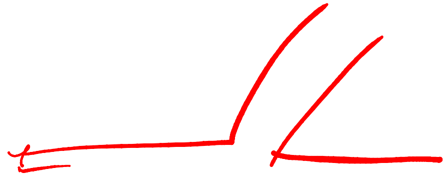
Why SCET?



- Advantage in understanding a physical system with energetic particles ($B \to \pi\pi, \pi K, \cdots$, deep inelastic scattering, Drell-Yan process)
- Separate consideration of collinear and soft interactions
- Easy to implement factorization
- Can apply rigorous effective field theoretic methods like operator product expansion, renormalization group equation, etc.

$\langle \pi | J^\mu | 0 \rangle \otimes \langle \pi | J_\mu | B \rangle$ $\langle \pi\pi | O_i | B \rangle$

Basic ideas of SCET

$n^\mu = (1,0,0,1)$
$\bar{n}^\mu = (1,0,0,-1)$

momentum of energetic particle $n^2 = \bar{n}^2 = 0,\ n \cdot \bar{n} = 2$

$p^2 \sim Q\Lambda \sim (Q\lambda)^2$

$$p^\mu = \frac{n^\mu}{2}\bar{n}\cdot p + p_\perp^\mu + \frac{\bar{n}^\mu}{2} n\cdot p \sim \mathcal{O}(Q) + \mathcal{O}(Q\lambda) + \mathcal{O}(Q\lambda^2)$$

$$\lambda = \begin{cases} \mathcal{O}(\sqrt{\Lambda_{\mathrm{QCD}}/Q}), & \mathrm{SCET_I} \\ \mathcal{O}(\Lambda_{\mathrm{QCD}}/Q), & \mathrm{SCET_{II}} \end{cases}$$

ultrasoft particle $p_{us}^\mu = (\bar{n}\cdot p_{us}, p_{us,\perp}, n\cdot p_{us}) \sim Q(\lambda^2, \lambda^2, \lambda^2)$

$p^2 \sim \Lambda^2 \sim (Q\lambda)^2$

soft particle $p_s^\mu = (\bar{n}\cdot p_s, p_{s,\perp}, n\cdot p_s) \sim Q(\lambda, \lambda, \lambda)$

$$\lambda \sim \frac{p_\perp}{\bar{n}\cdot p}$$

*Interaction with ultrasoft particles do not alter the scaling of collinear momenta.

*Soft momenta put the collinear particle off-shell and should be integrated out.

*We will call usoft (ultrasoft) momenta as soft momenta.

Effective field theory

- Integrate out the degrees of freedom of order Q.

- The dynamics describes the fluctuation of order $Q\lambda$ or $Q\lambda^2$.

- Match the coefficients between the full and the effective theory.

- Construct the RG eq. to determine the running in the effective theory.

The full QCD Lagrangian $\mathcal{L}_{\rm QCD} = \bar{\psi} i \not{D} \psi - \frac{1}{4} G_{\mu\nu} G^{\mu\nu}$

Decompose the collinear momentum into a label momentum and a residual momentum.

$$p^\mu = \tilde{p}^\mu + k^\mu \text{ where } \tilde{p}^\mu = \frac{n^\mu}{2} \bar{n} \cdot p + p_\perp^\mu$$

$$\xi_{n,p} = \frac{\not{n}\not{\bar{n}}}{4} \psi_{n,p}; \quad \xi_{\bar{n},p} = \frac{\not{\bar{n}}\not{n}}{4} \psi_{n,p}$$

$$\begin{aligned} \psi(x) &= \sum_{\tilde{p}} e^{-i\tilde{p}\cdot x} \psi_{n,p}(x) \\ &= \sum_{\tilde{p}} e^{-i\tilde{p}\cdot x} \left(\frac{\not{n}\not{\bar{n}}}{4} + \frac{\not{\bar{n}}\not{n}}{4} \right) \psi_{n,p}(x) \\ &= \sum_{\tilde{p}} e^{-i\tilde{p}\cdot x} (\xi_{n,p}(x) + \xi_{\bar{n},p}(x)) \end{aligned}$$

$$\frac{\not{n}\not{\bar{n}}}{4} \xi_{n,p} = \xi_{n,p}, \quad \not{n} \xi_{n,p} = 0,$$

$$\frac{\not{\bar{n}}\not{n}}{4} \xi_{\bar{n},p} = \xi_{\bar{n},p}, \quad \not{\bar{n}} \xi_{\bar{n},p} = 0.$$

