Quasi-stationary states in long-range interacting systems

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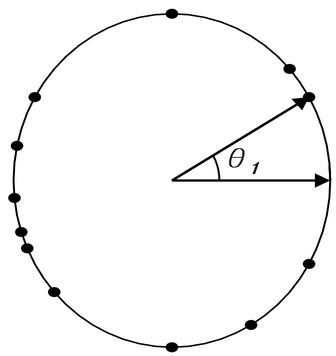
Plan

- Hamiltonian Mean Field (HMF) model
- Quasi-stationary-states (QSS)
- Klimontovich and Vlasov equations
- Vlasov equation on a lattice (with Bachelard, Dauxois, De Ninno and Staniscia)
- α-HMF model
- Linear response of quasistationary states (with Gupta, Nardini and Patelli)
- Inhomogeneous steady states (with de Buyl and Mukamel)

HMF model

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} (1 - \cos(\theta_i - \theta_j))$$



Magnetization $\mathbf{M} = \lim_{N\to\infty} \left(\frac{\sum_{i=1}^{N} \cos\theta_i}{N}, \frac{\sum_{i=1}^{N} \sin\theta_i}{N} \right) = (M_x, M_y)$

Energy $U = \lim_{N\to\infty} \frac{H}{N}$

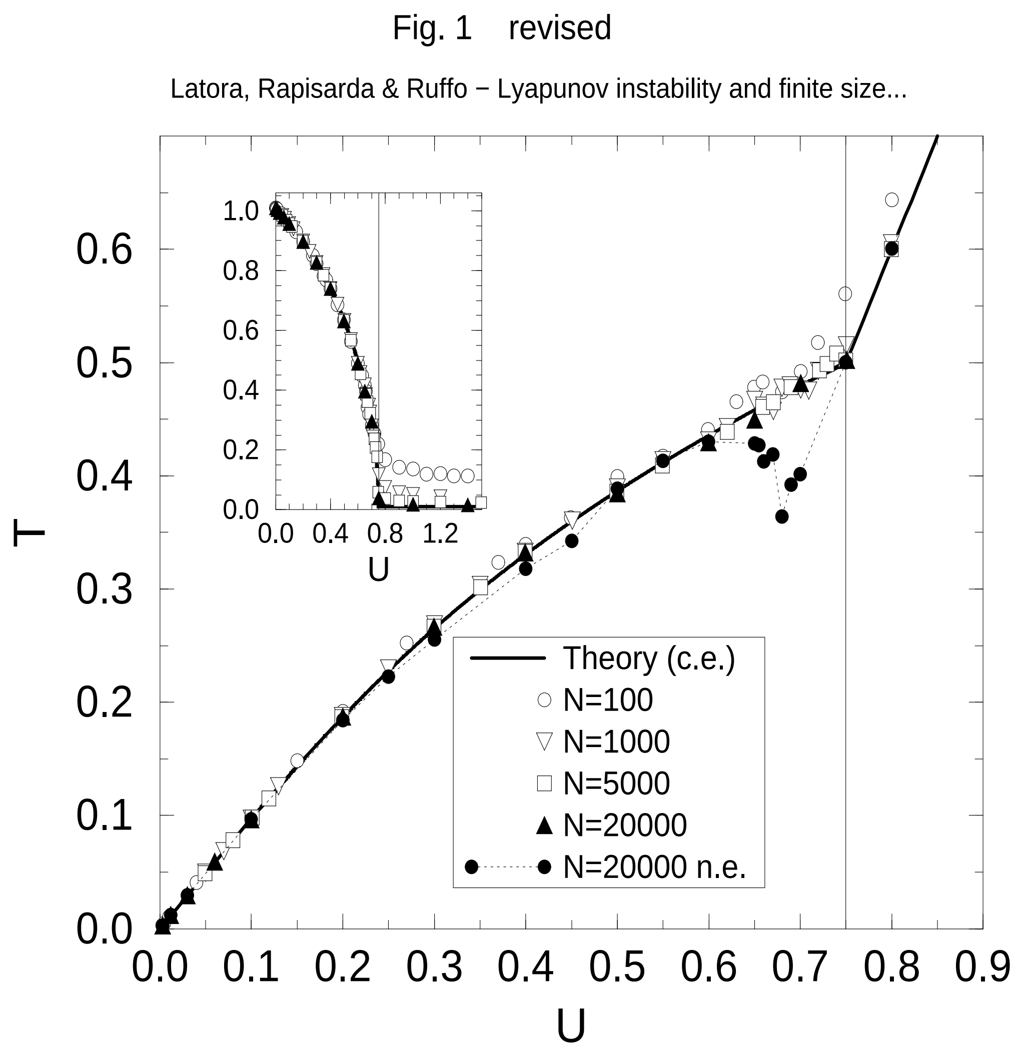
Equations of motion

$$\begin{aligned} \dot{\theta}_i &= \frac{\partial H}{\partial p_i} = p_i \\ \dot{p}_i &= -\frac{\partial H}{\partial \theta_i} = -\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j) \end{aligned}$$

$$\begin{aligned} \dot{\theta}_i &= p_i \\ \dot{p}_i &= -M\sin(\theta_i - \phi) \end{aligned}$$

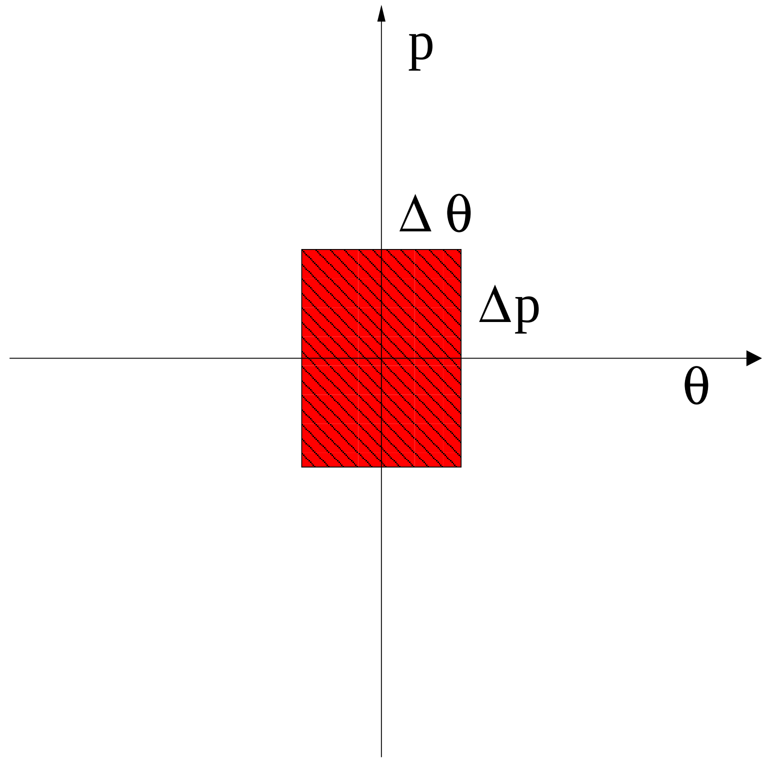
$\tan(\phi) = \frac{M_y}{M_x}$

Equilibrium phase transition



$$U_c = 3/4 = 0.75, T = (\partial S/\partial U)^{-1} = \lim_{N\to\infty} \sum_i p_i^2/N$$

