Current Fluctuations in the Open Exclusion Process

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Introduction

The statistical mechanics of a system at thermal equilibrium is encoded in the Boltzmann-Gibbs canonical law:

$$P_{\mathrm{eq}}(\mathcal{C}) = \frac{\mathrm{e}^{-E(\mathcal{C})/kT}}{Z}$$

the Partition Function Z being related to the Thermodynamic Free Energy F:

$$F = -kTLog\ Z$$

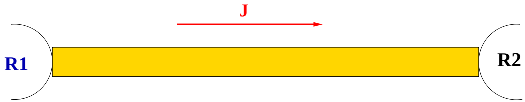
This provides us with a well-defined prescription to analyze systems *at equilibrium*:
(i) Observables are mean values w.r.t. the canonical measure.
(ii) Statistical Mechanics predicts fluctuations (typically Gaussian) that are out of reach of Classical Thermodynamics.

Systems far from equilibrium

No fundamental theory is yet available.

- What are the relevant macroscopic parameters?
- Which functions describe the state of a system?
- Do Universal Laws exist? Can one define Universality Classes?
- Can one postulate a general form for the microscopic measure?
- What do the fluctuations look like ('non-gaussianity')?

Example: Stationary driven systems in contact with reservoirs.



Rare Events and Large Deviations

Let $\epsilon_1, \ldots, \epsilon_N$ be N independent binary variables, $\epsilon_k = \pm 1$, with probability p (resp. $q = 1 - p$). Their sum is denoted by $S_N = \sum_1^N \epsilon_k$.

- The Law of Large Numbers implies that $S_N/N \to p - q$ a.s.
- The Central Limit Theorem implies that $[S_N - N(p-q)]/\sqrt{N}$ converges towards a Gaussian Law.

One can show that for $-1 < r < 1$, in the large N limit,

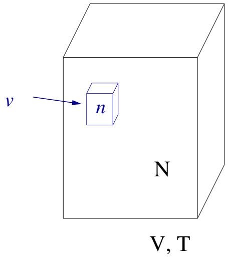
$$\mathrm{Pr}\left(\frac{S_N}{N} = r\right) \sim \mathrm{e}^{-N\,\Phi(r)}$$

where the positive function $\Phi(r)$ vanishes for $r = (p-q)$.

The function $\Phi(r)$ is a Large Deviation Function: it encodes the probability of rare events.

$$\Phi(r) = \frac{1+r}{2}\ln\left(\frac{1+r}{2p}\right) + \frac{1-r}{2}\ln\left(\frac{1-r}{2q}\right)$$

Density fluctuations in a gas



Mean Density $\rho_0 = \frac{N}{V}$

In a volume v s. t. $1 \ll v \ll V$
$\langle \frac{n}{v} \rangle = \rho_0$

The local density varies around ρ_0 . Typical fluctuations scale as $\sqrt{v/V}$.

The probability of observing large fluctuations is given by

$$\Pr\left(\frac{n}{v} = \rho\right) \sim \mathrm{e}^{-v\,\Phi(\rho)} \text{ with } \Phi(\rho_0) = 0$$

Thermodynamic Free Energy as a L. D. F.

The Large Deviation Function for density fluctuations is given by

$$\Phi(\rho) = f(\rho, T) - f(\rho_0, T) - (\rho - \rho_0)\frac{\partial f}{\partial \rho_0}$$

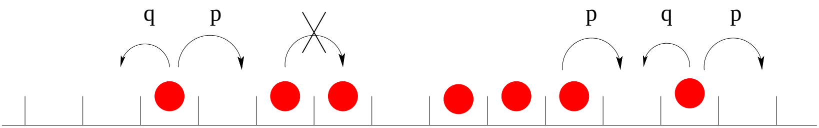
where $f = -\log Z(\rho, T)$ is the *free energy per unit volume* in units of kT.

The Free Energy of Thermodynamics can be viewed as a Large Deviation Function

Conversely, large deviation functions *may* play the role of potentials in non-equilibrium statistical mechanics.

Large deviation functions obey remarkable identities that remain valid far from equilibrium: *Fluctuation Theorem of Gallavotti and Cohen.*
In the vicinity of equilibrium the Fluctuation Theorem yields the fluctuation-dissipation relation (Einstein), Onsager's relations and linear response theory (Kubo).

Classical Transport in 1d: ASEP



Asymmetric Exclusion Process. A paradigm for non-equilibrium Statistical Mechanics.

