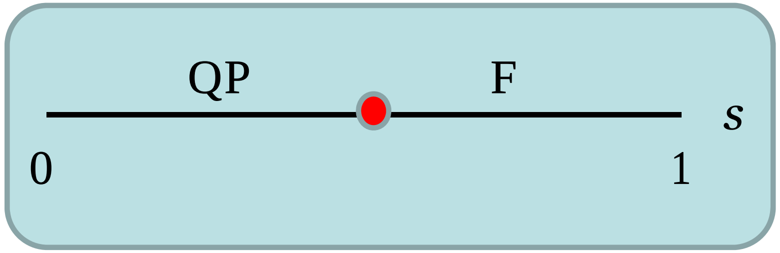


Gap scaling $$\Delta \propto \begin{cases} e^{-aN} \\ N^{-b} \end{cases}$$

Complexity $$\tau \propto \begin{cases} e^{2aN} & \text{(hard)} \\ N^{2b} & \text{(easy)} \end{cases}$$

p-spin ferromagnet

$$H(s) = -sN\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p - (1-s)\sum_{i=1}^{N}\sigma_i^x \quad (s = t/\tau)$$



- 1^{st} order transition at finite s
- Exponentially small energy gap. $\Delta \propto e^{-aN}$
- Exponentially large time for adiabatic computation. $\tau \propto e^{bN}$

Jörg *et al* (2010) “The problem that quantum annealing *cannot* solve”

Seki and Nishimori (2012) “The problem that quantum annealing CAN solve”

An additional quantum term

$$H(\lambda,s)=s\left(\;H_0\qquad\qquad\right)+(1-s)H_{\mathrm{TF}}$$

$$H_0=-N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p\;,\quad H_{\mathrm{TF}}=-\sum_{i=1}^{N}\sigma_i^x$$

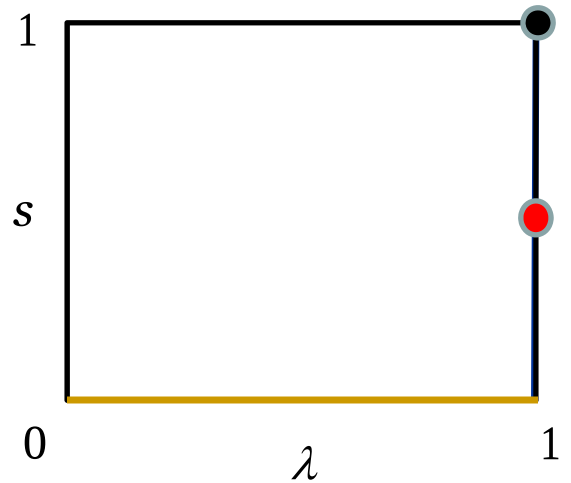
$$H(\lambda,s)=s\left(\lambda H_0+(1-\lambda)H_{\mathrm{AFF}}\right)+(1-s)H_{\mathrm{TF}}$$

$$H_{\mathrm{AFF}}=N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^x\right)^2$$