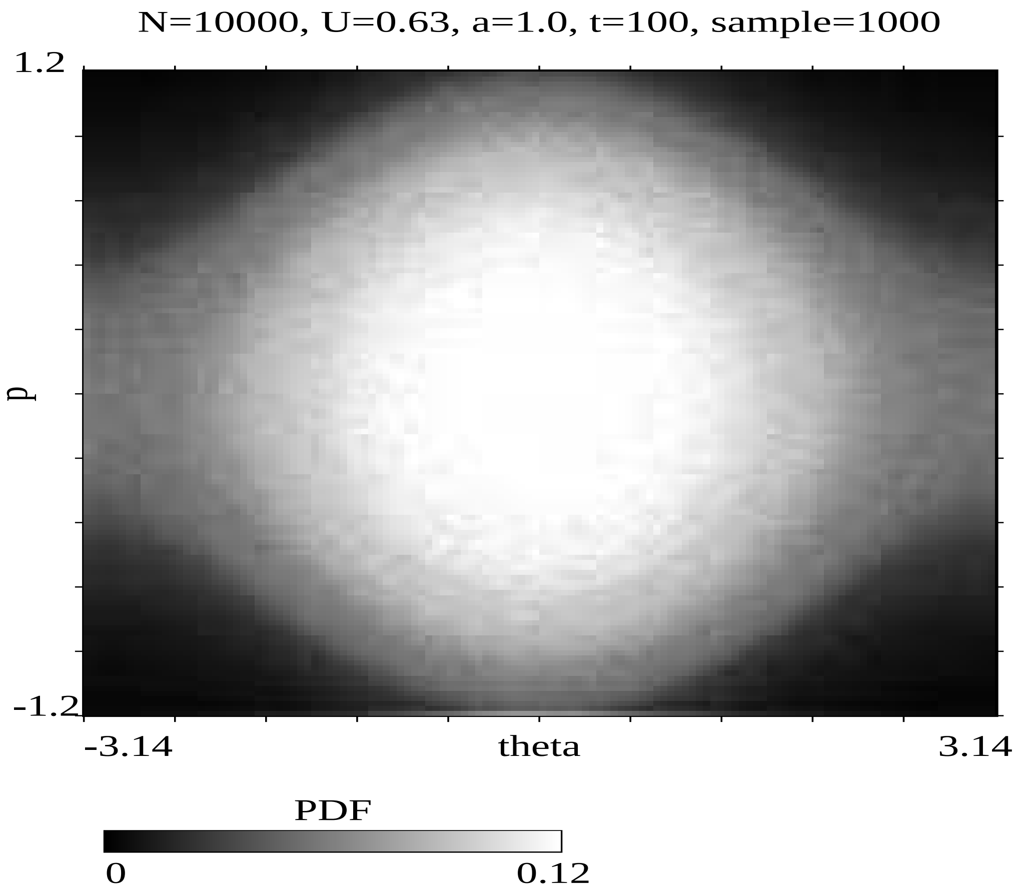
Waterbag



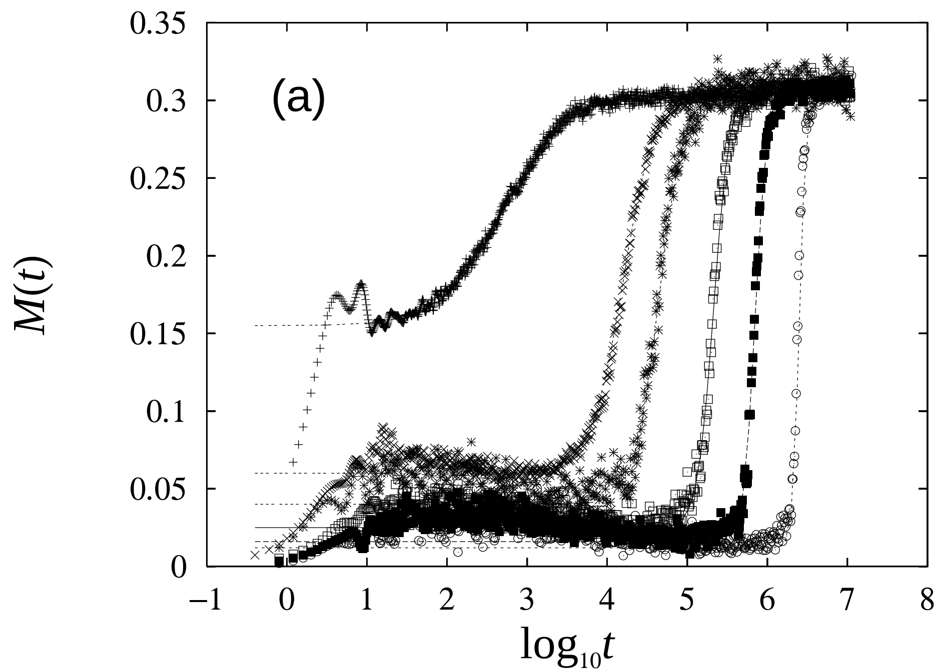
$$M_0 = \frac{\sin \Delta\theta}{\Delta\theta}$$

$$U = \lim_{N\to\infty} \frac{H}{N} = \frac{(\Delta p)^2}{6} + \frac{1-(M_0)^2}{2}$$

Equilibrium

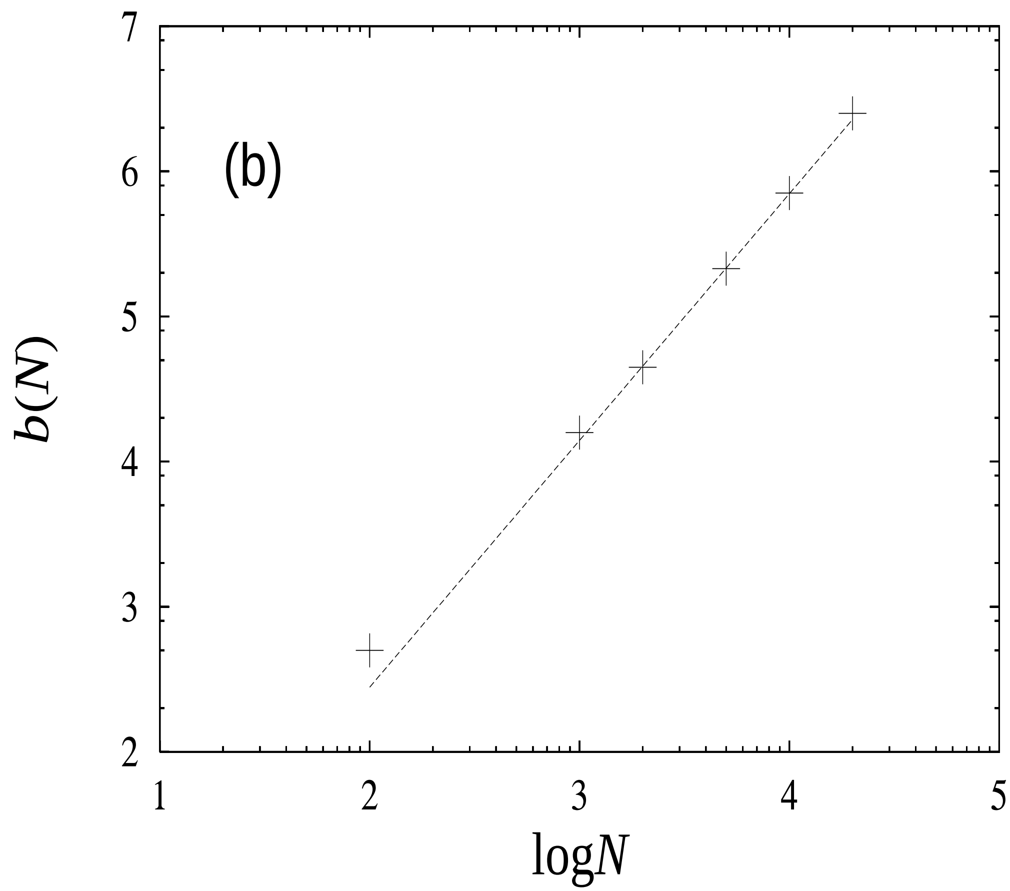


Quasi-stationary states



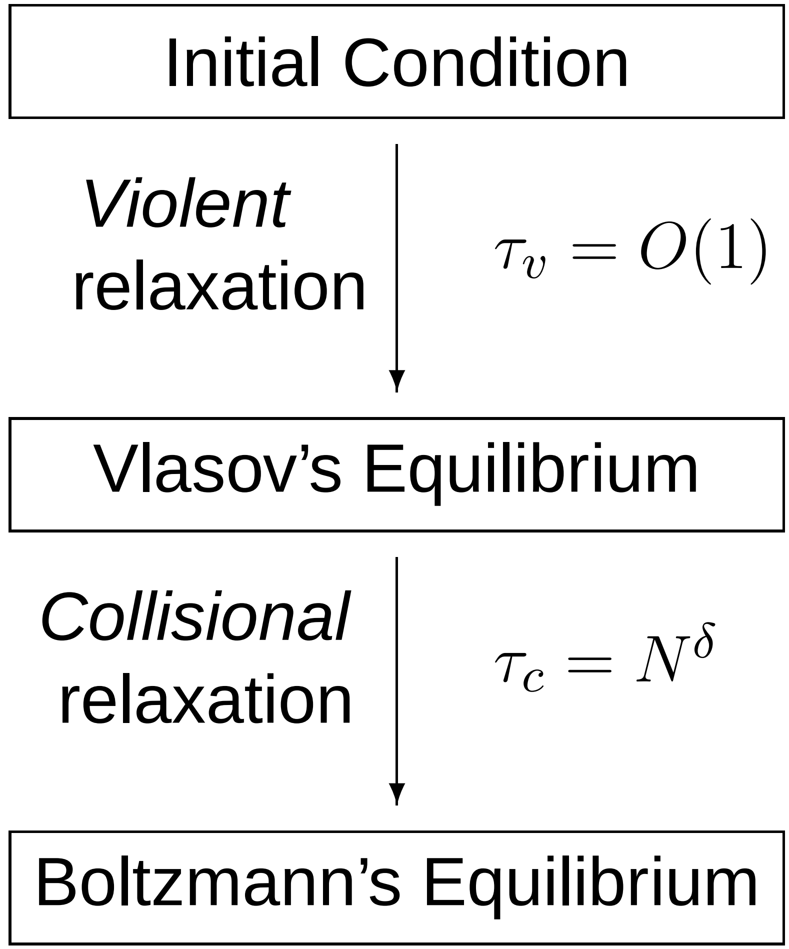
$U = 0.69$, from left to right
$N = 10^2, 10^3, 2 \times 10^3, 5 \times 10^3, 10^4, 2 \times 10^4$.
Initially $\Delta\theta = \pi$, hence $M_0 = 0$.

