The 9th Summer Institute for Theoretical Physics

Introduction to Cosmology

Excercises

(Mostly taken from Ch. 2, 3 and 6 of the textbook "Modern Cosmology", Scott Dodelson (Academic Press, 2003))

1. Using the action principle, derive the Einstein equation from the Einstein-Hilbert action

$$S_{\rm EH} = \int d^4x\sqrt{-g}(\kappa^{-2}R + \mathcal{L}_M),$$

where R is the Ricci scalar and $\mathcal{L}_M$ is the matter Lagrangian.

2. The three-dimensional sphere (pseudo-sphere) can be embedded in a four-dimensional Euclidean (Lorentzian) space as $\pm w^2 + x^2 + y^2 + z^2 = \pm a^2$ ($a^2 > 0$). Verify that the metric of a three-dimensional space of constant curvature can be written as

$$dl^2 = a^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right),$$

where $k = 0, \pm 1$. Introduce the rescaled radial coordinate $\bar{r}$, defined by

$$r = \frac{\bar{r}}{1 + k\bar{r}^2/4},$$

and show that this metric can then be rewritten in explicitly isotropic form:

$$dl^3 = a^2\,\frac{d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2}{(1 + k\bar{r}^2/4)^2},$$

where

$$\bar{x} = \bar{r}\sin\theta\cos\phi, \quad \bar{y} = \bar{r}\sin\theta\sin\phi, \quad \bar{z} = \bar{r}\cos\theta.$$

3. For the Robertson-Walker metric, verify

$$\Gamma^0_{ij} = \frac{\dot{a}}{a}\tilde{g}_{ij}, \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j, \quad \Gamma^i_{jk} = ??,$$

where $\tilde{g}_{ij}$ is the spatial metric. Also verify that the Ricci tensor and the Ricci scalar are given by

$$R_{00} = -3\frac{\dot{a}}{a}, \quad R_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2k)\tilde{g}_{ij}, R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right),$$

and derive the Friedmann equations.

4. Derive the conservation equation $T_{\mu\nu}{}^{;\nu} = 0$ from the Einstein equation and discuss its physical meaning.

5. Convert the following quantities by inserting the appropriate factors of c, $\hbar$, and k_B:

- $T_0 = 2.725$ K $\rightarrow$ eV
- $\rho_\gamma = \pi^2 T_0^4/15 \rightarrow$ eV^4 and g cm^{-3}
- $1/H_0 \rightarrow$ cm
- $m_P \equiv 1.2 \times 10^{19}$ GeV $\rightarrow$ K, cm^{-1}, sec^{-1}

6. **(a)** Compute the pressure of a relativistic species in equilibrium with temperature T. Show that $p = \rho/3$ for both Fermi-Dirac and Bose-Einstein statistics.
(b) Suppose the distribution function depends only on E/T as it does in equilibrium. Find dp/dT. A simple way to do this is to rewrite df/dT in the integral as $-(E/T)df/dE$ and then integrate the pressure integral by parts.
(c) Consider the entropy density $s \equiv (\rho+p)/T$. For a massless particle, $p = \rho/3$, so $s = 4\rho/3T$. Express s as a function of T for both bosons and fermions (assumed massless) in equilibrium with zero chemical potential. Show that the entropy density for a massive particle in equilibrium ($T \ll m$; $\mu = 0$) is exponentially small.

7. An important parameter for CMB anisotropies is the sound speed at decoupling. This is determined by the ratio of baryons to photons.
(a) Find

$$R \equiv \frac{3\rho_b}{4\rho_\gamma}$$

as a function of a. Evaluate it at decoupling. Your answer should depend on $\Omega_b h^2$.
(b) The sound speed of the combined photon/baryon fluid is

$$c_s = \sqrt{\frac{1}{3(1+R)}}.$$

Use your answer from (a) to plot the sound speed at decoupling as a function $\Omega_b h^2$.