From now on, summation over label momenta will be omitted, with the understanding that each term conserves the label momenta.

$$A^\mu = A_c^\mu + A_s^\mu + A_{us}^\mu \qquad A_c^\mu = \sum_{\tilde{q}} e^{-i\tilde{q}\cdot x} A_{n,q}(x)$$

Eq. of motion $\frac{\partial \mathcal{L}}{\partial \bar{\xi}_{\bar{n}}} = 0$

$$\mathcal{L} = \bar{\xi}_n \frac{\not{\bar{n}}}{2} n \cdot iD \xi_n + \bar{\xi}_n (\not{p}_\perp + i\not{D}_\perp) \xi_{\bar{n}}$$

$$+ \bar{\xi}_{\bar{n}} \frac{\not{n}}{2} (\bar{n} \cdot p + i\bar{n} \cdot D) \xi_{\bar{n}} + \bar{\xi}_{\bar{n}} (\not{p}_\perp + i\not{D}_\perp) \xi_n$$

$$(\bar{n} \cdot p + i\bar{n} \cdot D) \xi_{\bar{n}} = (\not{p}_\perp + i\not{D}_\perp) \frac{\not{\bar{n}}}{2} \xi_n$$

$\xi_{\bar{n}}$ is a small component.

$$\xi_{\bar{n}} = \frac{1}{\bar{n} \cdot p + i\bar{n} \cdot D} (\not{p}_\perp + i\not{D}_\perp) \frac{\not{\bar{n}}}{2} \xi_n$$

$$= \bar{\xi}_n \Big[n \cdot iD + g n \cdot A_n$$

$$+ \left(\not{p}_\perp + i\not{D}_\perp + g\not{A}_{n,\perp} \right) \frac{1}{\bar{n} \cdot p + \bar{n} \cdot iD + g\bar{n} \cdot A_n} \left(\not{p}_\perp + i\not{D}_\perp + g\not{A}_{n,\perp} \right) \Big] \frac{\not{\bar{n}}}{2} \xi_n$$

$$(\bar{\xi}_n + \bar{\xi}_{\bar{n}}) i\not{D} (\xi_n + \xi_{\bar{n}}) = \bar{\xi}_n i\not{D} \xi_n + \bar{\xi}_n i\not{D} \xi_{\bar{n}} + \bar{\xi}_{\bar{n}} i\not{D} \xi_n + \bar{\xi}_{\bar{n}} i\not{D} \xi_{\bar{n}}$$

$$\bar{\xi}_n (\not{p}_\perp + i\not{D}_\perp) \frac{1}{\bar{n} \cdot p + i\bar{n} \cdot D} (\not{p}_\perp + i\not{D}_\perp) \frac{\not{\bar{n}}}{2} \xi_n$$

Expand this in powers of λ and g .

$\int d^4x \, \bar{\psi} \, i\partial \psi$

Power counting

$$\int d^4x \, \bar{\xi}_n \, in \cdot \partial \, \xi_n$$

$d^4x \sim \lambda^{-4}$, $in \cdot \partial \sim \lambda^2$

h_v	ξ_n	$\bar{n} \cdot A_n$	$A_{n,\perp}$	$n \cdot A_n$	A_s
λ^3	λ	λ^0	λ	λ^2	λ^2

Gauge symmetries

- SU(3) gauge symmetry in full QCD
- Various subgroups of gauge symmetries:
  collinear, soft, usoft gauge symmetries

Collinear gauge symmetry $\partial^\mu U \sim Q(1,\lambda,\lambda^2), \quad U(x) = \sum_Q e^{-iQ\cdot x}\mathcal{U}_Q$