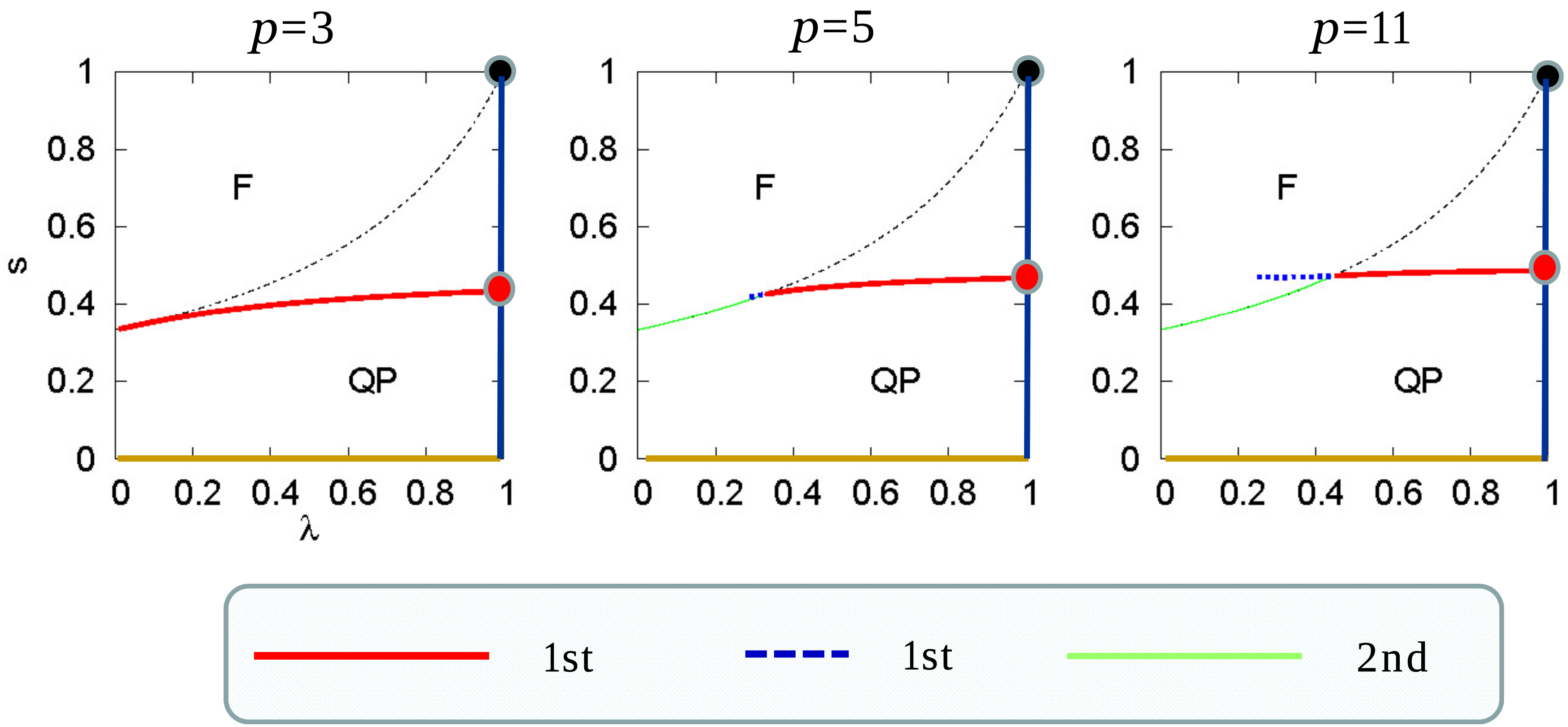
Conventional case: λ=1

Start: s=0, λ=any

Goal: s=1, λ=1 ●



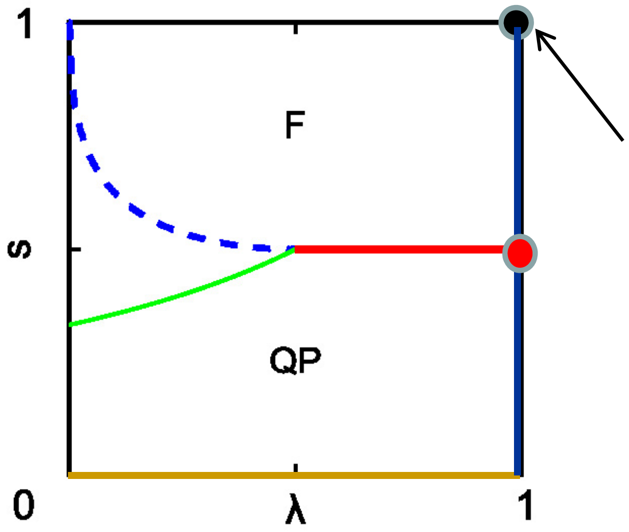
Result



$$H(\lambda, s) = s\left(\lambda H_0 + (1-\lambda)H_{\mathrm{AFF}}\right) + (1-s)H_{\mathrm{TF}}$$

Result (cont'd)

