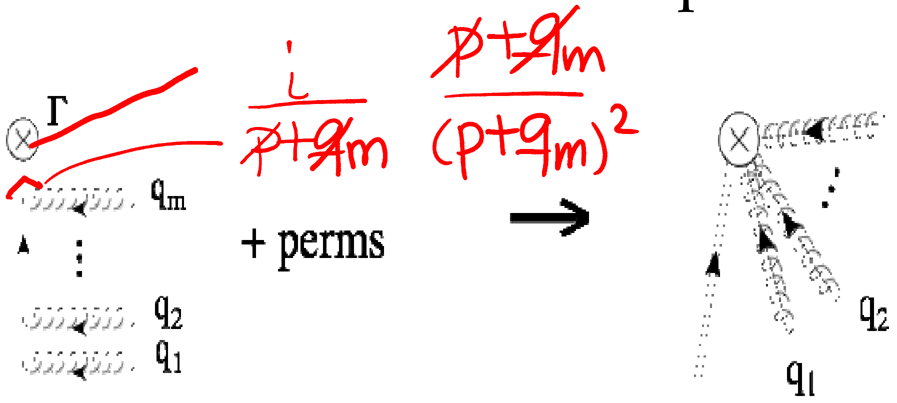
Define the operator $\overline{\mathcal{P}}$ $\quad \chi_n \equiv W_n^\dagger \xi_n$

$$f(\overline{\mathcal{P}})(\phi^\dagger_{q_1}\cdots\phi^\dagger_{q_m}\phi_{p_1}\cdots\phi_{p_n}) = f(\bar{n}\cdot p_1+\cdots+\bar{n}\cdot p_n-\bar{n}\cdot q_1-\cdots-\bar{n}\cdot q_m)(\phi^\dagger_{q_1}\cdots\phi^\dagger_{q_m}\phi_{p_1}\cdots\phi_{p_n})$$

Define the Wilson line operator as $\quad W = \left[\sum_{\text{perm}} \exp\left(-g\frac{1}{\overline{\mathcal{P}}}\bar{n}\cdot A_n\right)\right].$



$$f(\overline{\mathcal{P}}+g\bar{n}\cdot A_n) = W f(\overline{\mathcal{P}}) W^\dagger$$

$$W = \sum_{m=0}\sum_{\text{perm}} \frac{(-g)^m}{m!}\frac{\bar{n}\cdot A_{n,q_1}\cdots\bar{n}\cdot A_{n,q_m}}{\bar{n}\cdot q_1\bar{n}\cdot(q_1+q_2)\cdots\bar{n}\cdot(\sum q_i)}T^{a_m}\cdots T^{a_1}$$

Wilson line operator

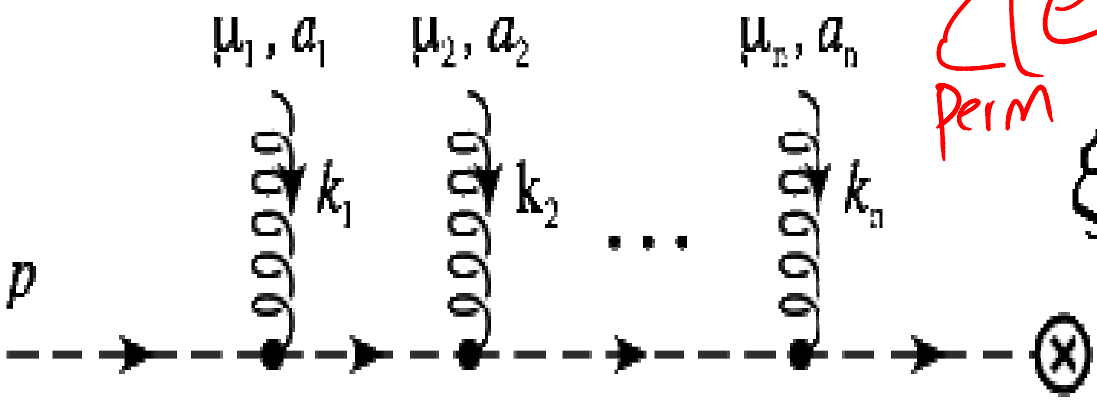
$$\begin{aligned}
W_n(x) &= P\exp\left(ig\int_{-\infty}^{x} ds\bar{n}\cdot A_n(s\bar{n})\right),\\
S(x) &= P\exp\left(ig\int_{-\infty}^{x} dsn\cdot A_s(sn)\right),\\
Y(x) &= P\exp\left(ig\int_{-\infty}^{x} dsn\cdot A_{us}(sn)\right).
\end{aligned}$$