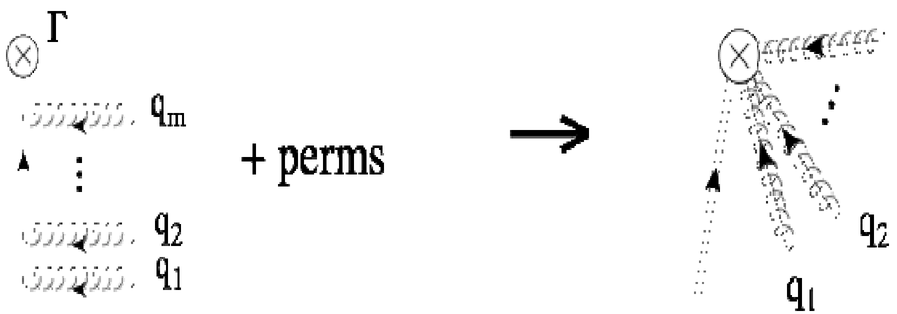
fields	collinear	soft	usoft
ξ_n	$U\xi_n$	ξ_n	$V_{us}\xi_n$
A_n^μ	$UA_n^\mu U^\dagger + U[i\mathcal{D}^\mu U^\dagger]/g$	A_n^μ	$V_{us}A_n^\mu V_{us}^\dagger$
q_s	q_s	$V_s q_s$	$V_{us}q_s$
A_s^μ	A_s^μ	$V_s(A_s^\mu + \mathcal{P}^\mu/g)V_s^\dagger$	$V_{us}A_s^\mu V_{us}^\dagger$
q_{us}	q_{us}	q_{us}	$V_{us}q_{us}$
A_{us}^μ	A_{us}^μ	A_{us}^μ	$V_{us}(A_{us}^\mu + i\partial^\mu/g)V_{us}^\dagger$
W	UW	W	$V_{us}WV_{us}^\dagger$
S	S	V_sS	$V_{us}SV_{us}^\dagger$
Y	Y	Y	$V_{us}Y$

$\chi_n = W^\dagger\xi_n$ is collinear gauge invariant.

Define the operator $\overline{\mathcal{P}}$

$$f(\overline{\mathcal{P}})(\phi^\dagger_{q_1}\cdots\phi^\dagger_{q_m}\phi_{p_1}\cdots\phi_{p_n}) = f(\bar{n}\cdot p_1+\cdots+\bar{n}\cdot p_n-\bar{n}\cdot q_1-\cdots-\bar{n}\cdot q_m)(\phi^\dagger_{q_1}\cdots\phi^\dagger_{q_m}\phi_{p_1}\cdots\phi_{p_n})$$

Define the Wilson line operator as $W=\Big[\sum_{\text{perm}}\exp\Big(-g\frac{1}{\overline{\mathcal{P}}}\bar{n}\cdot A_n\Big)\Big].$



$$f(\overline{\mathcal{P}}+g\bar{n}\cdot A_n)=Wf(\overline{\mathcal{P}})W^\dagger$$