Power law



Power law increase of the lifetime, exponent 1.7

Separation of time scales



Klimontovich equation

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + U(\theta_i) \quad , \quad U(\theta_1, .., \theta_N) = \sum_{i<j}^{N} V(\theta_i - \theta_j)$$

Discrete, one-particle, time dependent density function

$$f_d(\theta, p, t) = \sum_{i=1}^{N} \delta(\theta - \theta_i(t))\delta(p - p_i(t)) \quad , \quad \iint d\theta dp f_d = N$$

Using the equations of motion and the property $x\delta(x-y) = y\delta(x-y)$ one gets the Klimontovich equation

$$\frac{\partial f_d}{\partial t} + p\frac{\partial f_d}{\partial \theta} - \frac{\partial v}{\partial \theta}\frac{\partial f_d}{\partial p} = 0$$

where

$$v(\theta, t) = \int d\theta' dp' V(\theta - \theta') f_d(\theta', p', t)$$

From Klimontovich to Vlasov

Smooth distributions

$$f_d = \langle f_d(\theta, p, t) \rangle + \delta f(\theta, p, t) = f(\theta, p, t) + \delta f(\theta, p, t)$$

where the average $\langle \cdot \rangle$ is taken over initial conditions.

$$v(\theta, t) = \langle v \rangle(\theta, t) + \delta v(\theta, t)$$

where

$$\langle v \rangle(\theta, t) = \int d\theta' dp' V(\theta - \theta') f(\theta', p', t)$$

Using Klimontovic equation, one gets the exact evolution equation

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial \theta} - \frac{\partial \langle v \rangle}{\partial \theta} \frac{\partial f}{\partial p} = \langle \frac{\partial \delta v}{\partial \theta} \frac{\partial \delta f}{\partial p} \rangle$$

Scaling to infinite N

Let us impose that potential energy scales like kinetic energy (virial condition)

$$Np^2 \sim N^2 V_0$$

where V_0 is the typical value of the potential energy per pair. Hence, $E \sim N^2 V_0$, $T \sim p^2 \sim NV_0$, $t_d \sim R/p \sim R/\sqrt{NV_0}$, where R is the typical size of the system and t_d a dynamical time. Now, let us consider the thermodynamic limit in which $\varepsilon = E/(N^2 V_0)$, $T' = T/(NV_0)$ and $\tau_d = t_d \sqrt{NV_0}/R$ stay constant as $N \to \infty$. All this works if $N \to \infty$, $R \sim 1$, $E \sim N$, $T \sim 1$, $t_d \sim 1$ and $V_0 \sim 1/N$, this latter being the well known Kac trick. As for the reduced one-particle distribution function: $f/N \sim f_d/N \sim 1$, $< v > \sim 1$, $\delta f \sqrt{N}$, $\delta v \sim 1/\sqrt{N}$, which implies

$$\frac{1}{N} \langle \frac{\partial \delta v}{\partial \theta} \frac{\partial \delta f}{\partial p} \rangle \sim \frac{1}{N}$$

while f/N remains of order 1. This proves that in this scaling limit the r.h.s. of the exact evolution equation for f scales to zero faster than the l.h.s., proving in turn that Vlasov equation becomes exact in this scaling limit.

Adiabatic approximation

The perturbation δf obeys the following equation

$$\frac{\partial \delta f}{\partial t} + p\frac{\partial \delta f}{\partial \theta} - \frac{\partial \delta v}{\partial \theta}\frac{\partial f}{\partial p} - \frac{\partial \langle v \rangle}{\partial \theta}\frac{\partial \delta f}{\partial p} = \frac{\partial \delta v}{\partial \theta}\frac{\partial \delta f}{\partial p} - \langle \frac{\partial \delta v}{\partial \theta}\frac{\partial \delta f}{\partial p} \rangle$$

The l.h.s. of this equation is $O(\sqrt{N})$ while the r.h.s. is $O(1)$, hence we can neglect the r.h.s. in the large N limit (quasilinear theory). This corresponds to neglect higher order correlations (higher than those created by two-body collisions). If, moreover, we consider homogeneous states: $\partial f/\partial \theta = 0$, $\partial \langle v \rangle/\partial \theta = 0$, we obtain the coupled equations

$$\frac{\partial f}{\partial t} = \langle \frac{\partial \delta v}{\partial \theta}\frac{\partial \delta f}{\partial p} \rangle \quad , \quad \frac{\partial \delta f}{\partial t} + p\frac{\partial \delta f}{\partial \theta} - \frac{\partial \delta v}{\partial \theta}\frac{\partial f}{\partial p} = 0$$

These equations are valid if the spatially homogeneous distribution is Vlasov stable and the system evolves only under the effect of “collisions". In the adiabatic Bogoliubov approximation the time evolution of δf (and of course of δv) is considered to be much faster than the one of f itself. Therefore, one can neglect the time evolution of f when solving the equation for δf. Once δf and δv are so obtained, they are inserted in the first equation, which then becomes a kinetic evolution equation for f, the so-called Lenard-Balescu equation.

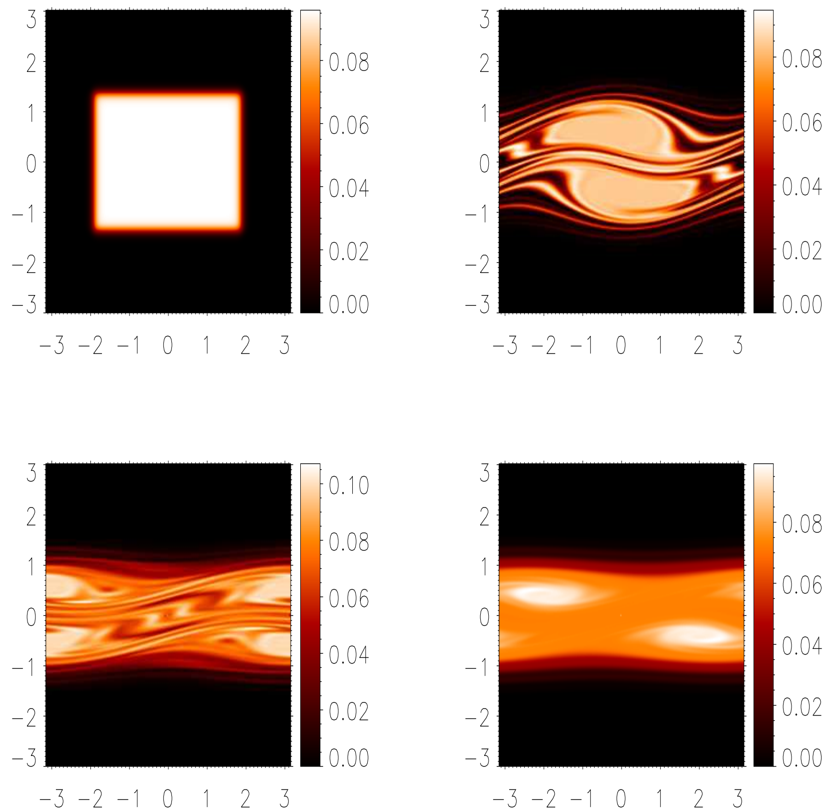
HMF Vlasov equation

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial \theta} - \frac{dV}{d\theta}\frac{\partial f}{\partial p} = 0 \quad ,$$

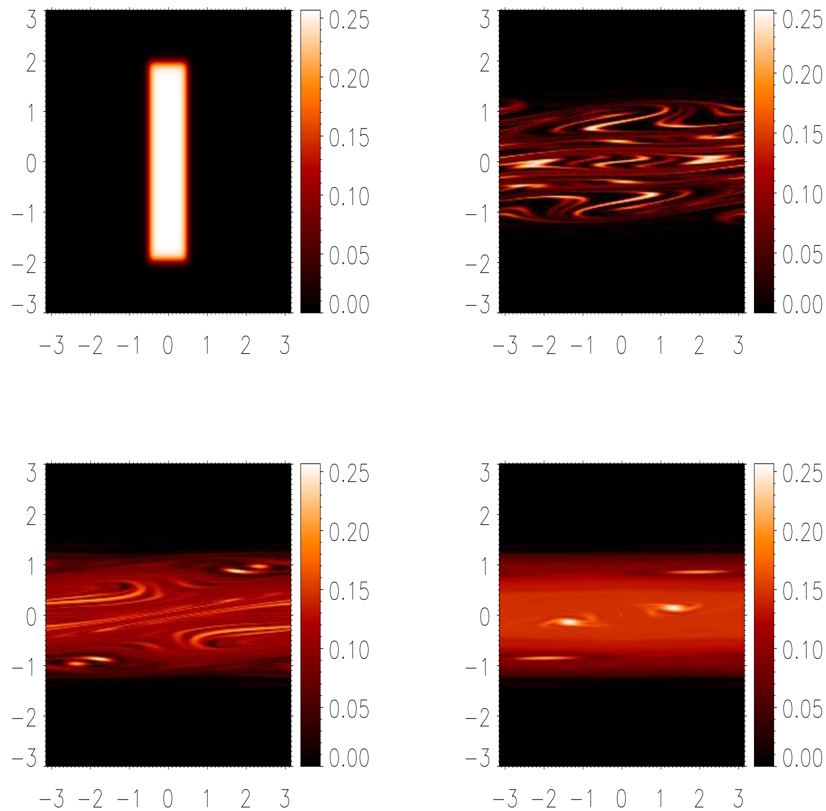
$$\begin{aligned} V(\theta)[f] &= 1 - M_x[f]\cos(\theta) - M_y[f]\sin(\theta) \ , \\ M_x[f] &= \int f(\theta,p,t)\ \cos\theta \mathrm{d}\theta\mathrm{d}p \quad , \\ M_y[f] &= \int f(\theta,p,t)\ \sin\theta \mathrm{d}\theta\mathrm{d}p \quad . \end{aligned}$$

Specific energy
$e[f] = \int (p^2/2) f(\theta,p,t)\mathrm{d}\theta\mathrm{d}p + 1/2 - (M_x^2 + M_y^2)/2$ and momentum $P[f] = \int p f(\theta,p,t)\mathrm{d}\theta\mathrm{d}p$ are conserved.

