Chern-Simons Theory and Its Applications

The 10th Summer Institute for Theoretical Physics

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Maxwell Theory

★ Maxwell Theory:

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

★ Gauge Transformation and Invariance

$$A_\mu \Rightarrow A_\mu + \partial_\mu \Lambda, \quad \partial_\mu J^\mu = 0$$

★ Gauss Law

$$-\partial_i \left(\frac{1}{g^2} F_{i0} \right) - J^0 = 0$$

★ Charge

$$Q = \int d^3x J^0 = -\int d\Sigma^i_{S^2_\infty} \frac{1}{g^2} F_{i0}$$

★ Degrees of Freedom: In 4-dimension, two helicity ± 1

★ In 3-dim, one massless scalar $\frac{1}{g^2} F_{\mu\nu} = \epsilon_{\mu\nu\rho} \partial^\rho \varphi$

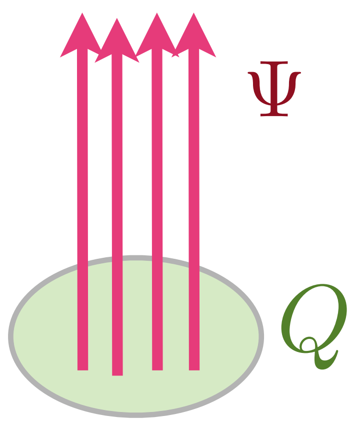
Chern-Simons Term

★ Chern-Simons Theory:

$$\mathcal{L} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu J^\mu$$

★ Gauge Transformation and Invariance: Total derivative

$$\delta\mathcal{L} = \frac{\kappa}{4}\epsilon^{\mu\nu\rho}\partial_\mu(\Lambda F_{\nu\rho})$$



★ Field Equation

$$F_{i0} = -\frac{1}{\kappa}\epsilon_{ij}J^j \qquad F_{12} = \frac{1}{\kappa}J^0$$

★ Gauss Law: Flux-Charge Composite

$$\Psi = \int d^2x F_{12} = \frac{1}{\kappa}Q$$

★ Without Current, the Gauge Field Becomes Pure Gauge $F_{\mu\nu} = 0$

★ Topological Theory: Independent of Metric

$$S = \frac{\kappa}{2}\int d^3\sqrt{g}\frac{1}{\sqrt{g}}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho = \frac{\kappa}{2}\int A\wedge dA$$

Topics to be covered

- ★ Anyons: Particles of Fractional Spin and Statistics
- ★ Maxwell-Chern-Simons -(Higgs)Theory
- ★ Non-Abelian Chern-Simons Theory
- ★ Quantization
- ★ Massive Fermion
- ★ Induced Chern-Simons Term
- ★ Knot Invariants, Jons Polynomial
- ★ Matter Coupling
- ★ Supersymmetry and Superconformal Symmetry
- ★ 2+1 dimensional Gravity and Chern-Simons Theory
- ★ Aharony-Bergman-Jafferis-Maldacena Theory of M2 Branes
- ★ Quantum Hall Effect

Anyons

★ Identical Charged Particles with Conserved Current:

$$J^\mu(x) = e\sum_a (1, \dot{\mathbf{q}}_a)\delta^2(\mathbf{x} - \mathbf{q}_a)$$

★ The Lagrangian for these Particles is

$$L = \frac{m}{2}\sum_a \dot{\mathbf{q}}_a^2 + \int d^2x\left[\frac{k}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu J^\mu\right]$$

★ Field Equation: $F_{12} = \frac{e}{\kappa}\sum_a \delta(\mathbf{x} - \mathbf{q}_a)$

★ Use the relation $\epsilon_{ij}\partial_i\partial_j \mathrm{Arg}(\mathbf{x}) = 2\pi\delta^2(\mathbf{x})$

★ Pure Gauge (Singular)

$$A_\mu = \frac{e}{2\pi\kappa}\partial_\mu \mathrm{Arg}\delta^2(\mathbf{x} - \mathbf{q}_a)$$

★ Momentum and Hamiltonian

$$\mathbf{p}_a = m\dot{\mathbf{q}}_a - e\mathbf{A}(\mathbf{x} = \mathbf{q}_a, t) \qquad \mathcal{H} = \frac{1}{2m}|\mathbf{p} + e\mathbf{A}(\mathbf{q}_a)|^2$$

Anyon (2)

- Two Body & Center of Mass and Relative Positions

$$\mathbf{R} = \frac{1}{2}(\mathbf{q}_1 + \mathbf{q}_2), \quad \mathbf{r} = \mathbf{q}_1 - \mathbf{q}_2$$

- Ignore Self-Interaction and the Lagrangian becomes

$$\mathcal{L} = \frac{M}{2}\dot{\mathbf{R}}^2 + \frac{\mu}{2}\dot{\mathbf{r}}^2 - \frac{e}{2\pi\kappa}\dot{\mathbf{r}} \cdot \nabla \mathrm{Arg}(\mathbf{r})$$

- The Relative Hamiltonian

$$\mathcal{H} = \frac{1}{2\mu}(\mathbf{p} + \frac{e}{2\pi\kappa}\nabla \mathrm{Arg}(\mathbf{r}))^2$$

- Quantize and Consider the Energy Eigen States

$$\mathbf{p} = -i\nabla, \quad \mathcal{H}\Psi(\mathbf{r}) = E\Psi(\mathbf{r})$$

- Assume the original particles are bosons, and so the wave function is symmetric under the exchange.