- EXCLUSION: Hard core-interaction; at most 1 particle per site.
- ASYMMETRIC: External driving; breaks detailed-balance
- PROCESS: Stochastic Markovian dynamics; no Hamiltonian.

The probability $P_t(\mathcal{C})$ to find the system in the microscopic configuration $\mathcal{C}$ at time t satisfies

$$\frac{dP_t(\mathcal{C})}{dt} = MP_t(\mathcal{C})$$

where the Markov Matrix M encodes the transitions rates amongst configurations.

ORIGINS

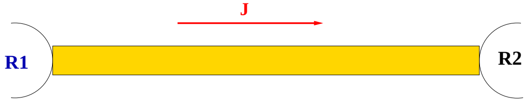
- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (Katz, Lebowitz and Spohn).
- Transport of Macromolecules through thin vessels.
  Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.

APPLICATIONS

- Traffic flow.
- Sequence matching.
- Brownian motors.

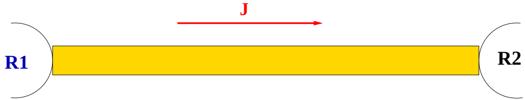
Total Current transported through an Open System

A paradigm of a non-equilibrium system

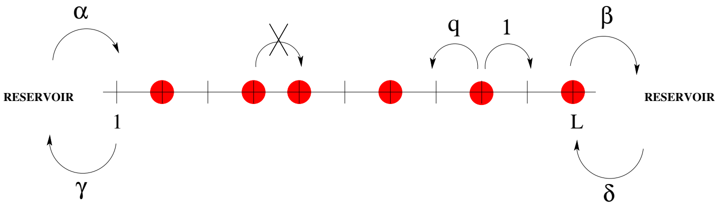


Total Current transported through an Open System

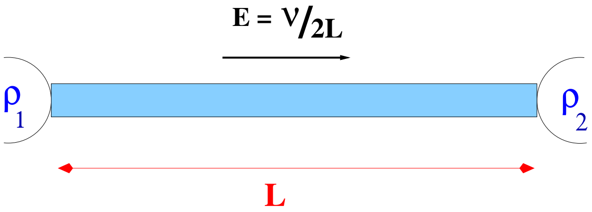
A paradigm of a non-equilibrium system



The asymmetric exclusion model with open boundaries



The Hydrodynamic Limit



Starting from the microscopic level, define local density $\rho(x,t)$ and current $j(x,t)$ with macroscopic space-time variables $x = i/L, t = s/L^2$ (diffusive scaling).

The typical evolution of the system is given by the hydrodynamic behaviour:

$$\partial_t \rho = \frac{1}{2}\nabla^2 \rho - \nu \nabla \sigma(\rho) \quad \text{with} \quad \sigma(\rho) = \rho(1-\rho)$$

(Lebowitz, Spohn, Varadhan)

Large Deviations at the Hydrodynamic Level

What is the probability to observe an atypical current $j(x,t)$ and the corresponding density profile $\rho(x,t)$ during $0 \leq s \leq L^2 T$?

$$\mathrm{Pr}\{j(x,t), \rho(x,t)\} \sim \mathrm{e}^{-L\,\mathcal{I}(j,\rho)}$$

where the Large-Deviation functional is given by macroscopic fluctuation theory (Jona-Lasinio et al.)

$$\mathcal{I}(j,\rho) = \int_0^T dt \int_0^1 dx \frac{\left(j - \nu\sigma(\rho) + \frac{1}{2}\nabla\rho\right)^2}{\sigma(\rho)}$$

with the constraint: $\partial_t \rho = -\nabla . j$
This leads to a variational procedure to control a deviation of the density and of the associated current: an optimal path problem.

- A general framework but the corresponding Euler-Lagrange equations can not be solved in general.
- For a non-vanishing external field, the M. F. T. does not apply (Jensen-Varadhan Large Deviation Theory).

Large Deviations: Profile vs Current

The probability of observing an atypical density profile in the steady state was calculated starting from the exact microscopic solution (B. Derrida, J. Lebowitz E. Speer, 2002). For the symmetric case $\nu = 0$, the Large Deviation Functional is given by

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left(B(\rho(x), F(x)) + \log \frac{F'(x)}{\rho_2 - \rho_1} \right)$$

where $B(u, v) = (1-u) \log \frac{1-u}{1-v} + u \log \frac{u}{v}$ and $F(x)$ satisfies

$$F\left(F'^2 + (1-F)F''\right) = F'^2 \rho \quad \text{with} \quad F(0) = \rho_1 \text{ and } F(1) = \rho_2 \,.$$

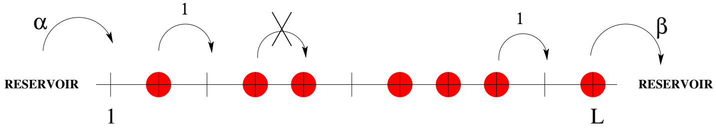
Our aim is to study the statistics of the current and its large deviations starting from the microscopic model.