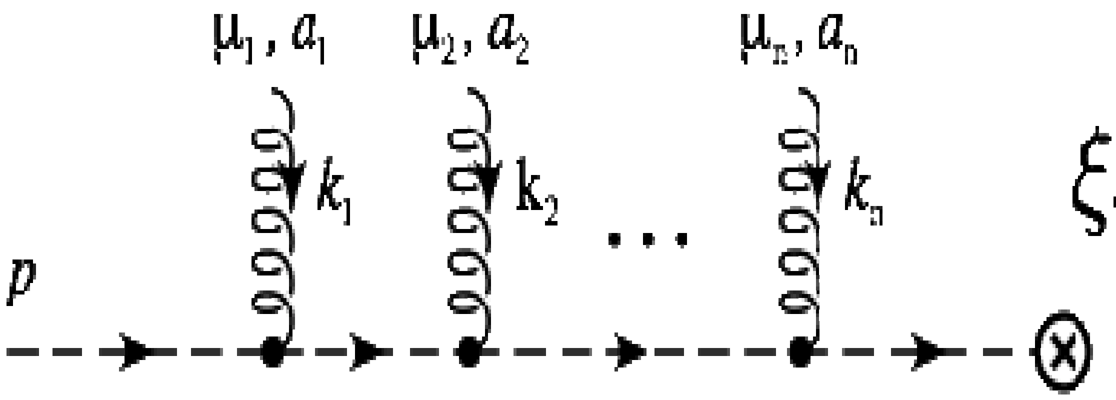
Wilson line operator

$$W=\sum_{m=0}\sum_{\text{perm}}\frac{(-g)^m}{m!}\frac{\bar{n}\cdot A_{n,q_1}\cdots\bar{n}\cdot A_{n,q_m}}{\bar{n}\cdot q_1\bar{n}\cdot(q_1+q_2)\cdots\bar{n}\cdot(\sum q_i)}T^{a_m}\cdots T^{a_1}$$

$$\begin{aligned}
W_n(x) &= P\exp\Big(ig\int_{-\infty}^{x} ds\bar{n}\cdot A_n(s\bar{n})\Big),\\
S(x) &= P\exp\Big(ig\int_{-\infty}^{x} dsn\cdot A_s(sn)\Big),\\
Y(x) &= P\exp\Big(ig\int_{-\infty}^{x} dsn\cdot A_{us}(sn)\Big).
\end{aligned}$$

Usoft factorization

collinear fermion



$$\xi_n = Y \xi_n^{(0)}$$

$$Y = 1 + \sum_{m=1}^{\infty} \sum_{\text{perm}} \frac{(-g)^m}{m!} \frac{n \cdot A_{us}^{a_1} \cdots n \cdot A_{us}^{a_m}}{n \cdot k_1 n \cdot (k_1 + k_2) \cdots n \cdot (\sum_{i=1}^{m} k_i)} T^{a_m} \cdots T^{a_1}$$

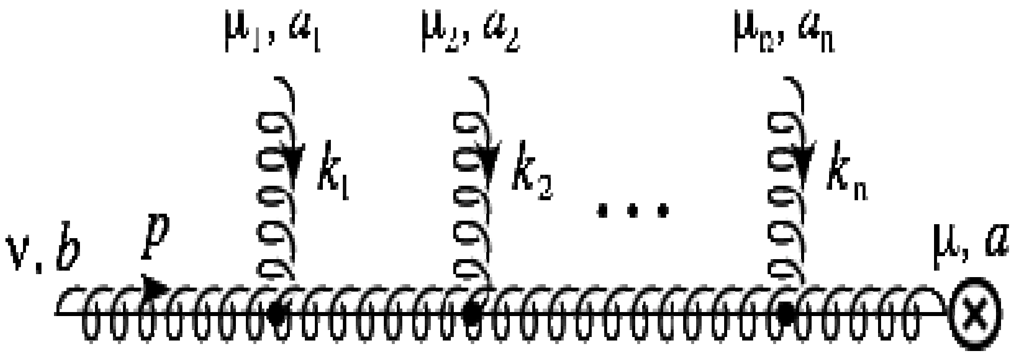
It is related to the path-ordered exponential

$$Y(x) = P \exp\Big(ig \int_{-\infty}^{x} n \cdot A_{us}(ns) T^a\Big).$$

(This will be discussed more in detail later.)