$$\Psi(\mathbf{r}) = \Psi(-\mathbf{r}) \qquad \Psi(r,\theta) = f(r)e^{2in\theta},$$

Anyon(3)

- ★ Make a Singular Gauge Transformation

$$\Psi(\mathbf{r}) = e^{-\frac{ie}{2\pi\kappa}\mathrm{Arg}(\mathbf{r})}\Psi_A(\mathbf{r}) \qquad \Psi_A(r,\theta) = f(r)e^{i(2n+\frac{e}{2\pi\kappa})\theta}$$

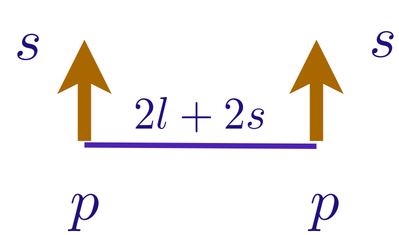
- ★ The Hamiltonian becomes free.

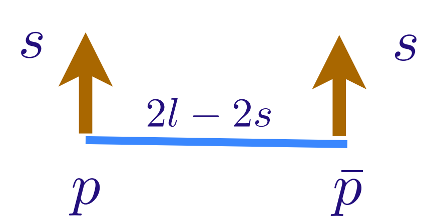
$$\mathcal{H}_{\mathrm{free}}\Psi_A(\mathbf{r}) = E\Psi_A(\mathbf{r}), \quad \mathcal{H}_{\mathrm{free}} = \frac{\mathbf{p}^2}{2\mu} = \frac{1}{2\mu}(p_r^2 + \frac{L^2}{r^2})$$

- ★ the Orbital Angular Momentum

$$L_{free} = -i\partial_\theta = 2n + \frac{e}{2\pi\kappa}$$

- ★ Spin-Statistics: Relativity + Positive Energy (Fiertz, Pauli)





$$s = \frac{e}{4\pi\kappa}$$

- ★ Exchange Statics of Anyons: $\Psi_A(r,\theta+\pi) = e^{i2\pi s}\Psi_A(r,\theta)$

New Kinds of Particles in 2+1 dim

- ★ 3-dim Rotation SO(3)=SU(2): |s,m>

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2} \cdots$$

- ★ 2-dim Rotation U(1): |s>

$$U(\theta)|s>= e^{is\theta}|s>$$

- ★
- ★ First Homotopy Group of Circle
- ★ Braid Group

Maxwell-Chern-Simons + Higgs

- 2+1 dim Maxwell Theory= A Single Massless Scalar Theory

$$\mathcal{L}_M = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} \Leftrightarrow \mathcal{L}_s = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \qquad F_{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\rho\phi$$

- Maxwell-Chern-Simons Theory

$$\mathcal{L}_{MCS} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} - \frac{\kappa}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$

- Field Equation

$$\partial_\sigma F^{\sigma\mu} - \frac{g^2\kappa}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} = 0 \qquad \Rightarrow (\partial_\mu^2 + m^2)F_{\mu\nu} = 0 \; with \; m = \frac{g^2\kappa}{2\pi}$$

- Massive Neutral Vector Boson of Mass $|m|$ and Spin $\frac{m}{|m|} = \frac{\kappa}{|\kappa|}$
- Maxwell + Higgs : $\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + v^2|\partial_\mu\varphi + A_\mu|^2$

- Parity Even and Massive Vector Boson of Mass $|gv|$ and Spin ± 1
- Chern-Simons-Higgs: Mass $\left|\frac{v^2}{\kappa}\right|$ and Spin $-\frac{\kappa}{|\kappa|}$

Non-Abelian Theory

- ★ The Lagrangian

$$\mathcal{L} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}\mathrm{Tr}(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho), \quad D_\mu\Phi = (\partial_\mu + A_\mu)\Phi$$

- ★ Under the gauge transformation $A_\mu \to A^g = \bar{g}A_\mu g + \bar{g}\partial_\mu g$

$$\delta\mathcal{L} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}\partial_\mu\mathrm{Tr}(\partial_\nu g\bar{g}A_\rho) + \frac{\kappa}{6}\epsilon^{\mu\nu\rho}\mathrm{Tr}(\bar{g}\partial_\mu g\bar{g}\partial_\nu g\bar{g}\partial_\rho g)$$

- ★ The third homotopy group of Lie group is interger. $\pi_3(G) = Z$
- ★ The winding number is $N = \frac{i}{24\pi^2}\int d^3x\epsilon^{\mu\nu\rho}\mathrm{Tr}(\bar{g}\partial_\mu g\bar{g}\partial_\nu g\bar{g}\partial_\rho g) \in Z$

$$S_{CS} \to S_{CS} + 2\pi N(2\pi\kappa) \qquad e^{iS_{CS}} \to e^{iS_{CS}+2\pi i N(2\pi\kappa)}$$

- ★ The Path Integral is gauge invariant and well defined if

$$2\pi\kappa = k = \text{integer}$$

- ★ The Chern-Simons Level is the integer k

Simple Exercise for SU(2) Case

★ SU(2) = Three Sphere

$$g = x_4 + i\sigma_i x_i, \quad \bar{g}g = 1, \mathrm{det} g = 1 \Rightarrow x_\mu \in R, \sum_\mu x_\mu^2 = 1$$

★ Near the Pole $x_0 = 0,$

$$\mathrm{Tr}(\bar{g}dg)^3 = i\mathrm{Tr}(\sigma_i dx_i)^3 = 12 dx \wedge dy \wedge dz$$