8. Consider the evolution of the free electron fraction governed by the equation

$$\frac{dX_e}{dt} = (1 - X_e)\beta - X_e^2 n_b \alpha^{(2)}$$

where the ionization rate is typically denoted

$$\beta \equiv \langle \sigma v \rangle \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T}$$

and the recombination rate

$$\alpha^{(2)} \equiv \langle \sigma v \rangle = 9.78 \frac{\alpha^2}{m_e^2} \left(\frac{\epsilon_0}{T}\right)^{1/2} \ln\left(\frac{\epsilon_0}{T}\right).$$

$\epsilon_0 = 13.6$ eV is the binding energy of hydrogen atom. Throughout, take parameters $\Omega_m = 1$, $\Omega_b = 0.06$, $h = 0.5$.
(a) Using as an evolution variable $x \equiv \epsilon_0/T$ instead of time, rewrite the equation in terms of

x and the Hubble rate at $T = \epsilon_0$.
(b) Find the final freeze-out abundance of the free electron fraction, $X_e(x = \infty)$.
(c) Numerically integrate the equation from **(a)** from $x = 1$ down to $x = 1000$. What is the final frozen-out X_e?
(d) Peebles argued that even captures to excited states would not be important except for the fraction of times that the $n = 2$ state decays into two photons or expansion redshifts the Lyman alpha photon so that it cannot pump up a ground-state atom. Qualitatively, he multiplied the right-hand side of the equation by the correction factor,

$$C = \frac{\Lambda_\alpha + \Lambda_{2\gamma}}{\Lambda_\alpha + \Lambda_{2\gamma} + \beta^{(2)}}$$

where the two-photon decay rate is $\Lambda_{2\gamma} = 8.227\ rmsec^{-1}$; Lyman alpha production is $\beta^{(2)} = \beta e^{3\epsilon_0/4T}$; and

$$\Lambda_\alpha = \frac{H(3\epsilon)^3}{(8\pi)^2}.$$

Do this and show how it changes your final answer. Now compare the freeze-out abundance with the result of **(c)** and plot the evolution.

9. Suppose that there were no baryon asymmetry so that the number density of baryons exactly equaled that of anti-baryons. Determine the final relic density of (baryons + anti-baryons). At what temperature is this asymptotic value reached?

10. There is a fundamental limitation on the annihilation cross section of a particle with mass m. Because of unitarity, $\langle \sigma v \rangle$ must be less than or equal to $1/m^2$, give or take a factor of order unity. Determine Ω_X for a particle which saturates this bound, i.e., for a particle with $\langle \sigma v \rangle = 1/m^2$. For what value of m is Ω_X equal to 1? (Keep x_f and g_* equal to the nominal values $x_f \sim 10$ and $g_* \sim 100$.)

11. Inflation also solves the *flatness* problem. This is the question of why the energy density today is so close to critical.
(a) Suppose that

$$\Omega(t) \equiv \frac{8\pi G\rho(t)}{3H^2(t)}$$

is equal to 0.3 today, where ρ counts the energy density in matter and radiation (assumed zero cosmological constant). From the Friedmann equation, plot $\Omega(t) - 1$ as a function of the scale factor. How close to one would $\Omega(t)$ have been back at the Planck epoch (assuming no inflation took place so that the scale factor at the Planck epoch was of order 10^{-32})? This fine-tuning of the initial conditions is the flatness problem. If not for the fine-tuning, an open universe would be *obviously* open (i.e., Ω would be almost exactly zero today.
(b) Now show that inflation solves the flatness problem. Extrapolate $\Omega(t) - 1$ back to the end of inflation, and then through 60 e-folds of inflation. What is $\Omega(t) - 1$ right before these 60 e-folds of inflation?

12. Consider a free, homogeneous scalar field ϕ with mass m. The potential for this field is $V = m^2\phi^2/2$. Show that, if $m \gg H$, the scalar field oscillates with frequency equal to its mass. Also show that its energy density falls off as a^{-3}, so it behaves exactly like ordinary non-relativistic matter.