Vlasov fluid-I



Vlasov fluid-II



Vlasov equation on a lattice

$$H = \sum_j \frac{p_j^2}{2} + \frac{1}{2\tilde{N}} \sum_{j,k=1}^{N} \frac{v(q_j, q_k)}{|x_j - x_k|^\alpha} \quad , \quad x_j = ja$$

Equations of motion

$$\dot{q}_j = p_j, \quad , \quad \dot{p}_j = -\frac{1}{\tilde{N}} \sum_k \frac{v'(q_j, q_k)}{|x_j - x_k|^\alpha}$$

Continuum limit

$$\dot{q} = p \quad , \quad \dot{p} = -\frac{\partial V_x[f](q,t)}{\partial q}$$

where

$$V_x[f](q,t) = \kappa_\alpha \iiint dq' dp' dx' f(q', p'; x', t) \frac{v(q, q')}{|x - x'|^\alpha} \ ,$$

$\kappa_\alpha^{-1} = \int_{-1/2}^{+1/2} dx/|x|^\alpha$. This allows us to write the following Vlasov equation

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial q} - \frac{\partial V_x'[f](q,t)}{\partial q} \frac{\partial f}{\partial p} = 0.$$

Linearized Vlasov equation

$$f(q,p;x,t) = f_0(p) + \delta f(q,p;x,t) \quad , \quad \delta f_t(q,p;x) = e^{\lambda t}\bar{f}(q,p;x)$$

$$\partial_t(\delta f) + p\partial_q(\delta f) - \partial_p f_0(p)\partial_q V_x[\delta f](q,t) = 0.$$

Dispersion relation

$$\bar{f}(q,p;x) - \partial_p f_0(p)\frac{e^{-\lambda \frac{q}{p}}}{p}\int_{q_0}^{q} e^{\lambda \frac{q'}{p}}\partial_{q'}V_x[\bar{f}](q')dq' = 0,$$

This is indeed a set of integro-differential equations for each position x, all coupled together by the $V_x[\bar{f}]$ term. These equations can be decoupled for periodic boundary conditions using a Fourier series expansion $\bar{f}(q,p;x) = \sum_k \hat{f}_k(q,p)\exp(2i\pi kx)$, giving

$$\hat{f}_k(q,p) - c_k(\alpha)\partial_p f_0(p)\frac{e^{-\lambda_k \frac{q}{p}}}{p}\int_{q_0}^{q} e^{\lambda_k \frac{q'}{p}} V[\hat{f}_k](q')dq' = 0$$

where

$$c_k(\alpha) = \kappa_\alpha \int_{-1/2}^{+1/2}\frac{e^{2i\pi ky}}{|y|^\alpha}dy \quad , \quad V[\hat{f}_k](q) = \iint dq'dp'\hat{f}_k(q',p')v(q,q')$$

α-HMF model-I

$$v(q,q') = -\cos(q-q') \quad , \quad V[\hat{f}_k](q) = -\kappa_\alpha \left(m_x[\hat{f}_k]\cos q + m_y[\hat{f}_k]\sin q \right),$$

$$m_x[\hat{f}_k] = \iint dq'\,dp'\,\hat{f}_k(q',p')\cos q' \quad , \quad m_y[\hat{f}_k] = \iint dq'\,dp'\,\hat{f}_k(q',p')\sin q'$$

The dispersion relation in Fourier space is then

$$\hat{f}_k(q,p) - c_k(\alpha)\partial_p f_0(p)\frac{e^{-\lambda_k \frac{q}{p}}}{p}\int_{q_0}^{q} e^{\lambda_k \frac{q'}{p}} \left(m_x[\hat{f}_k]\sin q' - m_y[\hat{f}_k]\cos q' \right) dq' = 0.$$

This equation can be solved by multiplying by $\cos q$ and $\sin q$ and integrating, giving

$$m_x[\hat{f}_k]\left(1 - c_k(\alpha)I^{\lambda_k}_{X,Y}[f_0]\right) + m_y[\hat{f}_k]c_k(\alpha)I^{\lambda_k}_{X,X}[f_0] = 0,$$

$$m_x[\hat{f}_k]c_k(\alpha)I^{\lambda_k}_{Y,Y}[f_0] - m_y[\hat{f}_k]\left(1 + c_k(\alpha)I^{\lambda_k}_{Y,X}[f_0]\right) = 0$$

α-HMF model-II

where

$$I^{\lambda}_{X,Y}[f_0] = \int dp \frac{\partial_p f_0(p)}{p} \oint dq e^{-\lambda \frac{q}{p}} X(q) \int_{q_0}^{q} dq' e^{\lambda \frac{q'}{p}} Y(q'),$$

with $X(q) = \cos q$ and $Y(q) = \sin q$. Assuming that the eigenmodes $\hat{f}$ have a non zero magnetization

$$\left(1 - c_k(\alpha) I^{\lambda_k}_{X,Y}[f_0]\right)\left(1 + c_k(\alpha) I^{\lambda_k}_{Y,X}[f_0]\right) + c_k(\alpha)^2 I^{\lambda_k}_{Y,Y}[f_0] I^{\lambda_k}_{X,X}[f_0] = 0.$$

Using the following formulae

$$\int^{q} dq' \, e^{\lambda_k \frac{q'}{p}} \sin q' \quad = \quad \frac{e^{\lambda_k \frac{q}{p}}}{1 + \frac{{\lambda_k}^2}{p^2}} \left(\frac{\lambda_k}{p} \sin q - \cos q\right), \tag{1}$$

$$\int^{q} dq' \, e^{\lambda_k \frac{q'}{p}} \cos q' \quad = \quad \frac{e^{\lambda_k \frac{q}{p}}}{1 + \frac{{\lambda_k}^2}{p^2}} \left(\sin q + \frac{\lambda_k}{p} \cos q\right). \tag{2}$$

α-HMF model-III

one can calculate $I^{\lambda}_{X,Y}$ obtaining the dispersion relation in explicit form

$$\left(1+\pi c_k(\alpha)\int dp\frac{\partial_p f_0(p)}{p\left(1+\frac{{\lambda_k}^2}{p^2}\right)}\right)^2+\left(\pi c_k(\alpha)\int dp\frac{\partial_p f_0(p)}{p^2\left(1+\frac{{\lambda_k}^2}{p^2}\right)}\right)^2.$$

For a waterbag distribution

$$f_0(p)=\frac{1}{2\pi}\frac{1}{2\Delta p}\left(\Theta(p+\Delta p)-\Theta(p-\Delta p)\right),$$

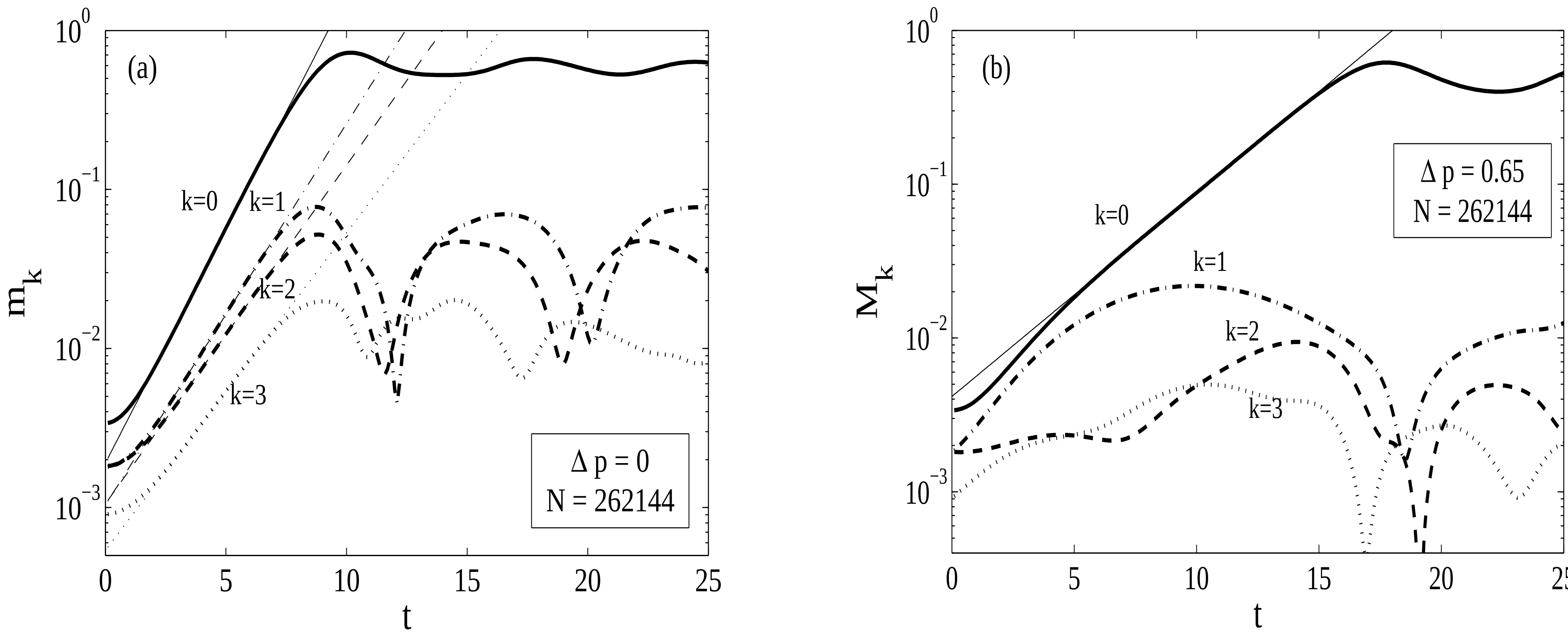
which is symmetric in p, the second term in the dispersion relation vanishes, and it simplifies into

$$1+\pi c_k(\alpha)\int dp\frac{\partial_p f_0(p)}{p\left(1+\frac{{\lambda_k}^2}{p^2}\right)}=1-\frac{c_k(\alpha)}{2\Delta p^2\left(1+\frac{{\lambda_k}^2}{\Delta p^2}\right)}=0.$$

Thus, the eigenvalue of the k-th Fourier mode is given by

$$\lambda_k=\sqrt{\frac{c_k(\alpha)}{2}-\Delta p^2}.$$

Numerical results



Perturbed hamiltonian

$$H(t) = H_0 + H_{\text{ext}} = H_0 - K(t) \sum_{i=1}^{N} b(q_i)$$

Vlasov equation as a Liouville equation for f

$$\frac{\partial f}{\partial t} - \mathcal{L}(q,p,t)[f] f = 0$$

$$\mathcal{L}(q,p,t)[f] \equiv -p \frac{\partial}{\partial q} + \frac{\partial \Phi(q,t)[f]}{\partial q} \frac{\partial}{\partial p} - K(t) \frac{\partial b}{\partial q} \frac{\partial}{\partial p}$$