Collinear fermions are decoupled from usoft gluons.

collinear gluon



$$A_n^{a,\mu} = \mathcal{Y}^{ab} A_n^{(0)b,\mu}$$

$$\mathcal{Y}^{ab} = \delta^{ab} + \sum_{m=1}^{\infty} \sum_{\text{perm}} \frac{(ig)^m}{m!} \frac{n\cdot A_{us}^{a_1} \cdots n\cdot A_{us}^{a_m}}{n\cdot k_1 n\cdot (k_1+k_2)\cdots n\cdot (\sum_{i=1}^m k_i)} f^{a_m a x_{m-1}} \cdots f^{a_2 x_2 x_1} f^{a_1 x_1 b}$$

$$\mathcal{Y}^{ab}(x) = \left[P\exp\left(ig\int_{-\infty}^{x} ds n\cdot A_{us}^{e}(ns)\mathcal{T}^e \right) \right]^{ab} \qquad (\mathcal{T}^e)^{ab} = -if^{eab}$$

$$A_n^{\mu} = A_n^{b,\mu} T^b = A_n^{(0)a,\mu}\mathcal{Y}^{ba} T^b = A_n^{(0)a,\mu} Y T^a Y^{\dagger} = Y A_n^{(0)a,\mu} Y^{\dagger}$$

$$W = \left[\sum_{\text{perm}} \exp\left(-g\frac{1}{\overline{\mathcal{P}}} Y\bar{n}\cdot A_n^{(0)} Y^{\dagger} \right) \right] = Y W^{(0)} Y^{\dagger}$$

Note that $YT^aY^{\dagger} = \mathcal{Y}^{ba}T^b$. Collinear gluons are also decoupled.

The SCET Lagrangian is written as

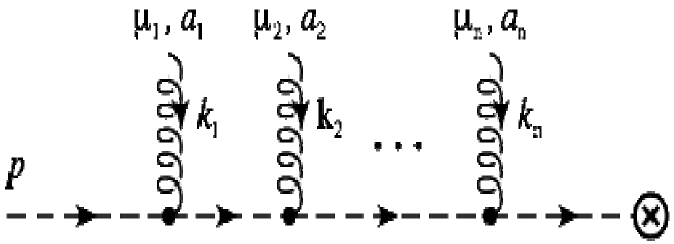
$$\begin{aligned}
\mathcal{L}_0 &= \bar{\xi}_n\Big[n\cdot iD + gn\cdot A_n + (\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W\frac{1}{\bar{\mathcal{P}}}W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})\Big]\frac{\bar{n}\!\!\!/}{2}\xi_n \\
&= \bar{\xi}_n^{(0)}Y^\dagger\Big[in\cdot D + gYn\cdot A_n^{(0)}Y^\dagger \\
&\quad +(\mathcal{P}\!\!\!/_\perp + gYA\!\!\!/_{n,\perp}^{(0)}Y^\dagger)YW^{(0)}Y^\dagger\frac{1}{\bar{\mathcal{P}}}YW^{(0)\dagger}Y^\dagger(\mathcal{P}\!\!\!/_\perp + gYA\!\!\!/_{n,\perp}^{(0)}Y^\dagger)\Big]\frac{\bar{n}\!\!\!/}{2}Y\xi_n^{(0)} \\
&= \bar{\xi}_n^{(0)}\Big[Y^\dagger n\cdot iDY + gn\cdot A_n^{(0)} + (\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp}^{(0)})W^{(0)}\frac{1}{\bar{\mathcal{P}}}W^{(0)\dagger}(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp}^{(0)})\Big]\frac{\bar{n}\!\!\!/}{2}\xi_n^{(0)} \\
&= \bar{\xi}_n^{(0)}\Big[n\cdot i\partial + gn\cdot A_n^{(0)} + (\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp}^{(0)})W^{(0)}\frac{1}{\bar{\mathcal{P}}}W^{(0)\dagger}(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp}^{(0)})\Big]\frac{\bar{n}\!\!\!/}{2}\xi_n^{(0)} \\
&\to \boxed{\bar{\chi}_n\Big[n\cdot i\partial + gn\cdot A_n + W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W\frac{1}{\bar{\mathcal{P}}}W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W\Big]\frac{\bar{n}\!\!\!/}{2}\chi_n.}
\end{aligned}$$

$$\mathcal{L}_1 = \bar{\chi}_n\Big[W^\dagger iD\!\!\!/_\perp W\frac{1}{\bar{\mathcal{P}}}W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W + W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W\frac{1}{\bar{\mathcal{P}}}W^\dagger iD\!\!\!/_\perp W\Big]\frac{\bar{n}\!\!\!/}{2}\chi_n.$$