★ Then as $\text{Volume of } S^3 = 2\pi^2$

$$N = \frac{1}{24\pi^2} 12 \int_{S^3} dx \wedge dy \wedge dz = 1$$

Quantization of Maxwell-Chern-Simons

- ★ Lagrangian $$\mathcal{L} = \frac{1}{2g^2}F_{0i}^2 - \frac{1}{2e^2}F_{12}^2 + \frac{\kappa}{2}\epsilon^{ij}\dot{A}_i A_j + \kappa A_0 F_{12}$$
- ★ Momenta and Gauss Law

$$\Pi_i = \frac{1}{e^2}F_{0i} + \frac{\kappa}{2}\epsilon^{ij}A_j \qquad \partial_i \Pi_i + \frac{\kappa}{2}F_{12} = 0$$

- ★ Hamiltonian

$$\mathcal{H} = \Pi_i \dot{A}_i - \mathcal{L} = \frac{e^2}{2}(\Pi_i - \frac{\kappa}{2}\epsilon_{ij}A_j)^2 + \frac{1}{2e^2}F_{12}^2 + A_0(\partial_i \Pi_i + \frac{\kappa}{2}F_{12})$$

- ★ Quantization

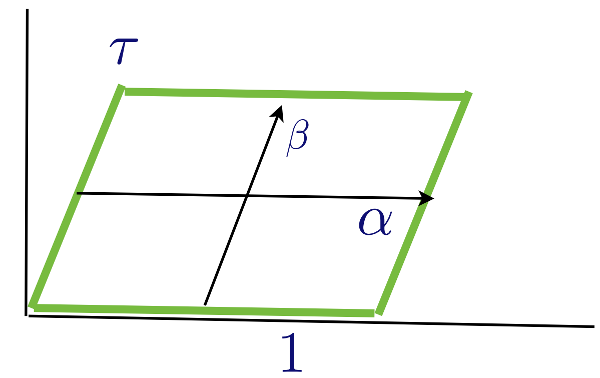
$$[A_i(\mathbf{x},t), \Pi_j(\mathbf{y},t)] = i\delta_{ij}\delta^2\mathbf{x} - \mathbf{y}) \qquad \Pi_i(\mathbf{x}) = -i\frac{\delta}{\delta A_i(\mathbf{x})}$$

- ★ The Wave Function $\mathit{\Psi}(A_1, A_2)$ $\quad (\partial_i \Pi_i + \frac{\kappa}{2}F_{12})\mathit{\Psi} = 0$
- ★ Decompose $A_i = \partial_i \lambda + A_i^T,\ \partial_i A_i^T = 0,\ \lambda = \nabla^{-2}\partial_i A_i$

$$\mathit{\Psi} = e^{-\frac{i\kappa}{2}\int d^2x \lambda F_{12}}\,\mathit{\Phi}(A_i^T), \quad \mathit{\Phi}(A_i^T + \partial_i \lambda) = \mathit{\Phi}(A_i^T)$$

Quantization of Pure Chern-Simons

- ★ The Commutation Relation $[A_1(\mathbf{x}), A_2(\mathbf{y})]_{ET} = \frac{i}{k}\delta^2(\mathbf{x}-\mathbf{y})$
- ★ Wave Function $\Psi(A_1)$
- ★ Complex or holomorphic coordinate $z = x^1 + ix^2$ $\quad A = \frac{1}{2}(A_1 + iA_2)$
- ★ Composition $A_i = \partial_i\omega + \epsilon_{ij}\partial_j\sigma \Rightarrow A = \partial_{\bar{z}}\chi, \quad \chi = \omega - i\sigma$
- ★ CS Theory on a Complex Riemann Surface Σ of ginus g
- ★ On torus with side 1 and τ and area Im(τ)
- ★ Nontrivial cohomology on torus $\omega(z)$



$$\int_\alpha \omega = 1, \int_\beta \omega = \tau, \int_{torus} |\omega|^2 = \mathrm{Im}(\tau)$$

- ★ The gauge field is the sum of exact and closed 1-form

$$A = \partial_{\bar{z}}\chi(z,\bar{z},t) + \frac{i\pi a(t)}{\mathrm{Im}(\tau)}\bar{\omega}$$

Massive (Majorana) Fermions

- ★ Gamma Matrices $\gamma^0 = \sigma_2, \gamma^1 = i\sigma_1, \gamma^2 = i\sigma_3$
- ★ Two Component Spinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$
- ★ Lagrangian and Field Equation

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \qquad i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

- ★ Majorana condition $\psi = \psi^*$
- ★ Quantization and Elementary Particles
- ★ Neutral Particles of Mass m and Spin $\frac{1}{2}\mathrm{sign}(m)$
- ★ The Mass Term is both P and CP violating
- ★ Dirac Fermions of Charge ± 1
- ★ Related to Massive Vector Boson of Maxwell-Chern-Simons by Supersymmetry and Quantum Corrections

Induced Chern-Simons Term

★ Gauged Charged Dirac Particles $\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$