Stationary state of the unperturbed Hamiltonian H_0

$$\mathcal{L}_0(q,p)[f_0] f_0 = 0$$

$$\mathcal{L}_0(q,p)[f_0] = -p \frac{\partial}{\partial q} + \frac{\partial \bar{\Phi}(q)[f_0]}{\partial q} \frac{\partial}{\partial p}$$

Linearized Vlasov equation

$$f(q,p,t) = f_0(q,p) + \Delta f(q,p,t)$$

$$\Delta f(q,p,0) = 0$$

$$\frac{\partial \Delta f}{\partial t} - \mathcal{L}_0(q,p)[f_0]\Delta f = \mathcal{L}_{\mathrm{ext}}(q,p,t)[\Delta f] f_0(q,p)$$

$$\mathcal{L}_{\mathrm{ext}}(q,p,t)[\Delta f] = \frac{\partial v_{\mathrm{eff}}(q,t)[\Delta f]}{\partial q} \frac{\partial}{\partial p}$$

$$v_{\mathrm{eff}}(q,t)[\Delta f] = \Phi(q,t)[\Delta f] - K(t) b(q)$$

Evolution of an observable

Formal solution of the linearized equation

$$\Delta f(q,p,t) = \int_0^t d\tau\; e^{(t-\tau)\mathcal{L}_0(q,p)[f_0]}\mathcal{L}_{\rm ext}(q,p,\tau)[\Delta f] f_0(q,p)$$

$$\langle \Delta a(q)\rangle(t) \equiv \langle a(q)\rangle(t) - \langle a(q)\rangle_{f_0} = \int dqdp\; a(q)\Delta f(q,p,t)$$

$$\langle \Delta a(q)\rangle(t) = -\int_0^t d\tau \left\langle \frac{\partial a(t-\tau)}{\partial p}\frac{\partial v_{\rm eff}(q,\tau)[\Delta f]}{\partial q}\right\rangle_{f_0}$$

with

$$\langle a(q)\rangle_{f_0} \equiv \iint dqdp\; a(q) f_0(q,p), \quad a(t-\tau) = e^{-(t-\tau)\mathcal{L}_0(q,p)[f_0]}a(q)$$

Solution in Laplace-Fourier

For homogeneous stationary states $f_0 = P(p)$

$$\widehat{\Delta f}(k,p,\omega)\frac{\partial P(p)}{\partial p}\frac{k}{kp-\omega}\left[2\pi\widetilde{v}(k)\int dp'\,\widehat{\Delta f}(k,p',\omega)-\widehat{K}(\omega)\widetilde{b}(k)\right]$$

where $\widetilde{v}(k)$ is the Fourier transform of the two-body potential and

$$\epsilon(k,\omega) = 1 - 2\pi k\widetilde{v}(k)\int \frac{dp}{kp-\omega}\frac{\partial P(p)}{\partial p}$$

is the so-called plasma response dielectric function.
Since we'll be interested in observables that depend only on q

$$\int dp\,\widehat{\Delta f}(k,p,\omega) = \frac{\widehat{K}(\omega)\widetilde{b}(k)}{2\pi\widetilde{v}(k)}\left[\frac{\epsilon(k,\omega)-1}{\epsilon(k,\omega)}\right]$$

Application to HMF

$$v(q) = 1 - \cos q, \quad \widetilde{v}(k) = \left[\delta_{k,0} - \frac{\delta_{k,-1} + \delta_{k,1}}{2}\right]$$

$$b(q) = \cos q, \quad \widetilde{b}(k) = \frac{\delta_{k,-1} + \delta_{k,1}}{2}$$

$$\widehat{K}(\omega) = -\frac{h}{i\omega}$$

$$\int dp\, \widehat{\Delta f}(\pm 1, p, \omega) = \frac{ih}{2\pi\omega}\Big[\frac{1 - \epsilon(\pm 1, \omega)}{\epsilon(\pm 1, \omega)}\Big]$$

$$\int dp\, \widetilde{\Delta f}(\pm 1, p, t) = \frac{ih}{4\pi^2}\int_L d\omega\, \frac{1}{\omega}\Big[\frac{1}{\epsilon(\pm 1, \omega)} - 1\Big] e^{-i\omega t}$$

$$\langle m_x \rangle(t) = \frac{ih}{2\pi}\int_L d\omega\, \frac{1}{\omega}\Big[\frac{1}{\epsilon(\pm 1, \omega)} - 1\Big] e^{-i\omega t}$$

while $\langle m_x \rangle(t) = 0$ for all times.

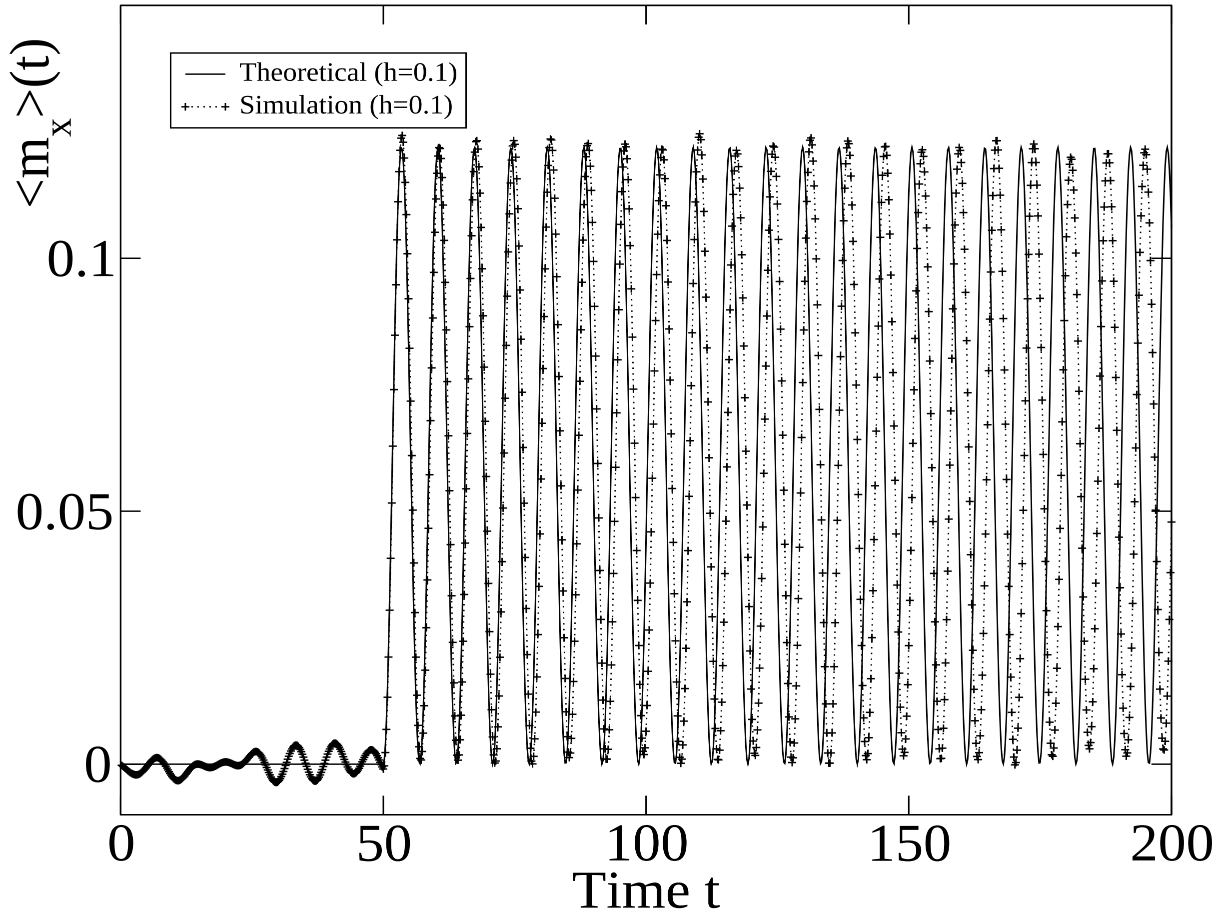
Homogeneous waterbag

$$P(p) = \frac{1}{2\pi}\frac{1}{2p_0}\Big[\Theta(p+p_0) - \Theta(p-p_0)\Big]; \quad p \in [-p_0, p_0]$$