1. Total Current Fluctuations:

The TASEP case

The Matrix Ansatz (DEHP, 1993)

The totally asymmetric exclusion model with open boundaries



The stationary probability of a configuration $\mathcal{C}$ is given by

$$P(\mathcal{C}) = \frac{1}{Z_L} \langle \alpha | \prod_{i=1}^{L} \left(\tau_i D + (1 - \tau_i) E \right) | \beta \rangle \, .$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty).
The normalization constant is $Z_L = \langle \alpha | \left(D + E \right)^L | \beta \rangle$

The operators D and E, the vectors $\langle \alpha |$ and $| \beta \rangle$ satisfy

$$D \, E \quad = \quad D + E$$

$$D \, | \beta \rangle = \frac{1}{\beta} | \beta \rangle \qquad \text{and} \qquad \langle \alpha | \, E = \frac{1}{\alpha} \langle \alpha |$$

Representations of the quadratic algebra

The algebra encodes combinatorial recursion relations between systems of different sizes.

The matrices D and E commute whenever they are finite-dimensional: $(D-1)(E-1) = 1$.

Infinite dimensional Representation:

$D = 1 + d$ where d =right-shift.

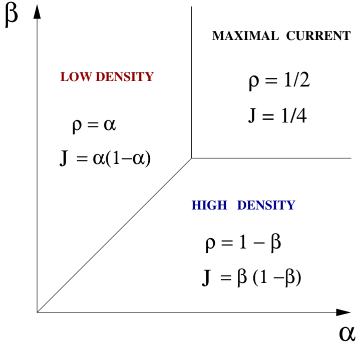
$E = 1 + e$ where e =left-shift.

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad E = D^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ & & & \ddots & \ddots \end{pmatrix}$$

We also have $\langle\alpha| = (1, a, a^2, a^3 \dots)$ and $|\beta\rangle = (1, b, b^2, b^3 \dots)$ with $a = (1-\alpha)/\alpha$ and $b = (1-\beta)/\beta$.

Phase Diagram

The matrix Ansatz allows one to calculate Stationary State Properties (currents, correlations, fluctuations) and to derive the Phase Diagram in the infinite size limit.



The Total Current

Let Y_t be the TOTAL NUMBER OF PARTICLES (time-integrated current) that have entered the system between 0 and t (from the left reservoir).

When a particle enters the system:

$$\boxed{Y_t = Y_t + 1}$$

- Expectation value: $\lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = J(\alpha, \beta, L) = \frac{Z_{L-1}}{Z_L}$

  The observable $J(\alpha, \beta, L)$ represents the average-current in the stationary state. It can be calculated by the steady-state matrix Ansatz.

- Variance: $\lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \Delta(\alpha, \beta, L)$

  The quantity $\Delta(\alpha, \beta, L)$ represents the fluctuations of the total current. It does not depend only on the stationary measure.

- Cumulant Generating Function: For $t \to \infty$, it can be proved that

$$\left\langle \mathrm{e}^{\mu Y_t} \right\rangle \simeq \mathrm{e}^{E(\mu) t}$$

  $E(\mu)$ encodes the statistical properties of the total current.

Large Deviations of the Current

The **Large-Deviation Function** $\Phi(j)$ of the total current is defined as

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

Using this asymptotic behaviour of the large-time PDF, one can evaluate the average of $\mathrm{e}^{\mu Y_t}$:

$$\langle \exp(\mu\, Y_t) \rangle \simeq \int dj\, \mathrm{e}^{\mu t j} e^{-t\Phi(j)}$$

Hence, by the saddle-point method, we obtain that the Large-Deviation Function and the Cumulant Generating Function $E(\mu)$ are related by a *Legendre transform:*

$$\boxed{E(\mu) = \max_j \left(\mu j - \Phi(j)\right)}$$

Current Statistics: Mathematical Framework

Let $P_t(\mathcal{C}, Y)$ be the joint probability of being at time t in configuration $\mathcal{C}$ with $Y_t = Y$. The time evolution of this joint probability can be deduced from the original Markov equation, by splitting the Markov operator

$$M = M_0 + M_+ + M_-$$

- M_0 corresponds to transitions that do not modify the value of Y.
- M_+ are transitions that increment Y by 1: a particle enters the system from the left reservoir.
- M_- encodes rates in which Y decreases by 1, if a particle exits the system from the left reservoir (does not happen in the simplest TASEP case).