Completely decoupled from usoft interactions

collinear gauge-invariant blocks $\chi_n = W^\dagger\xi_n,\ W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W,\ W^\dagger iD\!\!\!/_\perp W$

Soft Wilson lines



$$Y = 1 + \sum_{m=1}^{\infty} \sum_{\text{perm}} \frac{(-g)^m}{m!} \frac{n \cdot A_{us}^{a_1} \cdots n \cdot A_{us}^{a_m}}{n \cdot k_1 n \cdot (k_1 + k_2) \cdots n \cdot (\sum_{i=1}^{m} k_i)} T^{a_m} \cdots T^{a_1}$$

$$= \left[\sum_{\text{perm}} \exp\left(\frac{1}{n \cdot \mathcal{R} + i0} (-g n \cdot A_s) \right) \right]$$

The Fourier transform is given by $Y(x) = P \exp\left[ig \int_{-\infty}^{x} ds n \cdot A_s(ns) \right].$

Proof

$$n \cdot A(x) = \frac{1}{2\pi} \int dn \cdot q e^{-in \cdot q \bar{x}} n \cdot A_s(n \cdot q), \quad (\bar{x} = \bar{n} \cdot x/2),$$

$$n \cdot A_s(n \cdot q) = \int d\bar{x} e^{in \cdot q \bar{x}} n \cdot A_s(x).$$

At order g

$$-\frac{g}{2\pi} \int dn \cdot q \frac{e^{-in \cdot q \bar{x}}}{n \cdot q + i0} n \cdot A_s(n \cdot q)$$

$$= -\frac{g}{2\pi} \int_{\infty}^{\infty} d\bar{y} \int dn \cdot q \frac{e^{in \cdot q(\bar{y} - \bar{x})}}{n \cdot q + i0} n \cdot A_s(\bar{y}) = ig \int_{\infty}^{\bar{x}} d\bar{y} n \cdot A_s(\bar{y})$$

$$-2\pi i \theta(\bar{x} - \bar{y})$$

At order g^2

$$\frac{1}{n\cdot\mathcal{R}}gn\cdot A_s\frac{1}{n\cdot\mathcal{R}}gn\cdot A_s \quad\rightarrow\quad \int\frac{dn\cdot q_1 dn\cdot q_2}{(2\pi)^2}e^{-in\cdot(q_1+q_2)\overline{x}}\frac{gn\cdot A_s(n\cdot q_2)gn\cdot A_s(q_1)}{(n\cdot(q_1+q_2)+i0)(n\cdot q_1+i0)}$$

$$= \int d\overline{y}d\overline{z}\int\frac{dn\cdot q_1 dn\cdot q_2}{(2\pi)^2}e^{in\cdot q_1(\overline{y}-\overline{x})}e^{in\cdot q_2(\overline{z}-\overline{x})}\frac{gn\cdot A_s(\overline{z})gn\cdot A_s(\overline{y})}{(n\cdot(q_1+q_2)+i0)(n\cdot q_1+i0)}$$