★ Effective Action

$$e^{iS_{eff}(A)} = \int d\psi d\bar{\psi} e^{i\mathcal{L}}$$

★ Calculation

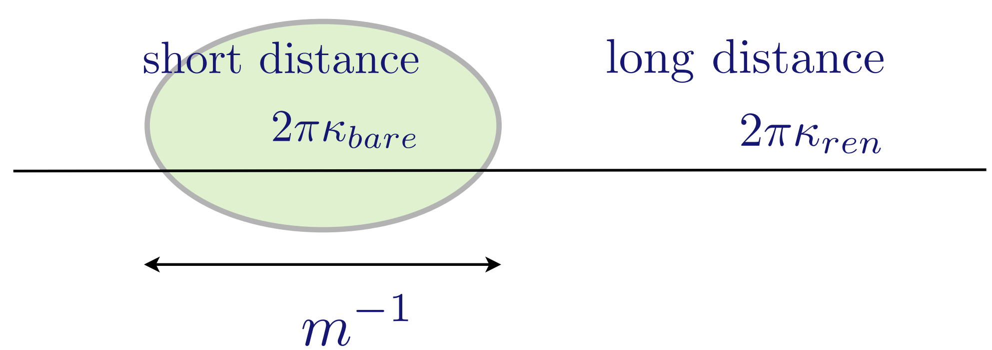
$$\begin{aligned} S_{eff}(A) &= N_f \mathrm{tr} \ln(i\partial\!\!\!/ + A\!\!\!/ - m) \\ &= N_f \mathrm{tr} \ln(i\partial\!\!\!/ - m) + N_f \mathrm{tr}\left(\frac{1}{i\partial\!\!\!/ - m} A\!\!\!/\right) \\ &\quad + N_f \mathrm{tr}\left(\frac{1}{i\partial\!\!\!/ - m} A\!\!\!/ \frac{1}{i\partial\!\!\!/ - m} A\!\!\!/\right) + \cdots \end{aligned}$$

★ Long Wave Length

$$S_{eff}(A) \approx \frac{iN_f}{2} \frac{m}{|m|} \frac{1}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

★ Renormalization of Chern-Simons

$$2\pi\kappa_{ren} = 2\pi\kappa_{bare} + \frac{N_f}{2} \frac{m}{|m|}$$



Supersymmetric Yang-Mills Chern-Simons

- ★ Yang-Mills-Chern-Simons:

$$\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho)$$

- ★ Mass $m = \frac{kg^2}{2\pi}$ Spin $\frac{k}{|k|}$
- ★ Low Energy Dynamics: $Energy << |m|$ => Pure Chern-Simons Theory
- ★ Renormalization of k $k_{ren} = k_{bare} + N$
- ★ Supersymmetry and Level Shift

spin	1	1/2	0	-1/2	-1	k_{ren}
N=0	1					k+N
N=1	1	1				k+N/2
N=2	1	2	1			k
N=3	1	3	3	1		k

SUSY YMCS Theory on Torus (Witten)

- Index Calculation on Two Torus of Size L $Z = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H}$
- Small Torus Limit and all KK Modes are Ignored. $g^2\kappa << 1/L$
- Low Energy Dynamics: Flat Diagonal Connections=constant

$$A_i = \sum_{a=1}^{r} c_i^a T^a, \quad \lambda_\pm = \sum_{a=1}^{r} \eta_\pm^a T^a,$$

- Supersymmetric Quantum Mechanics on the Vacuum Moduli Space

$$Q_\pm = \int_{T^2} \mathrm{Tr}\lambda_\mp \frac{D}{DA_{\bar{z},z}}, \quad \frac{D}{DA_i^a} = \frac{\delta}{\delta A_i^a(x)} + \frac{ik}{4\pi}\epsilon_{ij}A_J^a$$

- Supersymmetry is Broken when $N=1\ k<N/2, \quad N=2,3\ k<N$
- Vacuum Degeneracy for Other Cases..

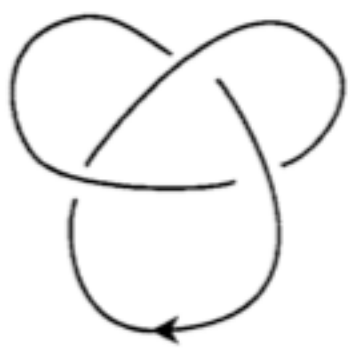
$$I(k) = \frac{(N+t-1)!}{t!(N-1)!}, \ t = k - \frac{N}{2} \text{ or } t = k - N$$

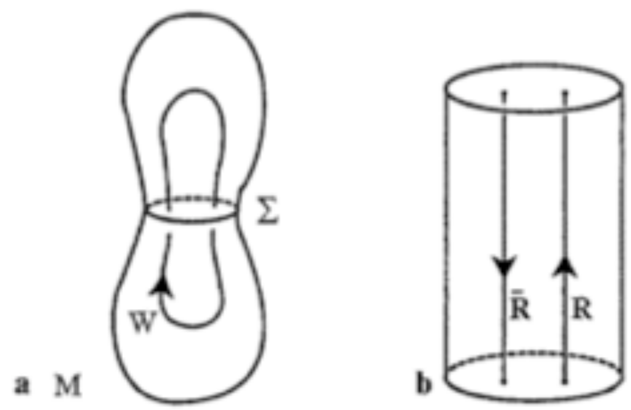
Knot and Jons Polynomial (Witten)

- The Partition Function

$$Z = \int [dA_\mu] e^{iS_{CS}} \prod_{i=1}^{r} W_{R_i}(C_i)$$

- Wilson Loop $\quad W_{R_i}(C_i) = \mathrm{Tr}_{R_i} P \exp(i \int dx^\mu A_\mu^a T_{R_i}^a)$





- Quantize on Space-Time $\quad \Sigma \times R$
- On Riemann Surface with Marked Points

$$\Sigma, \quad F_{12} = \sum_i T_{R_i} \delta^2(\mathbf{x} - \mathbf{q}_i)$$

- Conformal Blocks, Quantum-Groups,....