The Laplace transform of $P_t(\mathcal{C}, Y)$ with respect to Y is defined as

$$\hat{P}_t(\mathcal{C}, \mu) = \sum_Y \mathrm{e}^{\mu Y} P_t(\mathcal{C}, Y).$$

It satisfies an equation that can be derived from the fundamental Markov equation of the dynamics.

The Laplace transform $\hat{P}_t(\mathcal{C}, \mu)$ satisfies a dynamical equation governed by the deformation of the Markov Matrix M, obtained by adding a jump-counting *fugacity* μ:

$$\frac{d\hat{P}_t}{dt} = M(\mu)\hat{P}_t$$

with

$$\boxed{M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-}$$

In the long time limit, $t \to \infty$, this proves the asymptotic behaviour $\langle \mathrm{e}^{\mu Y_t} \rangle \simeq \mathrm{e}^{E(\mu)t}$ where the cumulant-generating function $E(\mu)$ is given by the eigenvalue of $M(\mu)$ with maximal real part.

The current statistics is reduced to an eigenvalue problem, solvable by Bethe Ansatz.

Analytic Procedure

Call $F_\mu(\mathcal{C})$ of the dominant eigenvector F_μ of $M(\mu)$. We have:

$$M(\mu).F_\mu = E(\mu)F_\mu$$

This dominant eigenvector can be formally expanded w. r. t. μ:

$$F_\mu(\mathcal{C}) = P(\mathcal{C}) + \mu R_1(\mathcal{C}) + \mu^2 R_2(\mathcal{C}) \dots$$

- For $\mu = 0$: $M(\mu = 0)$ is the original Markov operator, $E(\mu = 0) = 0$ and $P(\mathcal{C})$ is the stationary weight of the configuration $\mathcal{C}$: $M.P = 0.$
- The generalized weight vector $R_k(\mathcal{C})$ satisfies an inhomogeneous linear equation: $M.R_k = \Phi_k\left(P, R_1, \dots R_{k-1}\right)$, Φ_k being a linear functional.
- For each value of k, we show that F_μ can be represented by a matrix product Ansatz up to corrections of order μ^{k+1}.
- Knowing F_μ up to corrections of order μ^{k+1}, we calculate $E(\mu)$ to order μ^{k+1}.

Generalized Matrix Ansatz

One can prove that

$$F_\mu(\mathcal{C}) = \frac{1}{Z_L^{(k)}} \langle W_k | \prod_{i=1}^{L} (\tau_i D_k + (1-\tau_i) E_k) | V_k \rangle + \mathcal{O}\left(\mu^{k+1}\right)$$

The matrices D_k and E_k are defined recursively starting with $D_1 = D$ and $E_1 = E$ and

$$\begin{aligned} D_{k+1} &= (1 \otimes 1 + d \otimes e) \otimes D_k + (1 \otimes d + d \otimes 1) \otimes E_k \\ E_{k+1} &= (1 \otimes 1 + e \otimes d) \otimes E_k + (e \otimes 1 + 1 \otimes e) \otimes D_k \end{aligned}$$

Where $d = D - 1$ and $e = E - 1$
The boundary vectors $\langle W_k|$ and $|V_k\rangle$ are also constructed recursively.
Start with $|V_1\rangle = |\beta\rangle$ and $\langle W_1| = \langle \alpha|$ and iterate

$$|V_k\rangle = |\beta\rangle|\tilde{V}\rangle|V_{k-1}\rangle \quad \text{and} \quad \langle W_k| = \langle W^\mu|\langle \tilde{W}^\mu|\langle W_{k-1}|$$

where

$$\beta(1-d)\,|\tilde{V}\rangle = 0$$

$$\alpha\langle W^\mu|(1 + \mathrm{e}^\mu\, e) = \langle W^\mu| \quad \text{and} \quad \alpha\langle \tilde{W}^\mu|(1 - \mathrm{e}^\mu\, e) = 0$$

Representations of the tensor algebra

The tensor products define a systematic iterative construction, which can be visualized in the usual representation.

$$1 \otimes D_k + d \otimes E_k = \begin{pmatrix} D_k & E_k & 0 & 0 & \dots \\ 0 & D_k & E_k & 0 & \dots \\ 0 & 0 & D_k & E_k & \dots \\ & & & \ddots & \ddots \end{pmatrix}$$

$$1 \otimes E_k + e \otimes D_k = \begin{pmatrix} E_k & 0 & 0 & 0 & \dots \\ D_k & E_k & 0 & 0 & \dots \\ 0 & D_k & E_k & 0 & \dots \\ & & & \ddots & \ddots \end{pmatrix}$$

One has to calculate Matrix elements of these tensor products for arbitrary values of k.