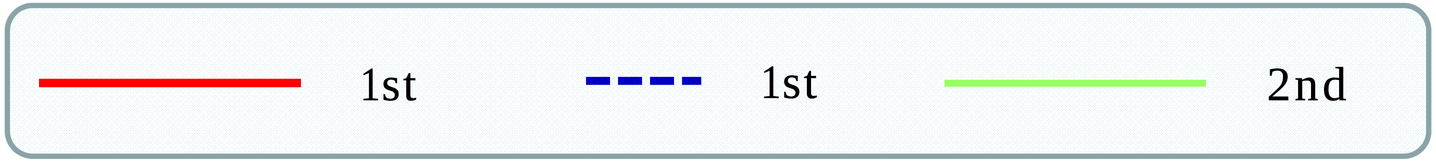
$p \to \infty$



Grover problem

$$\frac{H_0}{N} = -\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p = -1 \;\left(\sigma_i^z = 1,\ \forall i\right)$$

$$\frac{H_0}{N} = 0 \quad \text{otherwise}$$



Is $H_{\text{AFF}} = N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^x\right)^2$ essential?

Seoane and Nishimori (2012)

$$H(\lambda, s) = s\left(\lambda H_0 + (1-\lambda) H_{\text{AFF}}^{(k)}\right) + (1-s) H_{\text{TF}}$$

$$H_{\text{AFF}}^{(k)} = N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^x\right)^k$$

Even k (=2, 4,...) is OK .

Odd k (=3, 5,...) is OK, *partially*.

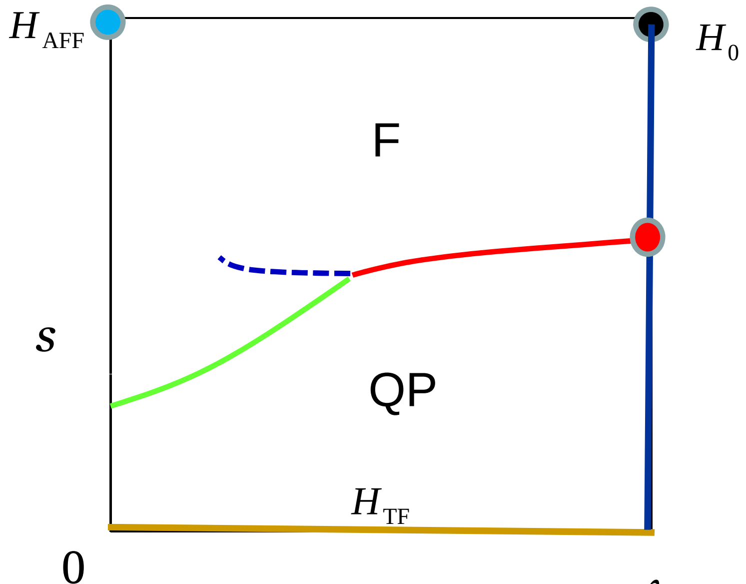
$$\left|\left\langle \Psi_{\text{ferro}} \,|\, \Psi_{\text{AFF}}^{(k)} \right\rangle\right|^2 \propto 1/\sqrt{N} \qquad\qquad \left|\left\langle \Psi_{\text{ferro}} \,|\, \Psi_{\text{AFF}}^{(k)} \right\rangle\right|^2 \propto e^{-aN}$$

k=2: Bapst and Semerjian (2012)