2+1 Dimensional Gravity (1) (Witten)

- ★ d-dim Einstein-Hilbert Action $I = \frac{1}{2}\int d^dx\sqrt{g}R$
- ★ Vierbein $e^a = e^a_\mu dx^\mu$ and Spin Connection $w^a{}_b = w^a{}_{b\mu}dx^\mu$
- ★ Relations $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ $de^a + w^a{}_b \wedge e^b = 0$
- ★ Curvature Tensor $R_{\mu\nu}{}^a{}_b = \partial_\mu w^a_{b\nu} - \partial_\nu w^a{}_{b\mu} + [w_\mu, w_\nu]^a{}_b$
- ★ 4-dim $I_4 = \frac{1}{2}\int d^4x\epsilon^{\mu\nu\rho}\epsilon_{abcd}e^a_\mu e^b_\nu R^{cd}{}_{\rho\sigma}$
- ★ 3-dim with Cosmological Constant

$$I_3 = \frac{1}{2}\int d^3x\epsilon^{\mu\nu\rho}\epsilon_{abc}\Big(e^a_\mu(\partial_\nu w^{bc}_\rho - \partial_\rho w^{bc}_\nu + [w_\nu, w_\rho]^{bc} + \lambda e^a_\mu e^b_\nu e^c_\rho\Big)$$

- ★ Anti-deSitter Space is the Solution with Isometry SO(2,2)
- ★ Generators of $J_a, P_a, \quad a = 1, 2, 3$ $J^{ab} = \epsilon^{abc}J_c$

$$[J_a, J_b] = \epsilon_{abc}J^c,\ [J_a, P_b] = \epsilon_{abc}P_c,\ [P_a, P_b] = \lambda\epsilon_{abc}J^c$$

- ★ Invariant quadratic form

$$\langle J_a, P_b\rangle = \delta_{ab},\ \langle J_a, J_b\rangle = 0 = \langle P_a, P_b\rangle$$

2+1 Dim Gravity (2)

- ★ Gauge Connection $w^a_\mu = \frac{1}{2}\epsilon^{abc} w_{bc\mu}$

$$A_\mu = e^A_\mu P_a + w^a_\mu J_a$$

- ★ Gauge Transformation =infinitesimal diffeomorphism and Lorentz transformation modulo Einstein equation.

$$\delta A_\mu = D_\mu u = \partial_\mu + [A_\mu u]$$

- ★ Zero Field Strength $F_{\mu\nu} = 0$ is identical to Einstein Eq..
- ★ Note $SO(2,2) = SL(2,R) \times SL(2,R)$
- ★ Decompose $J^\pm_a = \frac{1}{2}(J_a \pm \frac{1}{\sqrt{\lambda}} P_a)$ $A^{a\pm}_\mu = w^a_\mu \pm \sqrt{\lambda} e^a_\mu$
- ★ Action $I^\pm = \int \epsilon^{\mu\nu\rho}(2A^{a\pm}_\mu \partial_\nu A^{a\pm}_\rho + \frac{2}{3}\epsilon_{abc} A^{a\pm}_\mu A^{b\pm}_\nu A^{c\pm}_\rho)$
- ★ Einstein + lambda= $\frac{1}{2}(I^+ - I^-)$
- ★ 3-dim Graviational Chern-Simons = $\frac{1}{2}(I^+ + I^-) \sim \int w \wedge dw$

Supersymmetric CS Theory(1)

- ★ N=2 Chern-Simons Higgs $V, \Phi, \Sigma = \bar{D}^\alpha D_\alpha V$

$$S_{Ab}^{\mathcal{N}=2} = \int d^3x \int d^4\theta \left(\frac{k}{4\pi} V\Sigma + \bar{\Phi} e^V \Phi \right)$$

- ★ Nonabelian $S^{\mathcal{N}=2} = \int d^3x \int d^4\theta \left\{ \frac{k}{2\pi} \int_0^1 dt \mathrm{Tr}\left[V \bar{D}^\alpha (e^{-tV} D_\alpha e^{tV}) \right] + \bar{\Phi} e^V \Phi \right\}$

- ★ Pure Chern-Simons $S_{CS}^{\mathcal{N}=2} = \frac{k}{4\pi} \int \mathrm{Tr}(A \wedge dA + \frac{2}{3} A^3 - \bar{\chi}\chi + 2D\sigma)$

- ★ Matter

$$S_{matter} = \int d^4\theta \sum_{i=1}^{N_f} \bar{\Phi}^i e^V \Phi^i = \int \sum_{i=1}^{N_f} \left(D_\mu \bar{\phi}^i D^\mu \phi^i + i\bar{\psi}^i \gamma^\mu D_\mu \psi^i - \bar{\phi}^i \sigma^2 \phi^i + \bar{\phi}^i D \phi^i \right.$$
$$\left. - \bar{\psi}^i \sigma \psi^i + i\bar{\phi}^i \bar{\chi} \psi^i - i\bar{\psi}^i \chi \phi^i \right).$$

- ★ FI Term: $S_{FI} = \int d^3x \int d^4\theta \frac{k}{4\pi}(-2v^2)V = -\int d^3x \frac{k}{2\pi} v^2 D$

Supersymmetric CS Theory (2)