A Special Case

In the case $\alpha = \beta = 1$, a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)]!}{[k(L+1)]!\,[k(L+2)]!} \frac{B^k}{2k} \,,$$

$$E = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)-2]!}{[k(L+1)-1]!\,[k(L+2)-1]!} \frac{B^k}{2k} \,.$$

First cumulants of the current

- Mean Value : $J = \frac{L+2}{2(2L+1)}$
- Variance : $\Delta = \frac{3}{2} \frac{(4L+1)![L!(L+2)!]^2}{[(2L+1)!]^3(2L+3)!}$
- Skewness :
  $E_3 = 12 \frac{[(L+1)!]^2[(L+2)!]^4}{(2L+1)[(2L+2)!]^3} \left\{ 9 \frac{(L+1)!(L+2)!(4L+2)!(4L+4)!}{(2L+1)![(2L+2)!]^2[(2L+4)!]^2} - 20 \frac{(6L+4)!}{(3L+2)!(3L+6)!} \right\}$
  For large systems: $E_3 \to \frac{2187-1280\sqrt{3}}{10368}\pi \sim -0.0090978\ldots$

Full Current Statistics for the TASEP

For arbitrary (α, β), the parametric representation of $E(\mu)$ is

$$\mu = -\sum_{k=1}^{\infty} C_k(\alpha,\beta)\frac{B^k}{2k}$$

$$E = -\sum_{k=1}^{\infty} D_k(\alpha,\beta)\frac{B^k}{2k}$$

with

$$C_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi}\frac{F(z)^k}{z} \quad \text{and} \quad D_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi}\frac{F(z)^k}{(1+z)^2}$$

where

$$F(z) = \frac{-(1+z)^{2L}(1-z^2)^2}{z^L(1-az)(z-a)(1-bz)(z-b)}\,, \qquad a = \frac{1-\alpha}{\alpha}\,, \qquad b = \frac{1-\beta}{\beta}$$

Some explicit expressions

- Mean Current: (Same expression as in DEHP)

$$J = \frac{D_1(\alpha, \beta)}{C_1(\alpha, \beta)}$$

- Fluctuations: (an expression more compact than the one of 1995)

$$\Delta = \frac{D_1\, C_2 - D_2\, C_1}{C_1^3}$$

- Saddle point analysis in the low density phase: $(\rho = \alpha)$

$$\begin{aligned}
E_1 &= \rho(1-\rho) \\
E_2 &= \rho(1-\rho)(1-2\rho) \\
E_3 &= \rho(1-\rho)(1-6\rho+6\rho^2) \\
E_4 &= \rho(1-\rho)(1-2\rho)(1-12\rho+12\rho^2) \\
E_5 &= \rho(1-\rho)(1-30\rho+150\rho^2-240\rho^3+120\rho^4) \ \ldots
\end{aligned}$$

Asymptotics in the TASEP Phase Diagram

In the limit $L \to \infty$ of systems of large size, we have

- Maximal Current phase $\alpha > 1/2$ and $\beta > 1/2$: Cumulants are independent from α and β

$$E_k \sim \pi(\pi L)^{k/2-3/2} \text{ for } \quad k \geq 2$$

- Low Density phase $\alpha < \min(\beta, 1/2)$:

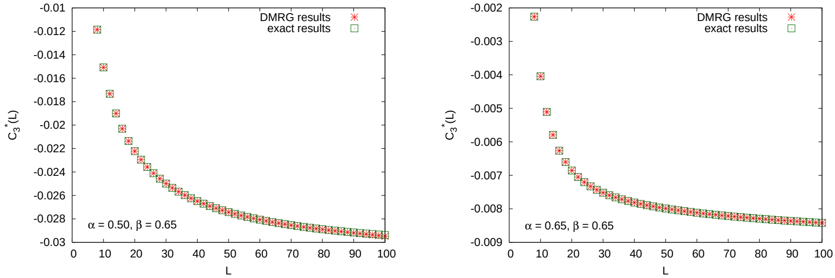
$$E(\mu) = \frac{a}{a+1} \frac{\mathrm{e}^{\mu} - 1}{\mathrm{e}^{\mu} + a}$$

Agrees with the Asymptotic result obtained using the Bethe Ansatz (J. de Gier and F. L. Essler).