Even k and odd k

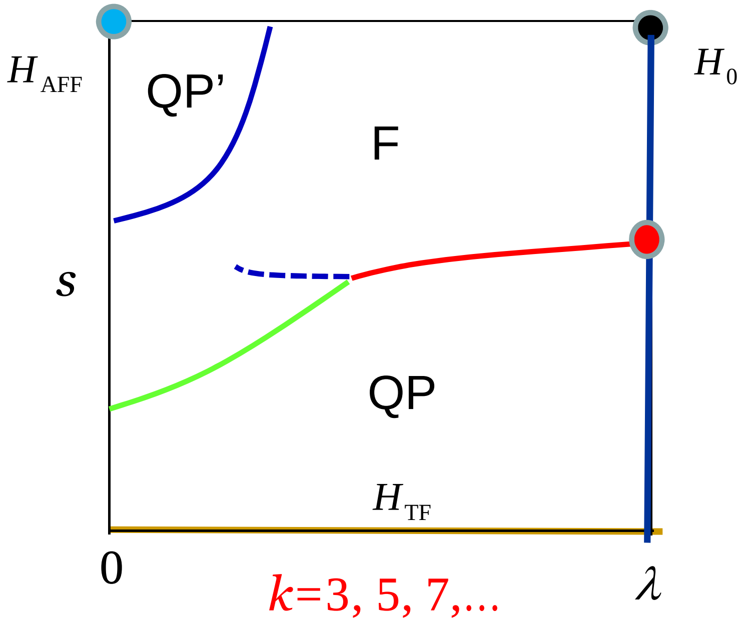
$$H(\lambda, s) = s\left(\lambda H_0 + (1-\lambda) H_{\mathrm{AFF}}^{(k)}\right) + (1-s) H_{\mathrm{TF}}$$

$$H_{\mathrm{AFF}}^{(k)} = N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^x\right)^k$$



k=2, 4, 6,…

$$\left|\langle \Psi_{\mathrm{ferro}} | \Psi_{\mathrm{AFF}} \rangle\right|^2 \propto 1/\sqrt{N}$$