- D-Term Eq $\sigma = \frac{2\pi}{k}(\sum_f e_f|\phi_f|^2 - v^2)$
- Potential $U = \sum_f e_f^2|\phi_f|^2\sigma^2$
- N=2 Superconformal Theory when v=0
- N=3 Theory: N=2+ Adjoint Chiral multiplet Φ, hypermultiplet $Q_i, \tilde{Q}_i$
- N=3 Lagrangian

$$S^{\mathcal{N}=2} = S_{CS}^{\mathcal{N}=2} + \int d^4\theta(\overline{Q}e^V Q + \tilde{Q}e^{-V}\overline{\tilde{Q}}) + \left[\int d^2\theta\left(-\frac{k}{4\pi}\mathrm{Tr}\Phi^2 + \tilde{Q}\Phi Q\right) + c.c.\right]$$

- Integration over Adjoint Chiral => Superpotential

$$W = \frac{2\pi}{k}(\tilde{Q}T^a Q)(\tilde{Q}T^a Q).$$

- N=3 $SO(3)_R$ R-Symmetry, Superconformal Symmetry,
- Susy preserving FI Term, Mass-Term breaks Conformal Symmetry

Maxwell-Matter

- ★ The Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - \frac{1}{4}(|\phi|^2 - v^2)^2$$

- ★ Energy Density $E = \frac{1}{2}F_{12}^2 + |D_i\phi|^2 + U$

- ★ BPS Bound

$$E = \frac{1}{2}(F_{12} - (v^2 - |\phi|^2)^2 + |(D_1 + iD_2)\phi|^2 + v^2 F_{12}$$

- ★ Energy is Bounded $\mathcal{H} = \int d^2xE \geq v^2\Psi = v^2 \int d^2F_{12}$

- ★ BPS Equation $(D_1 + iD_2)\phi = 0,\ F_{12} = \frac{1}{2}(v^2 - |\phi|^2)$

- ★ Vortices:

$$\phi = f(r)e^{in\theta}, \quad -\partial_i^2 \ln f^2 = (v^2 - f^2) + 4\pi n\delta^2(x)$$

Abelian CS Matter Theory (1)

- ★ Lagrangian

$$\mathcal{L} = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + D_\mu\bar{\phi}D^\mu\phi - U(|\phi|)$$

- ★ Gauss Law

$$\frac{k}{2\pi}F_{12} - i(\phi D_0\bar{\phi} - D_0\phi\bar{\phi}) = 0$$

- ★ Integration

$$\frac{k}{2\pi}\Psi_{flux} = Q = \int d^2x(i\phi\pi_\phi - i\bar{\phi}\pi_{\bar{\phi}})$$

- ★ Angular Momentum

$$J = -\int d^2x\epsilon_{ij}(D_0\bar{\phi}D_j\phi + D_j\bar{\phi}D_0\phi)$$

- ★ Q-Balls in Symmetric Phase and Vortices in Broken Phase

Abelian CS Matter(2)

★ Potential (N=2 Supersymmetry)

$$U(\phi) = \frac{4\pi^2}{k^2}|\phi|^2(|\phi|^2 - v^2)^2$$

★ Energy Density + Gauss Law

$$E = |D_0\phi \mp \frac{2\pi i}{k}(|\phi|^2 - v^2)|^2 + |(D_1 \mp iD_2)\phi|^2 \pm v^2 F_{12}$$

★ BPS Bound

$$\mathcal{H} = \int d^2x E \geq v^2|\Psi|$$

★ Solutons in Symmetric Phase: Q-balls and Q-balls with vortices

$$J = \frac{Q^2}{2k}$$

★ Solutions in Broken Phase: Vortices

$$\Psi = 2\pi n,\ Q = nk,\ J = -\frac{Q^2}{2k}$$

Nonrelativistic Limit of Symmetric Phase

★ Jackiw-Pi Model $\mathcal{L}_{\rm JP} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + i\psi^* D_0\psi - \frac{1}{2m}|\vec{D}\psi|^2 + \frac{g}{2}|\psi|^4$ $g = \frac{1}{m\kappa}$

★ Gauss Law $F_{12} = \frac{1}{\kappa}|\psi|^2$

★ Energy $\mathcal{E} = \int d^2x \frac{1}{2m}|(D_1 - iD_2)\psi|^2$

★ BPS Equation=Liouville Equation $\nabla^2 \ln\rho = -\frac{2}{\rho}$

$\rho = |\psi|^2$

★ Solution Found Explicitly $\rho = \frac{4\kappa N^2}{r_0^2}\frac{\left(\frac{r}{r_0}\right)^{2(N-1)}}{\left(1+\left(\frac{r}{r_0}\right)^{2N}\right)^2}$

★ Nonrelativistic Conformal Symmetry or Schrodinger Symmetry

On M2,D3,M5 Branes

- ★ Theory on D3 Branes: N=4 Supersymmetric Yang-Mills Theory, SCFT, dual to

$$AdS_5 \times S^5$$

- ★ Theory on M2 Branes: Strong Coupling Limit of N=8 Supersymmetric Yang-Mills Theory, dual to

$$AdS_4 \times S^7$$

- ★ Theory on M5 Branes: 6-dim (2,0) SCFT Theory with SO(5) R-symmetry dual to

$$AdS_7 \times S^4$$

- ★ Abelian (2,0) Theory $B_{\mu\nu}, \Psi, \Phi_I, I = 1, \cdots 5, , \quad H_{\mu\nu\rho} = \tilde{H}_{\mu\nu\rho}$

- ★ 2+1 Conformal Field Theory

$$\mathcal{L} \sim AdA + (D\phi)^2 + \bar{\psi} D\psi + \bar{\psi}\psi|\phi|^2 + |\phi|^6$$