- High Density phase is symmetrical to Low Density via $\alpha \leftrightarrow \beta$.
- Along the shock line $\alpha = \beta \leq 1/2$, fluctuations are enhanced

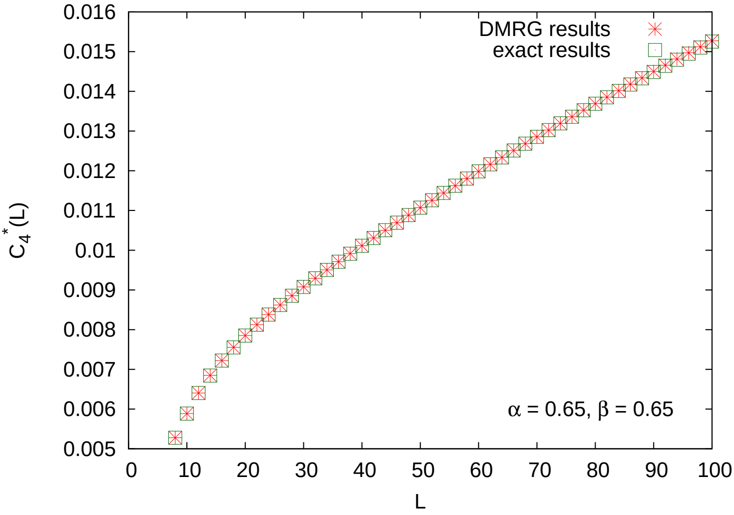
$$E_k \simeq \epsilon_k \alpha(1-\alpha)(1-2\alpha)^{k-1} L^{k-2} \text{ for } \; k \geq 2$$

DMRG Results (M. Gorissen, C. Vanderzande)



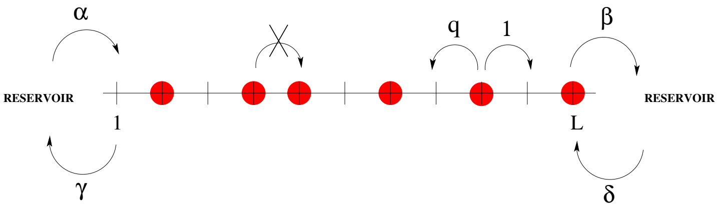
SKEWNESS

Fourth Cumulant (DMRG)



2. Current fluctuations: The general ASEP case

The Open ASEP



The stationary probability of a configuration $\mathcal{C}$ is given by

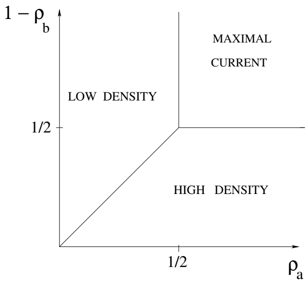
$$P(\mathcal{C}) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} \left(\tau_i D + (1-\tau_i) E \right) | V \rangle \, .$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty).

The operators D and E, the vectors $\langle W|$ and $|V\rangle$ now satisfy

$$\begin{aligned} D\,E - qE\,D &= D + E \\ (\beta\,D - \delta\,E)\,|V\rangle &= |V\rangle \\ \langle W|(\alpha\,E - \gamma\,D) &= \langle W| \end{aligned}$$

The Phase Diagram



$\rho_a = \frac{1}{a_+ + 1}$: effective left reservoir density.

$\rho_b = \frac{b_+}{b_+ + 1}$: effective right reservoir density.

$$a_\pm = \frac{(1-q-\alpha+\gamma) \pm \sqrt{(1-q-\alpha+\gamma)^2 + 4\alpha\gamma}}{2\alpha}$$

$$b_\pm = \frac{(1-q-\beta+\delta) \pm \sqrt{(1-q-\beta+\delta)^2 + 4\beta\delta}}{2\beta}$$

Total Current

The observable Y_t counts the total number of particles exchanged between the system and the left reservoir between times 0 and t.

Hence, $Y_{t+dt} = Y_t + y$ with

- $y = +1$ if a particle enters at site 1 (at rate α),
- $y = -1$ if a particle exits from 1 (at rate γ)
- $y = 0$ if no particle exchange with the left reservoir has occurred during dt.

These three mutually exclusive types of transitions lead to a three parts decomposition of the Markov Matrix: $M = M_+ + M_- + M_0$.