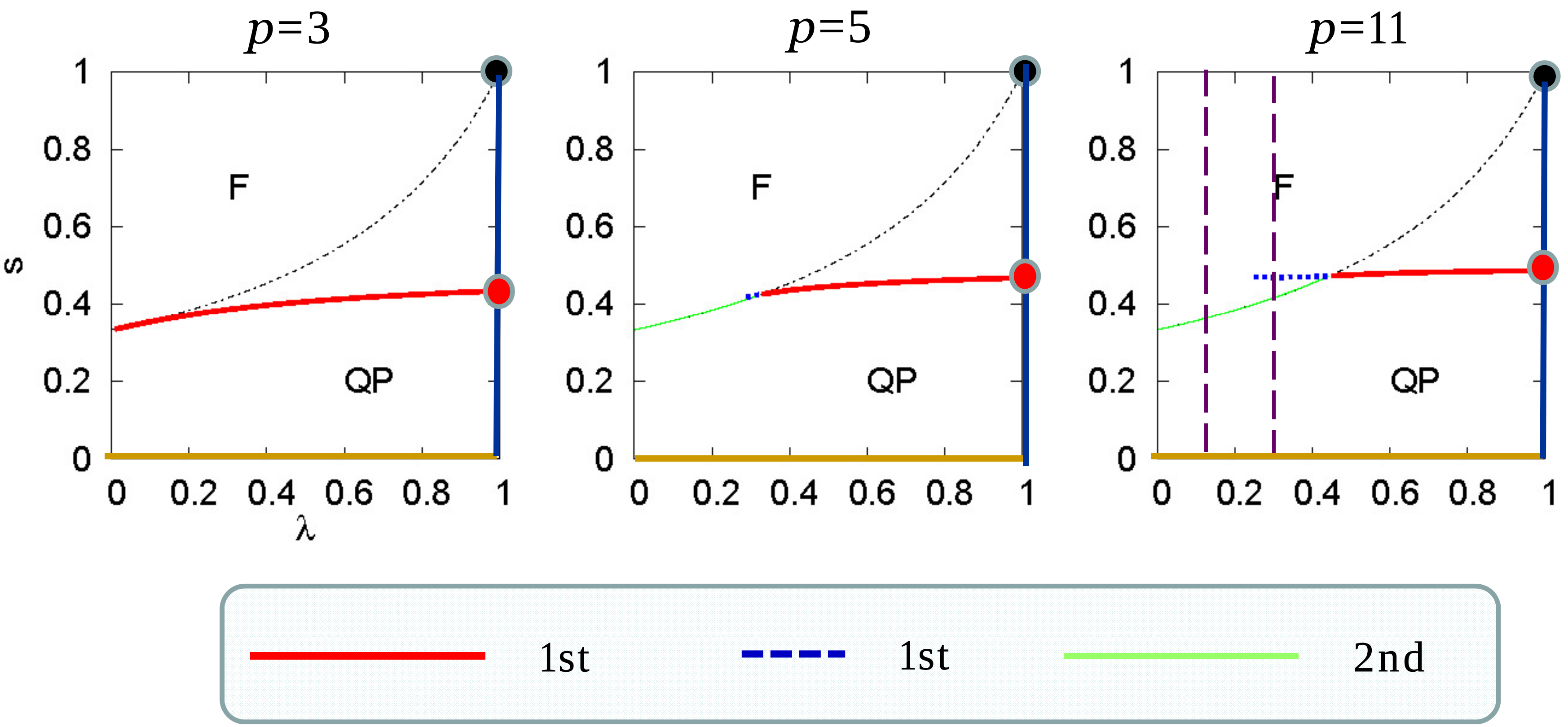


k=3, 5, 7,…

$$\left|\langle \Psi_{\mathrm{ferro}} | \Psi_{\mathrm{AFF}} \rangle\right|^2 \propto e^{-aN}$$

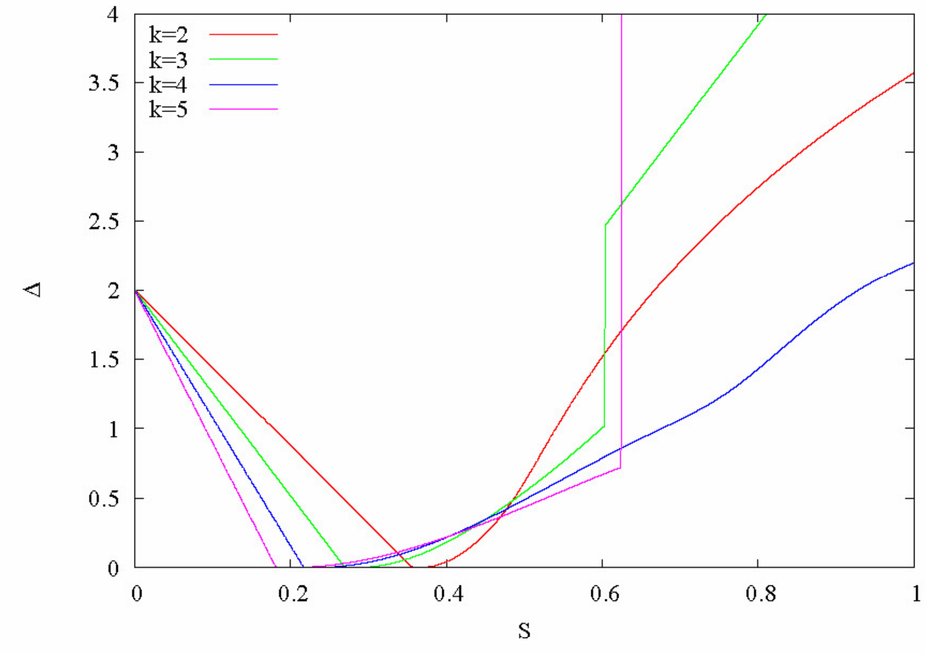
Bapst and Semerjian (2012)