Usoft factorization

eikonal approx.

$$\sum_{perm}\left[\exp\frac{-g\,n\cdot A_{us}}{n\cdot k}\right]$$

collinear fermion



$$\xi_n = Y\xi_n^{(0)}$$

$$Y = 1 + \sum_{m=1}^{\infty}\sum_{\text{perm}}\frac{(-g)^m}{m!}\frac{n\cdot A_{us}^{a_1}\cdots n\cdot A_{us}^{a_m}}{n\cdot k_1 n\cdot(k_1+k_2)\cdots n\cdot(\sum_{i=1}^{m}k_i)}T^{a_m}\cdots T^{a_1}$$

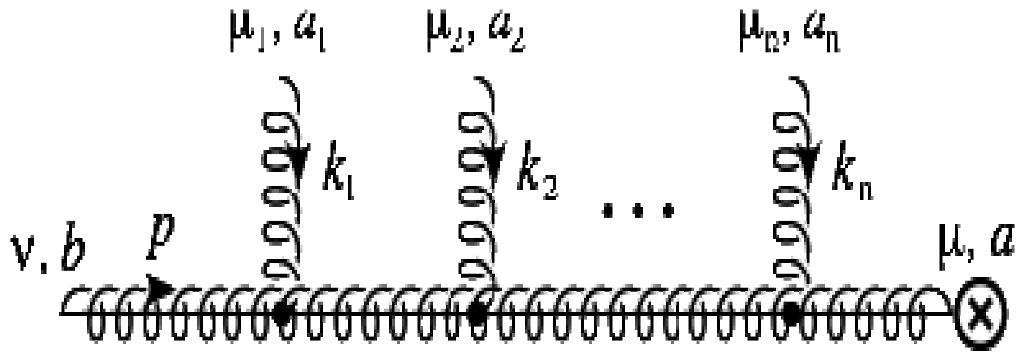
It is related to the path-ordered exponential

$$Y(x) = P\exp\left(ig\int_{-\infty}^{x} n\cdot A_{us}(ns)T^a\right).$$

(This will be discussed more in detail later.)

Collinear fermions are decoupled from usoft gluons.

collinear gluon



$$A_n^{a,\mu} = \mathcal{Y}^{ab} A_n^{(0)b,\mu}$$

$$\mathcal{Y}^{ab} = \delta^{ab} + \sum_{m=1}^{\infty} \sum_{\text{perm}} \frac{(ig)^m}{m!} \frac{n \cdot A_{us}^{a_1} \cdots n \cdot A_{us}^{a_m}}{n \cdot k_1 n \cdot (k_1 + k_2) \cdots n \cdot (\sum_{i=1}^m k_i)} f^{a_m a x_{m-1}} \cdots f^{a_2 x_2 x_1} f^{a_1 x_1 b}$$

$$\mathcal{Y}^{ab}(x) = \left[P \exp\left(ig \int_{-\infty}^{x} ds n \cdot A_{us}^{e}(ns) \mathcal{T}^e \right) \right]^{ab} \qquad (\mathcal{T}^e)^{ab} = -i f^{eab}$$

$$A_n^{\mu} = A_n^{b,\mu} T^b = A_n^{(0)a,\mu} \mathcal{Y}^{ba} T^b = A_n^{(0)a,\mu} Y T^a Y^{\dagger} = Y A_n^{(0)a,\mu} Y^{\dagger}$$

$$W = \left[\sum_{\text{perm}} \exp\left(-g \frac{1}{\overline{\mathcal{P}}} Y \bar{n} \cdot A_n^{(0)} Y^{\dagger} \right) \right] = Y W^{(0)} Y^{\dagger}$$

Note that $Y T^a Y^{\dagger} = \mathcal{Y}^{ba} T^b$. Collinear gluons are also decoupled.