The cumulant-generating function $E(\mu)$ when $t \to \infty$, $\langle e^{\mu Y_t} \rangle \simeq e^{E(\mu) t}$, is the dominant eigenvalue of the deformed matrix

$$\boxed{M(\mu) = M_0 + e^{\mu} M_+ + e^{-\mu} M_-}$$

Generalized Matrix Ansatz

We have proved that the dominant eigenvector of the deformed matrix $M(\mu)$ is given by the following matrix product representation:

$$F_\mu(\mathcal{C}) = \frac{1}{Z_L^{(k)}} \langle W_k | \prod_{i=1}^{L} (\tau_i D_k + (1-\tau_i) E_k) | V_k \rangle + \mathcal{O}\left(\mu^{k+1}\right)$$

The matrices D_k and E_k are the same as above

$$\begin{aligned} D_{k+1} &= (1 \otimes 1 + d \otimes e) \otimes D_k + (1 \otimes d + d \otimes 1) \otimes E_k \\ E_{k+1} &= (1 \otimes 1 + e \otimes d) \otimes E_k + (e \otimes 1 + 1 \otimes e) \otimes D_k \end{aligned}$$

The boundary vectors $\langle W_k|$ and $|V_k\rangle$ are constructed recursively:

$$|V_k\rangle = |\beta\rangle|\tilde{V}\rangle|V_{k-1}\rangle \quad \text{and} \quad \langle W_k| = \langle W^\mu|\langle \tilde{W}^\mu|\langle W_{k-1}|$$

$$[\beta(1-d) - \delta(1+e)]\,|\tilde{V}\rangle = 0$$

$$\langle W^\mu|[\alpha(1+\mathrm{e}^{\mu}\, e) - \gamma(1+\mathrm{e}^{-\mu}\, d)] = (1-q)\langle W^\mu|$$

$$\langle \tilde{W}^\mu|[\alpha(1-\mathrm{e}^{\mu}\, e) - \gamma(1-\mathrm{e}^{-\mu}\, d)] = 0$$

Structure of the solution I

For arbitrary values of q and $(\alpha, \beta, \gamma, \delta)$, and for any system size L the parametric representation of $E(\mu)$ is given by

$$\mu = -\sum_{k=1}^{\infty} C_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k}$$

$$E = -\sum_{k=1}^{\infty} D_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k}$$

The coefficients C_k and D_k are given by contour integrals in the complex plane:

$$C_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{(z+1)^2}$$

There exists an auxiliary function

$$W_B(z) = \sum_{k \geq 1} \phi_k(z) \frac{B^k}{k}$$

that contains the full information about the statistics of the current.

Structure of the solution II

This auxiliary function $W_B(z)$ solves a functional Bethe equation:

$$W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

- The operator X is a integral operator

$$X[W_B](z_1) = \oint_{\mathcal{C}} \frac{dz_2}{\imath 2\pi\, z_2} W_B(z_2) K\left(\frac{z_1}{z_2}\right)$$

$$\text{with kernel} \quad K(z) = 2\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \left\{z^k + z^{-k}\right\}$$

- The function $F(z)$ is given by

$$F(z) = \frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(a_+z)_\infty(a_+z^{-1})_\infty(a_-z)_\infty(a_-z^{-1})_\infty(b_+z)_\infty(b_+z^{-1})_\infty(b_-z)_\infty(b_-z^{-1})_\infty}$$

where $(x)_\infty = \prod_{k=0}^{\infty}(1-q^kx)$ and $a_\pm$, $b_\pm$ depend on the boundary rates.

- The complex contour $\mathcal{C}$ encircles 0, $q^ka_+, q^ka_-, q^kb_+, q^kb_-$ for $k \geq 0$.

Discussion

- These results are of *combinatorial nature*: *valid for arbitrary values of the parameters and for any system sizes with no restrictions.*
- *For $q = 0$, the TASEP case is retrieved.*
- *Average-Current:*

$$J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = (1-q)\frac{D_1}{C_1} = (1-q)\frac{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{z}}{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{(z+1)^2}}$$

(cf. T. Sasamoto, 1999.)