Higher Super Symmetric CS Theories

- ★ Supersymmetric Algebra: Central Term

$$\{Q,Q\} \sim P + Z, \quad [R,Q] \sim Q, \quad [Z,Q] = 0 = [Z,R]$$

- ★ Non-Central Term for d < 4

$$\{Q,Q\} \sim P + R, \quad [R,Q] \sim Q,$$

- ★ Yang-Mills Chern-Simons (N=4 is possible.)
- ★ Bagger-Lambert-Gustavsson Theory: $SU(2)_k$ x $SU(2)_{-k}$ =SO(4), N=8 SO(8) R-symmetry

$$X_I^a, \Psi_A^a, \; a = 1,2,3,4, I = 1,2,\cdots 8, A = 1,\cdots 8_c$$

- ★

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}D_\mu X_I^a D^\mu X_I^a + \frac{i}{2}\bar{\Psi}^a\Gamma^\mu D_\mu \Psi^a \\ &-\frac{i}{4\kappa} f^{abcd}\bar{\Psi}^a \Gamma_{IJ}\Psi^b X_I^c X_J^d - \frac{1}{12\kappa^2}\sum_{a,I,J,K}\left(f^{abcd}X_I^b X_J^c X_K^d\right)^2 \\ &+\frac{\kappa}{2}\epsilon^{\mu\nu\rho}\left(f^{abcd}A_\mu^{ab}\partial_\nu A_\rho^{cd} + \frac{2}{3}f^{acde}f^{bcfg}A_\mu^{ab}A_\nu^{de}A_\rho^{fg}\right),\end{aligned}$$

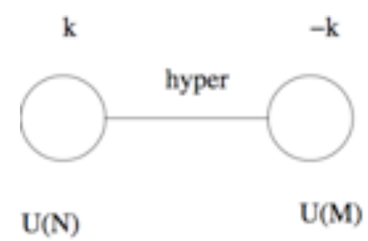
N=4 Supersymmetric Theory

- Gaiotto-Witten: Gauge Field + Hypermultitplet

$$\mathcal{L} = \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\left(k_{mn}A_\mu^m\partial_\nu A_\lambda^n + \frac{1}{3}f_{mnp}A_\mu^m A_\nu^n A_\lambda^p\right) + \frac{1}{2}\omega_{AB}\left(-\epsilon^{\alpha\beta}Dq_\alpha^A Dq_\beta^B + i\epsilon^{\dot\alpha\dot\beta}\psi_{\dot\alpha}^A \not{D}\psi_{\dot\beta}^B\right)$$
$$-i\pi k_{mn}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}j_{\alpha\dot\gamma}^m j_{\beta\dot\delta}^n - \frac{\pi^2}{6}f_{mnp}(\mu^m)^\alpha{}_\beta(\mu^n)^\beta{}_\gamma(\mu^p)^\gamma{}_\alpha, \quad (j_{\alpha\dot\gamma}^m \equiv q_\alpha^A t_{AC}^m \psi_{\dot\gamma}^C)\,. \qquad (2.15)$$

$$\mu_{\alpha\beta}^{mn} \equiv (\omega t^m t^n)_{AB}q_\alpha^A q_\beta^B\,.$$

- Gauge Group U(N)x U(M), O(N)x Sp(2M)
- Linear Quiver



- Adding Twisted Hyper (HLLLP) includes BLG

$$\mathcal{L} = \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\left(k_{mn}A_m^m\partial_\nu A_\lambda^n + \frac{1}{3}f_{mnp}A_\mu^m A_\nu^n A_\lambda^p\right)$$
$$+\frac{1}{2}\omega_{AB}\left(-\epsilon^{\alpha\beta}Dq_\alpha^A Dq_\beta^B + i\epsilon^{\dot\alpha\dot\beta}\psi_{\dot\alpha}^A\not{D}\psi_{\dot\beta}^B\right) + \frac{1}{2}\tilde\omega_{AB}\left(-\epsilon^{\dot\alpha\dot\beta}D\tilde q_{\dot\alpha}^A D\tilde q_{\dot\beta}^B + i\epsilon^{\alpha\beta}\tilde\psi_\alpha^A\not{D}\tilde\psi_\beta^B\right)$$
$$-i\pi k_{mn}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}j_{\alpha\dot\gamma}^m j_{\beta\dot\delta}^n - i\pi k_{mn}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\gamma\delta}\tilde j_{\dot\alpha\gamma}^m \tilde j_{\dot\beta\delta}^n + 4\pi i k_{mn}\epsilon^{\alpha\gamma}\epsilon^{\dot\beta\dot\delta}j_{\alpha\dot\beta}^m \tilde j_{\dot\delta\gamma}^n$$
$$+i\pi k_{mn}\left(\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\delta}\tilde\mu_{\dot\alpha\dot\beta}^m\psi_{\dot\gamma}^A t_{AB}^n\psi_{\dot\delta}^B + \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\mu_{\alpha\beta}^m\tilde\psi_\gamma^A\tilde t_{AB}^n\tilde\psi_\delta^B\right)$$
$$-\frac{\pi^2}{6}f_{mnp}(\mu^m)^\alpha{}_\beta(\mu^n)^\beta{}_\gamma(\mu^p)^\gamma{}_\alpha - \frac{\pi^2}{6}f_{mnp}(\tilde\mu^m)^\alpha{}_\beta(\tilde\mu^n)^\beta{}_\gamma(\tilde\mu^p)^\gamma{}_\alpha$$
$$+\pi^2(\tilde\mu^{mn})^{\dot\gamma}{}_{\dot\gamma}(\mu_m)^\alpha{}_\beta(\mu_n)^\beta{}_\alpha + \pi^2(\mu^{mn})^\gamma{}_\gamma(\tilde\mu_m)^{\dot\alpha}{}_{\dot\beta}(\tilde\mu_n)^{\dot\beta}{}_{\dot\alpha}, \qquad (2.31)$$