13. Determine the predictions of an inflationary model with a quartic potential,

$$V(\phi) = \lambda\phi^4.$$

(a) Compute the slow roll parameters ϵ and δ in terms of ϕ.
(b) Determine ϕ_e, the value of the field at which inflation ends, by setting $\epsilon = 1$ at the end of inflation.
(c) To determine the spectrum, you will need to evaluate ϵ and δ at $-k\eta = 1$. Choose the wavenumber k to be equal a_0H_0, roughly the horizon today. Show that the requirement $-k\eta = 1$ then corresponds to

$$e^{60} = \int_0^N dN' \frac{e^{N'}}{(H(N')/H_e)}$$

where H_e is the Hubble rate at the end of inflation, and N is defined to be the number of e-folds before the end of inflation:

$$N \equiv \ln\left(\frac{a_e}{a}\right).$$

(d) Take the Hubble rate to be a constant in the above with H/H_e equal to 1. This implies that $N \simeq 60$. Turn this into an expression for ϕ. The simplest way to do this is to note that $N = \int_t^{t_e} dt' H(t')$ and assume that H is dominated by potential energy. Show that this mode leaves the horizon when $\phi^2 = 60m_P^2/\pi$.
(e) Determine the predicted values of n and n_T.
(f) Estimate the scalar amplitude in terms of λ. As a rough estimate, assume that $k^3P_\Phi(k)$ for this mode is equal to 10^{-8}. What value does this imply for λ?

Short Term Project: The Redshift - Luminosity Distance Relation

The best-known way to trace the evolution of the universe observationally is to look into the redshift - luminosity distance relation [1, 2]. The well-measured quantity of a far distant object is the redshift of light it emitted due to the expansion of the universe. The redshift z is related to the scale factor a by

$$\frac{\lambda_0}{\lambda} \equiv 1 + z = \frac{a_0}{a}.$$

From now on, the quantity with the subscript 0 means the value at present. Another important observational quantity is the distance to the object. There are several ways of measuring distances in the expanding universe. The luminosity distance d_L is defined by the relation

$$d_L^2 \equiv \frac{L}{4\pi F},$$

where L is the luminosity of the object and F is the measured flux from the object. For the object whose luminosity is know in some way, we can determine its luminosity distance from the measured flux.

What you will do in this project is to derive the relation between the redshift and the luminosity distance in a few cosmological models and compare it with the data obtained from the observations of type Ia supernovae.

The expanding universe is described by the FRW metic

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right],$$

where $K = 0, \pm 1$ depending on the spatial curvature of the universe.

(1) Show that the measured flux at the origin from the object of luminosity L located at $r = r_1$ is given by

$$F = \frac{L}{4\pi (a_0 r_1)^2 (1+z)^2},$$

thus the luminosity distance to the object is $d_L = a_0 r_1 (1 + z)$. Consider why we have two factors of $(1 + z)$ in the numerator.

(2) r_1 is a function of the time t at which the light we see today was emitted by the object. From the fact that the light travels satisfying $ds^2 = 0$, derive

$$r_1 = f_K(z) \equiv \begin{pmatrix} \sin f(z), & \text{for } K = +1, \\ f(z), & \text{for } K = 0, \\ \sinh f(z), & \text{for } K = -1, \end{pmatrix}$$

where

$$f(z) = \frac{1}{a_0 H_0} \int_0^z \frac{dz'}{h(z')},$$

with the Hubble parameter $H = \dot{a}/a$ and $h(z) = H(z)/H_0$.

(3) The scale factor $a(t)$ satisfies the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3M_P^2}\sum_i \rho_i,$$

where ρ_i is the energy density of each component that fills the universe. Assume that the i-th component has the the equation of state $p_i = w_i \rho_i$ where w_i is a constant. When $w_i = 1/3,\ 0,\ -1$, it is called Radiation($i = R$), Matter($i = M$), and Cosmological Constant($i = \Lambda$), respectively. Then the energy density evolves as

$$\rho_i = \rho_{i0}\left(\frac{a}{a_0}\right)^{-3(1+w_i)}.$$

The Friedmann equation is rewritten as

$$H^2 = H_0^2\left[\Omega_K(1+z)^2 + \sum_i \Omega_i(1+z)^{3(1+w_i)}\right],$$

where $\Omega_i \equiv \rho_i/3M_P^2H_0^2$ and $\Omega_K = 1 - \sum_i \Omega_i$. Using this equation, find the expression for the luminosity distance $d_L = a_0(1+z)f_K(z)$ as a function of the redshift z.