- *Diffusion Constant:*

$$\Delta = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = (1-q)\frac{D_1\,C_2 - D_2\,C_1}{2\,C_1^3}$$

where C_2 and D_2 are obtained using

$$\phi_1(z) = \frac{F(z)}{2} \quad \textit{and} \quad \phi_2(z) = \frac{F(z)}{2}\left(F(z) + \oint_\Gamma \frac{dz_2 F(z_2) K(z/z_2)}{2\imath\pi z_2}\right)$$

(cf. the TASEP case: B. Derrida, M. R. Evans, K. M., 1995)

Asymptotic behaviour

- Maximal Current Phase: *(Similar Scaling as TASEP).*

$$\mu = -\frac{L^{-1/2}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+3/2)}} B^k$$

$$\mathcal{E} - \frac{1-q}{4}\mu = -\frac{(1-q)L^{-3/2}}{16\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+5/2)}} B^k$$

- Low Density (and High Density) Phases:
  Dominant singularity at a_+: $\phi_k(z) \sim F^k(z)$. By Lagrange Inversion:

$$E(\mu) = (1-q)(1-\rho_a)\frac{e^{\mu}-1}{e^{\mu}+(1-\rho_a)/\rho_a}$$

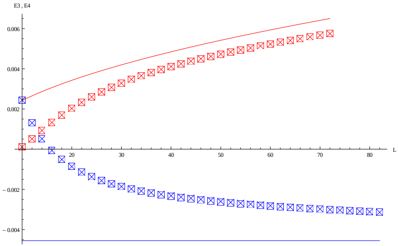
(cf de Gier and Essler, 2011).

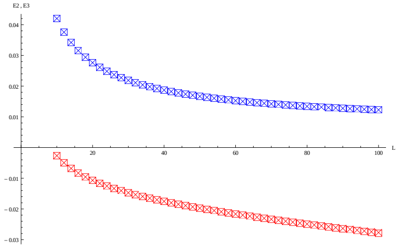
Current Large Deviation Function:

$$\Phi(j) = (1-q)\left\{\rho_a - r + r(1-r)\ln\left(\frac{1-\rho_a}{\rho_a}\frac{r}{1-r}\right)\right\}$$

where the current j is parametrized as $j = (1-q)r(1-r)$.
Matches the predictions of Macroscopic Fluctuation Theory, as observed by T. Bodineau and B. Derrida.

Numerical results (DMRG)





Left: **Max. Current** ($q = 0.5$, $a_+ = b_+ = 0.65$, $a_- = b_- = 0.6$), Third and Fourth cumulant.

Right: **High Density** ($q = 0.5$, $a_+ = 0.28$, $b_+ = 1.15$, $a_- = -0.48$ and $b_- = -0.27$), Second and Third cumulant.

Remarks

- The function $W_B(z)$ also contains information on the 6-vertex model associated with the ASEP.
- The periodic case falls to the same scheme (S. Prolhac, 2010):

$$\boxed{F(z) = \frac{(1+z)^L}{z^N}}$$

where L is the size of the system and N the conserved number of particles. The Kernel $K(z_1, z_2)$ has the same expression. Here, the coefficients C_k and D_k are combinatorial factors enumerating some tree structures (S. Prolhac, 2010).

- A striking coincidence: the double-series for the *open* TASEP of size L for $\alpha = 1$ and $\beta = 1/2$ are *identical to* the formulas for the half-filled *periodic* TASEP of size $2L + 2$.

TASEP CASE (Derrida Lebowitz 1998)

$E(\mu)$ is calculated by Bethe Ansatz to all orders in μ, thanks to the decoupling property of the Bethe equations.
The structure of the solution is given by a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = -\frac{1}{L}\sum_{k=1}^{\infty} \frac{[kL]!}{[kN]!\,[k(L-N)]!}\frac{B^k}{k}\,,$$

$$E = -\sum_{k=1}^{\infty} \frac{[kL-2]!}{[kN-1]!\,[k(L-N)-1]!}\frac{B^k}{k}\,.$$

Mean Total current:

$$J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = \frac{N(L-N)}{L-1}$$

Diffusion Constant:

$$D = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \frac{LN(L-N)}{(L-1)(2L-1)} \frac{C_{2L}^{2N}}{\left(C_L^N\right)^2}$$

Exact formula for the large deviation function.

Conclusion

Exact solutions of the asymmetric exclusion process are paradigms for the behaviour of systems far from equilibrium in low dimensions: Dynamical phase transitions, Non-Gibbsean measures, Large deviations, Fluctuations Theorems...

Large deviation functions (LDF) appear as the right generalization of the thermodynamic potentials: convex, optimized at the stationary state, and non-analytic features can be interpreted as phase transitions. The LDF's are very likely to play a key-role in the future of non-equilibrium statistical mechanics.

Tensor products of quadratic algebras provides us with an efficient tool to solve challenging problems: multispecies models; current fluctuations in the open TASEP. In particular, the tensor matrix Ansatz gives access to density profiles that generate atypical currents.