Energy gap

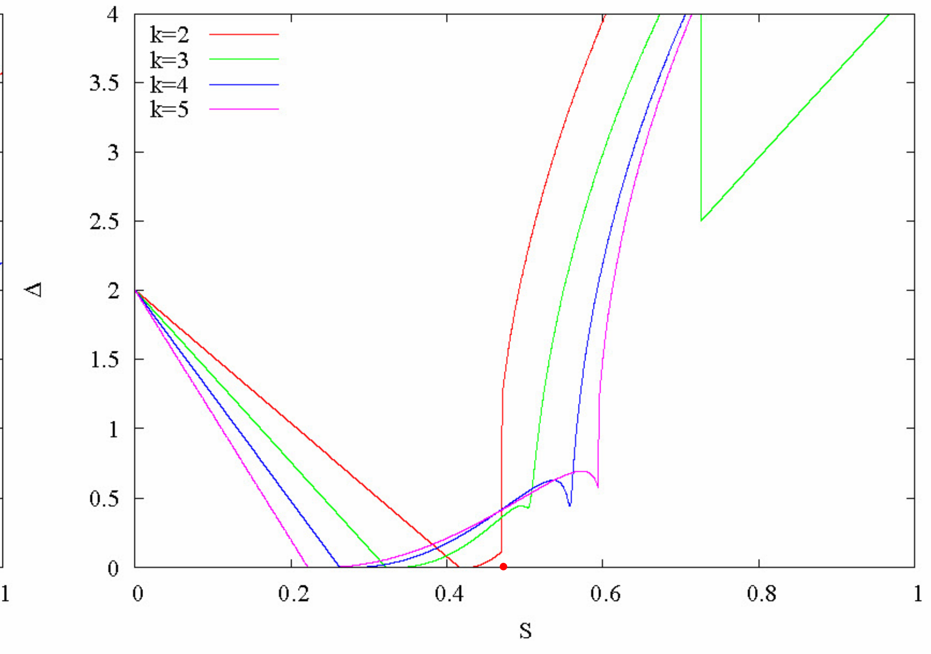


$$H(\lambda, s) = s\left(\lambda H_0 + (1-\lambda)H_{AFF}\right) + (1-s)H_{TF}$$

Energy gap



p=11, λ=0.1



p=11, λ=0.3

Method

1. Suzuki-Trotter + Static approximation
2. Classical approximation (+ quantum correction)

Botet-Julien (1983), Filiponne et al (2011)

$$H_0 = -N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p, \quad H_{\mathrm{TF}} = -\sum_{i=1}^{N}\sigma_i^x$$

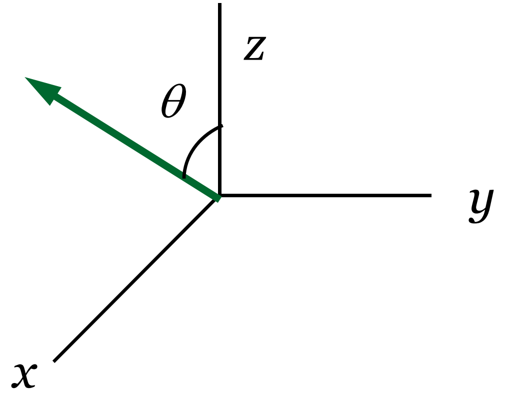
$$H_{\mathrm{AFF}} = N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^x\right)^2$$

$$\frac{1}{2}\sum_{i=1}^{N}\sigma_i^z = S^z \approx \frac{N}{2}\cos\theta, \quad \frac{1}{2}\sum_{i=1}^{N}\sigma_i^x = S^x \approx \frac{N}{2}\sin\theta\cos\varphi$$

$$\frac{H}{N} = -s\lambda(\sin\theta\cos\varphi)^p + s(1-\lambda)\cos^2\theta - (1-s)\cos\theta$$

Minimize with respect to θ

$$\varphi = 0$$



Method (Gap)

Classical approximation (+ quantum correction)

Holstein-Primakoff transformation + Bogoliubov transformation

$$S^z = \frac{N}{2} - a^+ a$$

$$S^+ = (N - a^+ a)^{1/2} a \approx Na, \; S^- = a^+ (N - a^+ a)^{1/2} \approx Na^+$$

$$H = NE_{\mathrm{g}} + c + \Delta(\theta) b^+ b$$

$\Delta(\theta)$: Gap in the thermodynamic limit

Summary

Summary

1. QA works fine as a generic, approximate algorithm.
2. "Better" than SA. Numerical, analytical
3. Different quantum terms:

   Useful to avoid 1^{st} order transitions.
4. Useful for other, more complex, problems?

   (under investigation)

Collaborators

- Tadashi Kadowaki
- Helmut G. Katzgraber
- Yoshiki Matsuda
- Satoshi Morita
- Yuya Seki
- Beatriz Seoane