ABJM Model

- U(N)k X U(N)-k Chern-Simons Theory N=6 Theory
- Matter $A_\mu, \tilde{A}_\mu, Z_\alpha, \Psi^\alpha$
- U(1)-Global, SU(4) R-Symmetry, $\xi_{\alpha\beta} = (\xi^{\alpha\beta})^* = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\xi^{\gamma\delta}$,
- Lagrangian

$$\mathcal{L}_{\text{Yukawa}} = \frac{2\pi i}{k}\text{Tr}\Big(\bar{Z}^\alpha Z_\alpha \bar{\Psi}_\beta \Psi^\beta - Z_\alpha \bar{Z}^\alpha \Psi^\beta \bar{\Psi}_\beta + 2 Z_\alpha \bar{Z}^\beta \Psi^\alpha \bar{\Psi}_\beta - 2\bar{Z}^\alpha Z_\beta \bar{\Psi}_\alpha \Psi^\beta + \epsilon_{\alpha\beta\gamma\delta}\bar{Z}^\alpha \Psi^\beta \bar{Z}^\gamma \Psi^\delta - \epsilon^{\alpha\beta\gamma\delta} Z_\alpha \bar{\Psi}_\beta Z_\gamma \bar{\Psi}_\delta\Big) . \quad (2.3)$$

$$\mathcal{L}_{\text{potential}} = -U = -\frac{4\pi^2}{3k^2}\text{Tr}\Big(6 Z_\alpha \bar{Z}^\alpha Z_\beta \bar{Z}^\gamma Z_\gamma \bar{Z}^\beta - 4 Z_\alpha \bar{Z}^\beta Z_\gamma \bar{Z}^\alpha Z_\beta \bar{Z}^\gamma - Z_\alpha \bar{Z}^\alpha Z_\beta \bar{Z}^\beta Z_\gamma \bar{Z}^\gamma - Z_\alpha \bar{Z}^\beta Z_\beta \bar{Z}^\gamma Z_\gamma \bar{Z}^\alpha\Big) . \quad (2.4)$$

$$\mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{kin}} = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\text{Tr}\Big(A_\mu \partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho - \tilde{A}_\mu \partial_\nu \tilde{A}_\rho + \frac{2i}{3}\tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho\Big) - \text{Tr}\left(D_\mu \bar{Z}^\alpha D^\mu Z_\alpha + i\bar{\Psi}_\alpha \gamma^\mu D_\mu \Psi^\alpha\right) ,$$

- Vacuum Moduli Space $C^4/Z_k, \quad (C^4/Z_k)^N/S_N$
- k=1,2 => N=8 Enhancement

$U(1)_k$ X $U(1)_{-k}$ Case

- Lagrangian
$$b_\mu = A_\mu - \tilde{A}_\mu, \;\; c_\mu = \frac{1}{2}(A_\mu + \tilde{A}_\mu)$$
$$\mathcal{L} = \frac{k}{2\pi}\epsilon^{\mu\nu\rho} b_\mu \partial_\nu c_\rho + |(\partial_\mu - i b_\mu) Z_\alpha|^2$$

- Auxiliary Field
$$\mathcal{L} = \frac{k}{2\pi}\epsilon^{\mu\nu\rho}(b_\mu + \frac{\sigma}{k}) f_{\nu\rho} + |(\partial_\mu - i b_\mu) Z_\alpha|^2$$

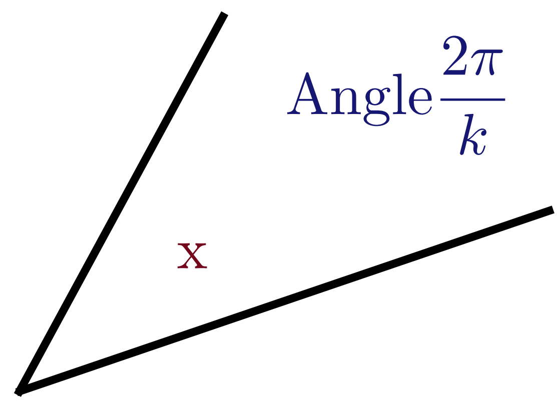
- Monopole or flux change $\quad \epsilon^{\mu\nu\rho}\partial_\mu f_{\nu\rho} = 2\pi\delta^3(x)$

- Periodic $\quad \sigma \sim \sigma + 2\pi$

- Integration over f: $\quad b_\mu = \frac{1}{k}\partial_\mu \sigma$

- Gauge Invariant $\quad \hat{Z}_\alpha = e^{-\frac{i}{k}\sigma} Z_\alpha \qquad \hat{Z}_\alpha \sim e^{2\pi i/k} \hat{Z}_\alpha$

- Moduli Space $\quad C^4/Z_k$



Chern-Simons Terms Everywhere

- ★ 5-dim Yang-Mills
- ★ 11-dim Supergravity action (Parity Even)
- ★ Mixed Chern-Simons Term
- ★ Noncommutative Plane and Fluid Mechanics
- ★ Anomaly and Axions in 4-dim
- ★ Qauntum Hall Effect
- ★ More to appear