The SCET Lagrangian is written as

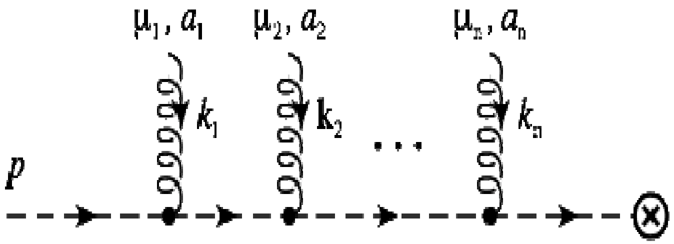
$$\begin{aligned}\mathcal{L}_0 &= \bar{\xi}_n\left[n\cdot iD + gn\cdot A_n + (\mathcal{P}\!\!\!/_\perp + g A\!\!\!/_{n,\perp})W\frac{1}{\bar{\mathcal{P}}}W^\dagger(\mathcal{P}\!\!\!/_\perp + g A\!\!\!/_{n,\perp})\right]\frac{\bar{n}\!\!\!/}{2}\xi_n \\
&= \bar{\xi}_n^{(0)}Y^\dagger\Big[in\cdot D + gYn\cdot A_n^{(0)}Y^\dagger \\
&\quad +(\mathcal{P}\!\!\!/_\perp + gYA\!\!\!/_{n,\perp}^{(0)}Y^\dagger)YW^{(0)}Y^\dagger\frac{1}{\bar{\mathcal{P}}}YW^{(0)\dagger}Y^\dagger(\mathcal{P}\!\!\!/_\perp + gYA\!\!\!/_{n,\perp}^{(0)}Y^\dagger)\Big]\frac{\bar{n}\!\!\!/}{2}Y\xi_n^{(0)} \\
&= \bar{\xi}_n^{(0)}\left[Y^\dagger n\cdot iDY + gn\cdot A_n^{(0)} + (\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp}^{(0)})W^{(0)}\frac{1}{\bar{\mathcal{P}}}W^{(0)\dagger}(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp}^{(0)})\right]\frac{\bar{n}\!\!\!/}{2}\xi_n^{(0)} \\
&= \bar{\xi}_n^{(0)}\left[n\cdot i\partial + gn\cdot A_n^{(0)} + (\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp}^{(0)})W^{(0)}\frac{1}{\bar{\mathcal{P}}}W^{(0)\dagger}(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp}^{(0)})\right]\frac{\bar{n}\!\!\!/}{2}\xi_n^{(0)} \\
&\to \boxed{\bar{\chi}_n\left[n\cdot i\partial + gn\cdot A_n + W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W\frac{1}{\bar{\mathcal{P}}}W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W\right]\frac{\bar{n}\!\!\!/}{2}\chi_n.}\end{aligned}$$

$$\mathcal{L}_1 = \bar{\chi}_n\left[W^\dagger iD\!\!\!/_\perp W\frac{1}{\bar{\mathcal{P}}}W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W + W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W\frac{1}{\bar{\mathcal{P}}}W^\dagger iD\!\!\!/_\perp W\right]\frac{\bar{n}\!\!\!/}{2}\chi_n.$$

Completely decoupled from usoft interactions

collinear gauge-invariant blocks $\chi_n = W^\dagger\xi_n,\ W^\dagger(\mathcal{P}\!\!\!/_\perp + gA\!\!\!/_{n,\perp})W,\ W^\dagger iD\!\!\!/_\perp W$

Soft Wilson lines



$$Y = 1 + \sum_{m=1}^{\infty} \sum_{\text{perm}} \frac{(-g)^m}{m!} \frac{n \cdot A_{us}^{a_1} \cdots n \cdot A_{us}^{a_m}}{n \cdot k_1 n \cdot (k_1 + k_2) \cdots n \cdot (\sum_{i=1}^{m} k_i)} T^{a_m} \cdots T^{a_1}$$

$$= \Big[\sum_{\text{perm}} \exp\Big(\frac{1}{n \cdot \mathcal{R} + i0} (-g n \cdot A_s) \Big) \Big]$$

The Fourier transform is given by $Y(x) = P \exp\Big[ig \int_{-\infty}^{x} ds\, n \cdot A_s(ns) \Big].$

Proof

$$n \cdot A(\bar{x}) = \frac{1}{2\pi} \int dn \cdot q\, e^{-in \cdot q \bar{x}} n \cdot A_s(n \cdot q), \; (\bar{x} = \bar{n} \cdot x/2),$$

$$n \cdot A_s(n \cdot q) = \int d\bar{x}\, e^{in \cdot q \bar{x}} n \cdot A_s(x).$$