(4) For simplicity, we consider the flat universe ($K = 0$), filled with Matter and Cosmological Constant. Note that $\Omega_M + \Omega_\Lambda = 1$ in this case. Develop the Mathematica code which does the integration and using it, draw $d_L(z)$ as a function of z for the cases $\Omega_\Lambda = 0,\ 0.3,\ 1$, respectively.

(5) The type Ia supernovae are so bright that they can be observed at very high redshifts. They have roughly a common luminosity independent of the redshift which is well calibrated by their light curves. Hence they are very good standard candles, which can be used to measure luminosity distances. Using the data given in Table 6 of Ref. [3], draw the figure like Figure 1 in which the predictions of cosmological models and the observational data are compared. Note that the luminosity distance data are given as distance moduli

$$\mu_0 = m - M = 5\log\left(\frac{d_L}{\text{Mpc}}\right) + 25,$$

where apparent magnitude m and absolute magnitude M are logarithmic measure of flux and luminosity, respectively.

References

[1] A. R. Liddle and D. H. Lyth, "Cosmological Inflation and Large-Scale Structure", Cambridge University Press (2000).

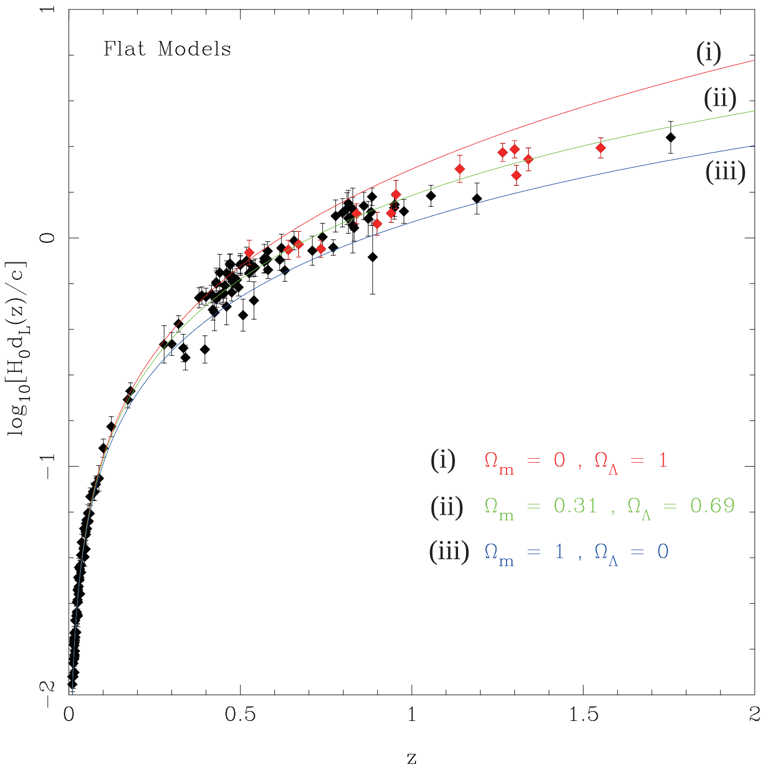

Figure 1: The luminosity distance $H_0 d_L$ versus the redshift z for a flat cosmological model, compared with the observational data. Taken from Ref. [4]

[2] E. J. Copeland, M. Sami and S. Tsujikawa, "Dynamics of dark energy," arXiv:hep-th/0603057.

[3] A. G. Riess *et al.*, "New Hubble Space Telescope Discoveries of Type Ia Supernovae at $z > 1$: Narrowing Constraints on the Early Behavior of Dark Energy," arXiv:astro-ph/0611572.

[4] T. R. Choudhury and T. Padmanabhan, Astron. Astrophys. **429**, 807 (2005) [arXiv:astro-ph/0311622].