$$\epsilon(\pm 1, \omega) = 1 - \frac{1}{2(p_0^2 - \omega^2)}$$

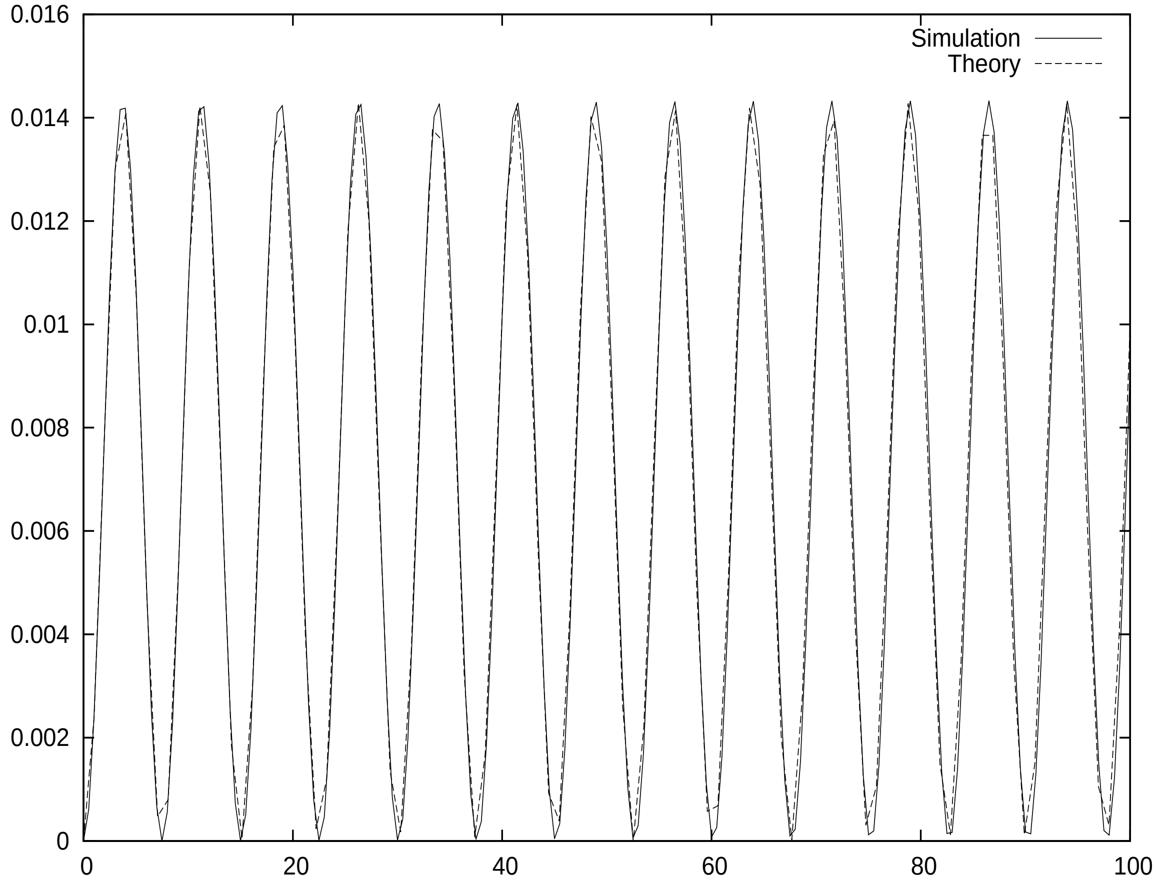
$$\langle m_x \rangle(t) = \frac{2h}{2p_0^2 - 1} \sin^2\left(\frac{t}{2}\sqrt{p_0^2 - \frac{1}{2}}\right)$$

Numerical results-I



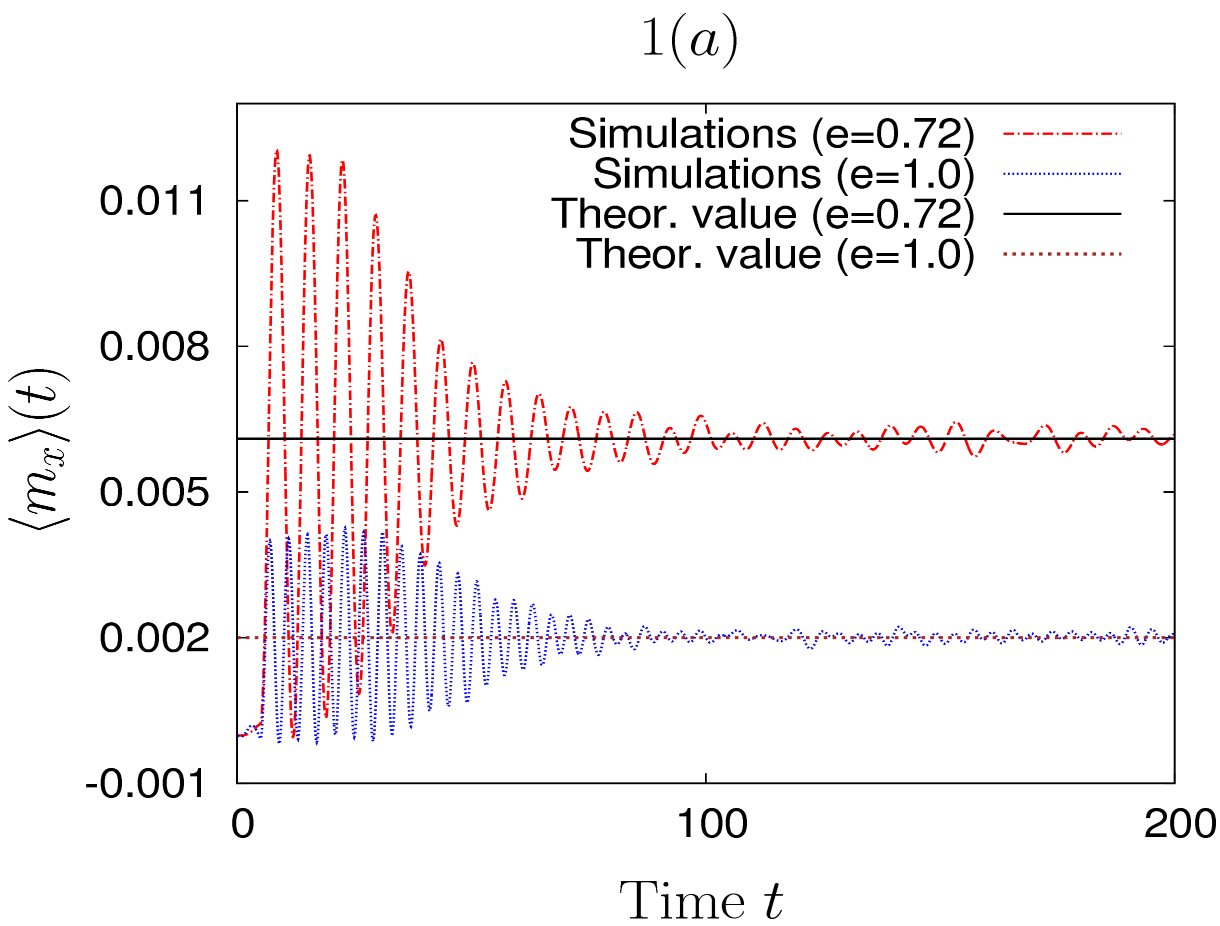
N-body simulation, $N = 10^5$, $p_0 = 1.1$, $U = .7$

Numerical results-II



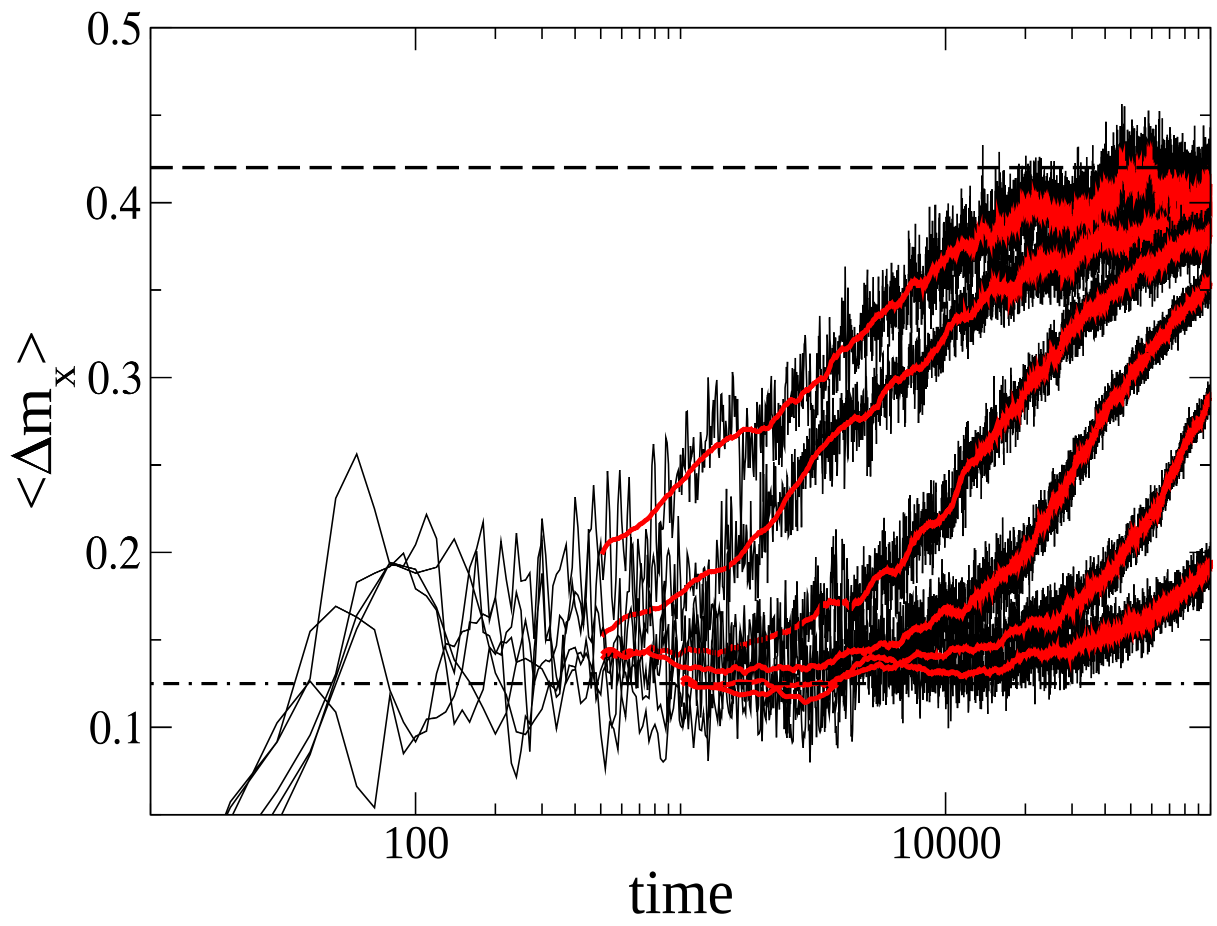
Vlasov simulation ($N = \infty$), $U = 0.7$.