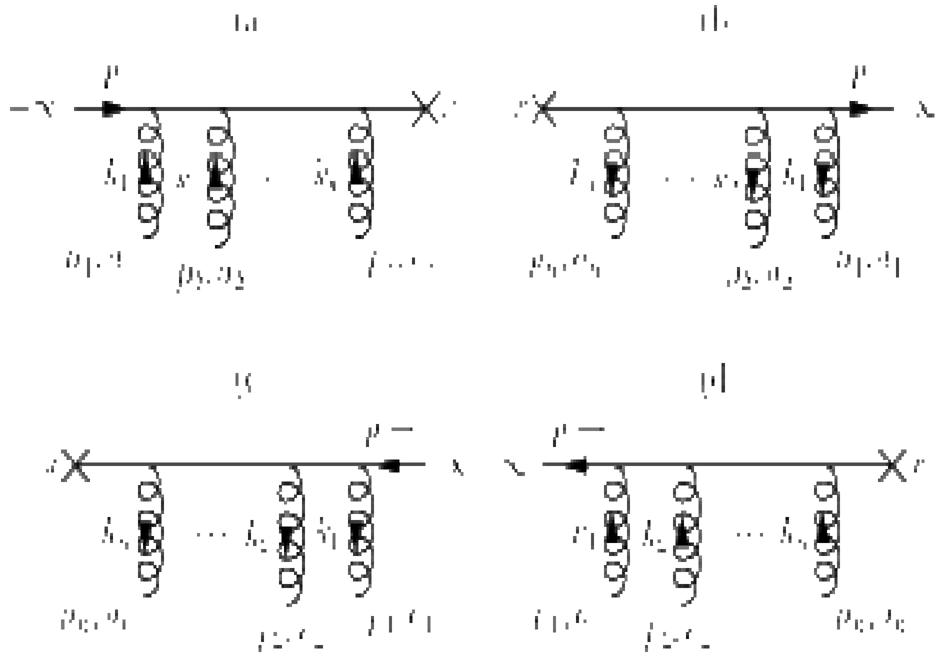
$$= \int d\overline{y}d\overline{z}gn\cdot A_s(\overline{z})gn\cdot A_s(\overline{y})\int\frac{dn\cdot q_1}{2\pi}\frac{e^{in\cdot q_1(\overline{y}-\overline{x})}}{n\cdot q_1+i0}\int\frac{dn\cdot q_2}{2\pi}\frac{e^{in\cdot q_2(\overline{z}-\overline{x})}}{n\cdot(q_1+q_2)+i0}$$

$$= (-i)^2\int d\overline{y}d\overline{z}gn\cdot A_s(\overline{z})gn\cdot A_s(\overline{y})\theta(\overline{x}-\overline{z})\theta(\overline{z}-\overline{y}) = (-i)^2\int_{-\infty}^{\overline{x}}d\overline{z}gn\cdot A_s(\overline{z})\int_{-\infty}^{\overline{z}}d\overline{y}gn\cdot A_s(\overline{y})$$

$$= \frac{(-i)^2}{2!}\int_{-\infty}^{\overline{x}}d\overline{z}\int_{-\infty}^{\overline{x}}d\overline{y}P[gn\cdot A_s(\overline{z})gn\cdot A_s(\overline{y})]$$

For the m-th term,

$$\left[\frac{1}{n\cdot\mathcal{R}+i0}gn\cdot A_s\right]^m \rightarrow \frac{(-1)^m}{m!}\int_{-\infty}^{\overline{x}}d\overline{y}_1\cdots d\overline{y}_m P[gn\cdot A_s(\overline{y}_1)\cdots gn\cdot A_s(\overline{y}_m)]$$



type	Wilson line	Fourier transform
(a):	$Y = \exp[\frac{1}{n\cdot\mathcal{P}+i\epsilon}(-gn\cdot A_s)]$	$Y(x) = P\exp\left[ig\int_{-\infty}^{x} dsn\cdot A_s(ns)\right]$
(b):	$\tilde{Y}^\dagger = \exp\left[-gn\cdot A_s\frac{1}{n\cdot\mathcal{P}^\dagger+i\epsilon}\right]$	$\tilde{Y}^\dagger(x) = P\exp\left[ig\int_{x}^{\infty} dsn\cdot A_s(ns)\right]$
(c):	$\tilde{Y} = \exp\left[\frac{1}{n\cdot\mathcal{P}-i\epsilon}(-gn\cdot A_s)\right]$	$\tilde{Y}(x) = \overline{P}\exp\left[-ig\int_{x}^{\infty} dsn\cdot A_s(ns)\right]$
(d):	$Y^\dagger = \exp\left[-gn\cdot A_s\frac{1}{n\cdot\mathcal{P}^\dagger-i\epsilon}\right]$	$Y^\dagger(x) = \overline{P}\exp\left[-ig\int_{-\infty}^{x} dsn\cdot A_s(ns)\right]$