Numerical results-III



$N = 10^5$

Relaxation to equilibrium



Exact inhomogeneous steady states

Vlasov equation for the HMF model

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial \theta} - \frac{\partial V[f](\theta,t)}{\partial \theta}\frac{\partial f}{\partial p} = 0$$

where

$$V[f](\theta,t) = \iint d\theta' dp' f(\theta',p',t)\left(1-\cos(\theta'-\theta)\right)$$

Any function

$$f_S(\theta,p) = F(h(\theta,p)) \text{ with}$$

with

$$h(\theta,p) = \frac{p^2}{2} + V[f_S](\theta).$$

and

$$V(\theta)[f_S] = 1 - M_x[f_S]\cos(\theta) - M_y[f_S]\sin(\theta)$$

is a stationary solution of the Vlasov equation, once we require that it has the correct properties of a one-particle distribution (Bernstein-Green-Kruskal (BGK) modes.

Non interacting particles

Consider the dynamics of an ensemble of *uncoupled particles* moving in a fixed external field H

$$\epsilon(\theta, p) = \frac{p^2}{2} - H\cos\theta$$

For an arbitrary function $F(\epsilon(\theta, p))$ to be a steady state of the interacting model, H has to satisfy the followign self-consistency condition

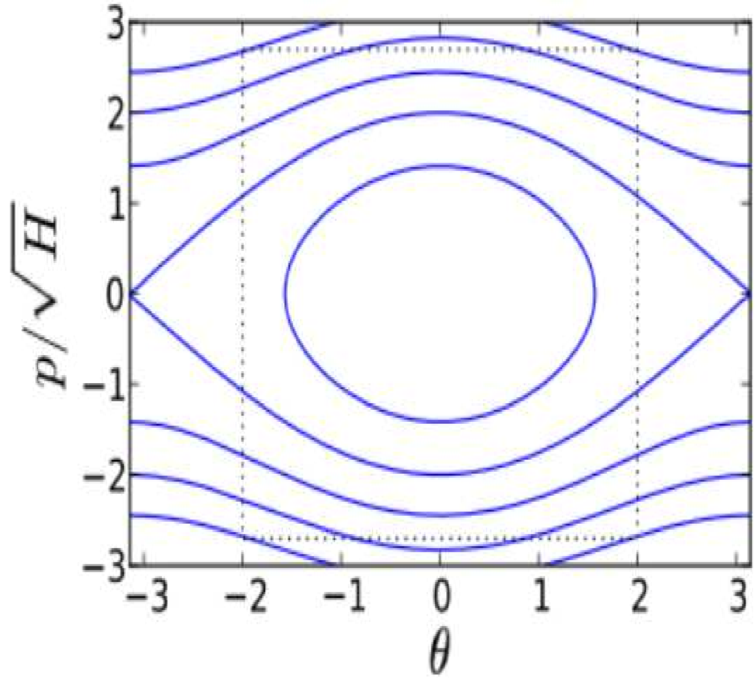
$$H = m_x = \iint d\theta dp F(\epsilon(\theta, p)) \cos\theta \quad ; \quad m_y = 0$$

To relate an initial distribution to the steady state to which it evolves, consider an initial distribution $f_0(\theta, p)$. The dynamics of the uncoupled model is such that particles in a given energy shell $[\epsilon, \epsilon + d\epsilon]$ keep moving inside that shell, eventually reaching a homogeneous distribution within it. As a result, the system attains the following steady state distribution

$$P(\theta, p) = \frac{\iint d\theta' dp' f_0(\theta', p')\, \delta\left(\epsilon(\theta', p') - \epsilon(\theta, p)\right)}{\iint d\theta' dp'\, \delta\left(\epsilon(\theta', p') - \epsilon(\theta, p)\right)}$$

Initial waterbag

$$f_0(\theta, p) = \begin{cases} (4\Delta\theta\Delta p)^{-1} & , \text{ for } |\theta| \leq \Delta\theta \text{ and } |p| \leq \Delta p \ , \\ 0 & , \text{ otherwise.} \end{cases}$$



Energy distribution

In order to evaluate $P(\theta, p)$, it is convenient to first consider the energy distribution $P_\epsilon(\epsilon)$. For the waterbag initial state it is given by

$$P_\epsilon(\epsilon) = \frac{1}{4\Delta\theta\Delta p} \int d\theta \int_{-\Delta p}^{\Delta p} dp \; \delta(\frac{p^2}{2} - H\cos\theta - \epsilon)$$

Integrating over p

$$P_\epsilon(\epsilon) = \frac{1}{2\Delta\theta\Delta p} \int d\theta \; \frac{1}{\sqrt{2(\epsilon + H\cos\theta)}} \; ,$$

for $-H \leq \epsilon \leq \Delta p^2/2 - H\cos\Delta\theta$ and zero outside this range.
The integration over θ need to be done in the domain enclosed by the initial waterbag

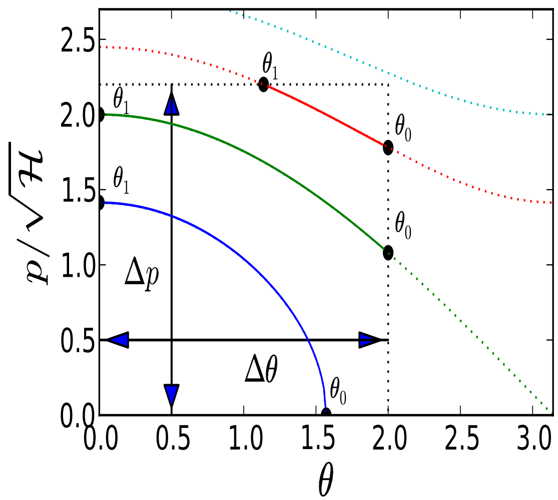
$$0 \leq \epsilon + H\cos\theta \leq \frac{\Delta p^2}{2}.$$

Thus,

$$P_\epsilon(\epsilon) = \frac{\sqrt{2}}{2\Delta\theta\Delta p} \int_{\theta_1}^{\theta_0} d\theta \frac{1}{\sqrt{(\epsilon + H\cos\theta)}} \qquad (3)$$

where θ_0 and θ_1 satisfy the conditions described in the next slide.

Integration limits



$$\theta_0 = \begin{cases} \arccos(-\epsilon/H) & \text{, for } -H < \epsilon < -H\cos\Delta\theta \\ \Delta\theta & \text{, for } \epsilon \geq -H\cos\Delta\theta \text{ ,} \end{cases}$$

$$\theta_1 = \begin{cases} 0 & \text{, for } -H < \epsilon \leq \Delta p^2/2 - H \\ \arccos(\frac{\Delta p^2/2-\epsilon}{H}) & \text{, for } \Delta p^2/2 - H \leq \epsilon < \\ & \quad \Delta p^2/2 - H\cos\Delta\theta \\ \Delta\theta & \text{, for } \epsilon \geq \Delta p^2/2 - H\cos\Delta\theta \text{ .} \end{cases}$$

Steady state distribution

In the steady state, the distribution is such that all the microstates corresponding to a given energy are equally probable. The boundaries on (θ, p) imposed by the initial waterbag are no longer valid. Thus, the steady state distribution $P(\theta, p)$ may be expressed as

$$P(\theta, p) = \frac{1}{4\Delta\theta\Delta p} \frac{P_\epsilon(\epsilon(\theta, p))}{Q_\epsilon(\epsilon(\theta, p))} \equiv \bar{P}_\epsilon(\epsilon(\theta, p)) \ ,$$

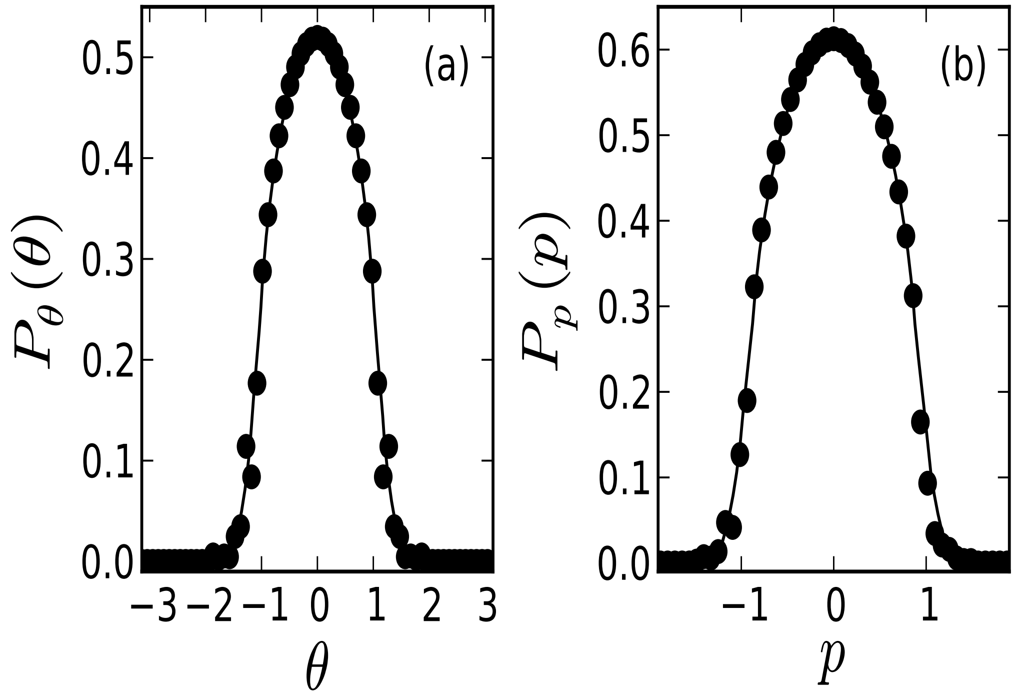
where $Q(\epsilon)$ is given by $P(\epsilon)$ with the bounds given by the waterbag removed. Integrating over p, it is straightforward to express, without any approximation, the marginal in θ as

$$P_\theta(\theta, H) = \sqrt{2} \int_{-H\cos\theta}^{\infty} d\epsilon \ \frac{1}{\sqrt{(\epsilon + H\cos\theta)}} \ \bar{P}_\epsilon(\epsilon) \ .$$

and then impose the *consistency* relation

$$H = \int_{-\pi}^{+\pi} d\theta P_\theta(\theta, H) \cos\theta \ .$$

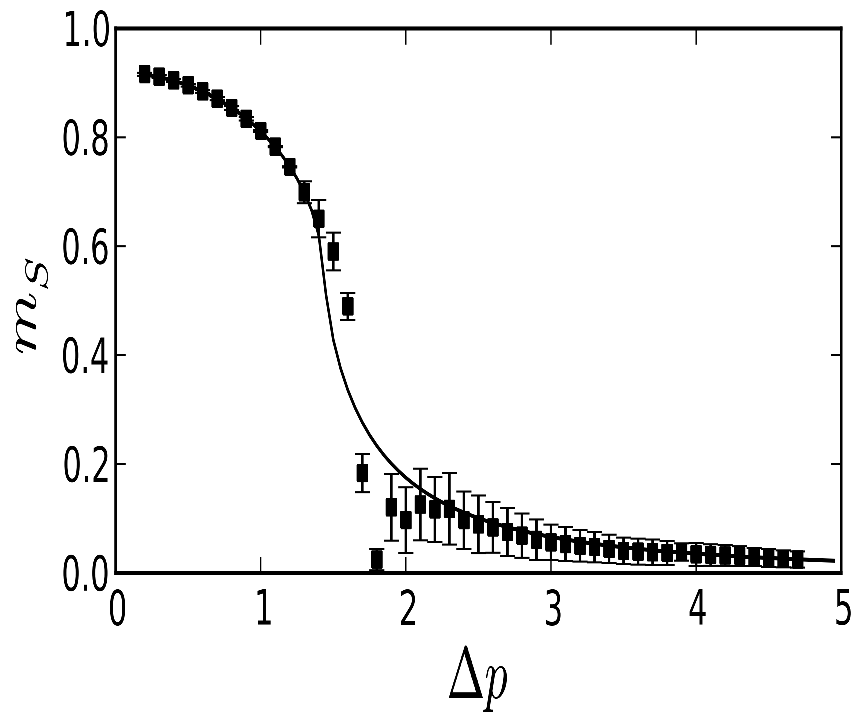
Marginals



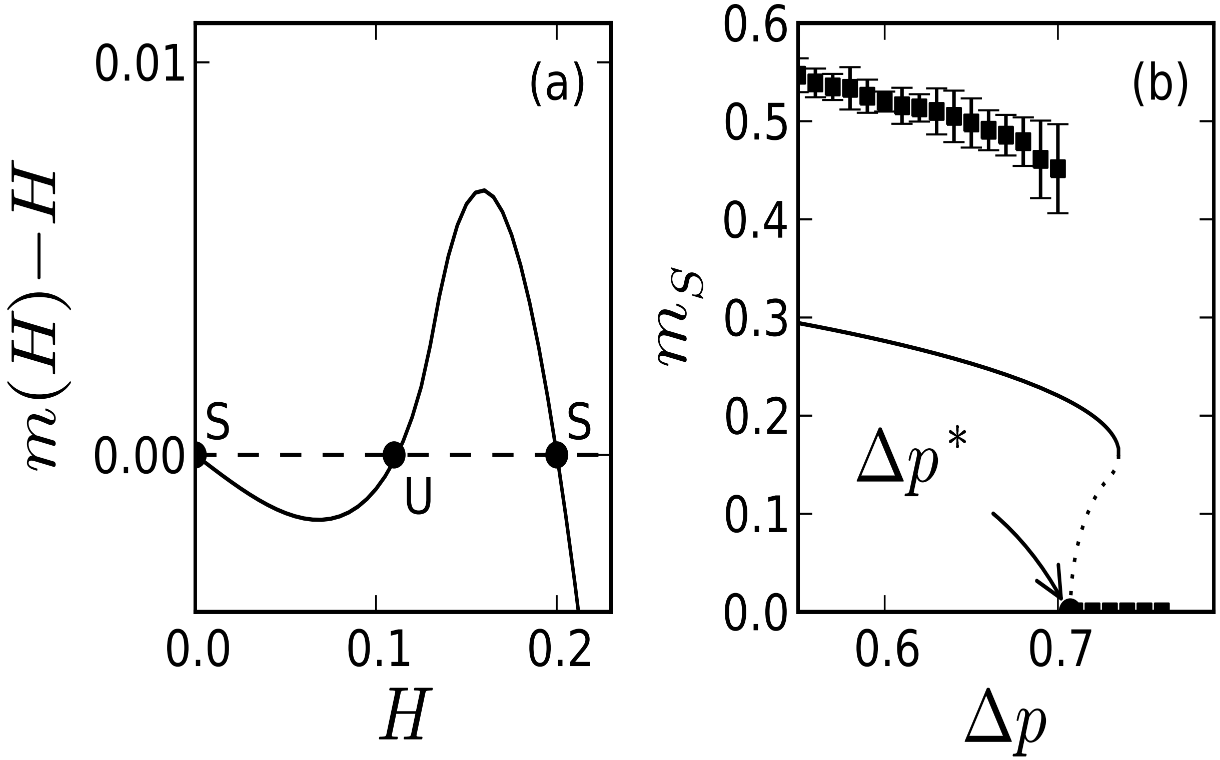
Marginals in θ and in p of the steady state distribution for $\Delta\theta = 1$ and $\Delta p = 1$.

Results

Unless $\Delta\theta = \pi$, the r.h.s of the consistency relations is proportional to $\sqrt{H}$. This implies that there is always a tail in magnetization at large values of Δp. Here, $\Delta\theta = 1$.



$\Delta\theta = \pi$



Generic homogeneous distribution

$$f_0(\theta, p) = \frac{\phi_0(p)}{2\pi}$$

$$P_{QSS}(\epsilon) = \frac{1}{2\pi} \frac{\int d\theta' \phi_0(\sqrt{2(H\cos\theta' + \epsilon)})(H\cos\theta' + \epsilon)^{-1/2}}{\int d\theta' (H\cos\theta' + \epsilon)^{-1/2}}$$

Formally expanding around $H = 0$

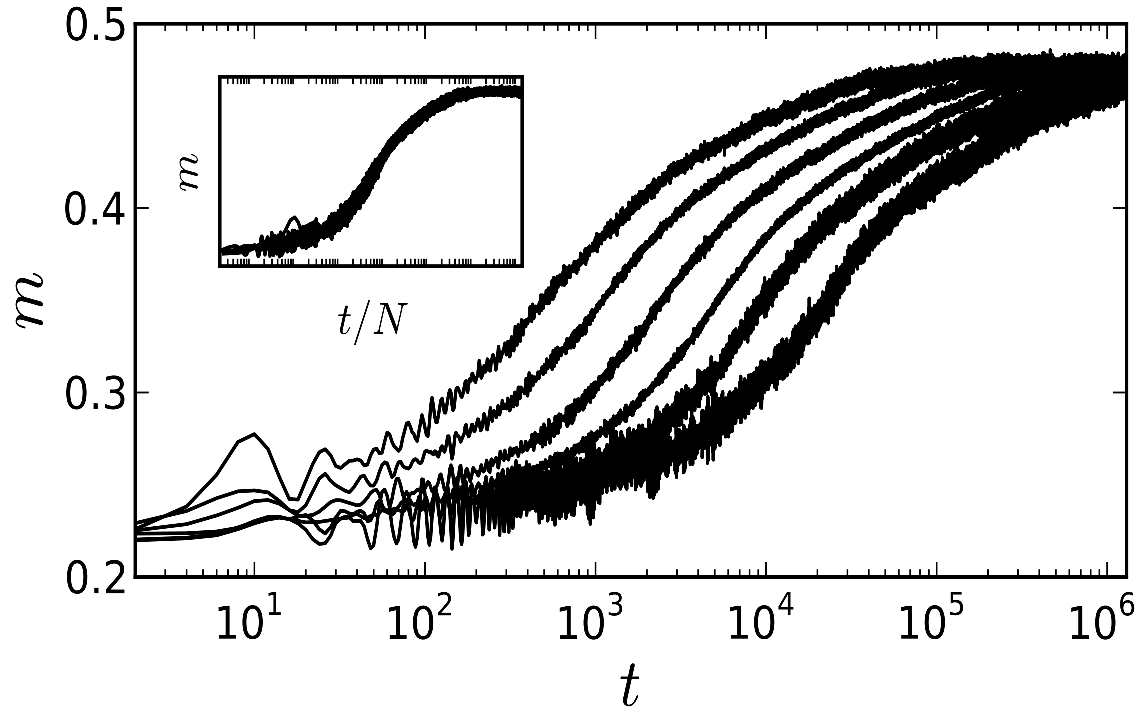
$$P_{QSS}(\theta, p) = \frac{\phi_0(p)}{2\pi} - \frac{\phi_0^{'}(p)\cos\theta}{2\pi p} H + \mathrm{O}(H^2)$$

$$H = \int d\theta dp P_{QSS}(\theta, p)\cos\theta = \int d\theta dp \frac{\phi_0(p)}{2\pi}\cos\theta - \int d\theta dp \frac{\phi_0^{'}\cos^2\theta}{2\pi p} H + O(H^2)$$

which gives

$$1 + \frac{1}{2}\int dp \frac{\phi_0^{'}(p)}{p} = 0$$

Relaxation time scales



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