At order g

$$-\frac{g}{2\pi} \int dn \cdot q \frac{e^{-in \cdot q \bar{x}}}{n \cdot q + i0} n \cdot A_s(n \cdot q)$$

$$= -\frac{g}{2\pi} \int_{\infty}^{\infty} d\bar{y} \int dn \cdot q \frac{e^{in \cdot q(\bar{y} - \bar{x})}}{n \cdot q + i0} n \cdot A_s(\bar{y}) = ig \int_{\infty}^{\bar{x}} d\bar{y} n \cdot A_s(\bar{y})$$

$$-2\pi i \theta(\bar{x} - \bar{y})$$

At order g^2

$$\frac{1}{n\cdot\mathcal{R}}gn\cdot A_s\frac{1}{n\cdot\mathcal{R}}gn\cdot A_s \quad\rightarrow\quad \int\frac{dn\cdot q_1 dn\cdot q_2}{(2\pi)^2}e^{-in\cdot(q_1+q_2)\bar{x}}\frac{gn\cdot A_s(n\cdot q_2)gn\cdot A_s(q_1)}{(n\cdot(q_1+q_2)+i0)(n\cdot q_1+i0)}$$

$$= \int d\bar{y}d\bar{z}\int\frac{dn\cdot q_1 dn\cdot q_2}{(2\pi)^2}e^{in\cdot q_1(\bar{y}-\bar{x})}e^{in\cdot q_2(\bar{z}-\bar{x})}\frac{gn\cdot A_s(\bar{z})gn\cdot A_s(\bar{y})}{(n\cdot(q_1+q_2)+i0)(n\cdot q_1+i0)}$$

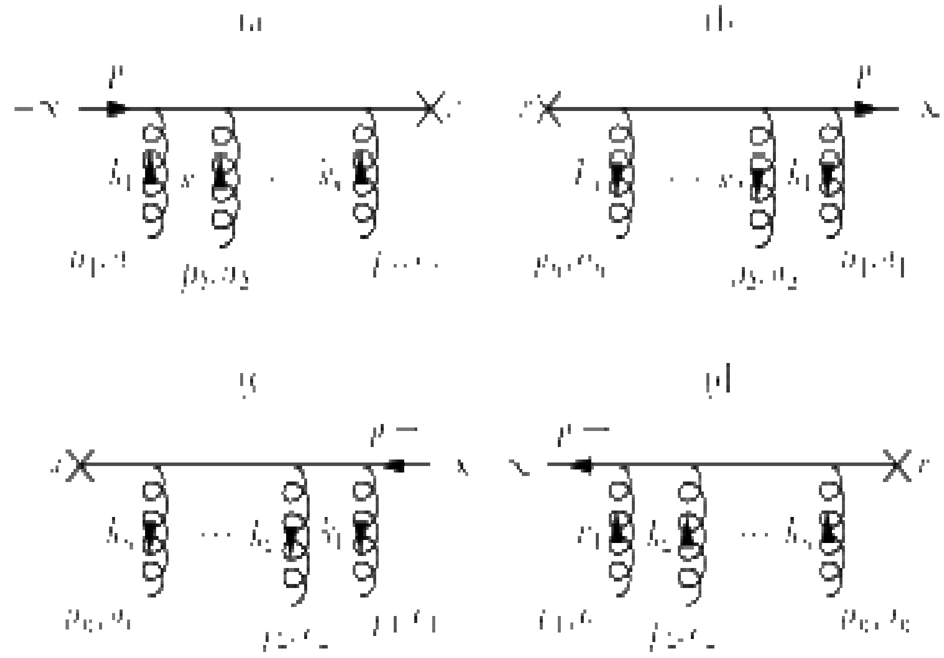
$$= \int d\bar{y}d\bar{z}gn\cdot A_s(\bar{z})gn\cdot A_s(\bar{y})\int\frac{dn\cdot q_1}{2\pi}\frac{e^{in\cdot q_1(\bar{y}-\bar{x})}}{n\cdot q_1+i0}\int\frac{dn\cdot q_2}{2\pi}\frac{e^{in\cdot q_2(\bar{z}-\bar{x})}}{n\cdot(q_1+q_2)+i0}$$

$$= (-i)^2\int d\bar{y}d\bar{z}gn\cdot A_s(\bar{z})gn\cdot A_s(\bar{y})\theta(\bar{x}-\bar{z})\theta(\bar{z}-\bar{y}) = (-i)^2\int_{-\infty}^{\bar{x}}d\bar{z}gn\cdot A_s(\bar{z})\int_{-\infty}^{\bar{z}}d\bar{y}gn\cdot A_s(\bar{y})$$

$$= \frac{(-i)^2}{2!}\int_{-\infty}^{\bar{x}}d\bar{z}\int_{-\infty}^{\bar{x}}d\bar{y}P[gn\cdot A_s(\bar{z})gn\cdot A_s(\bar{y})]$$

For the m-th term,

$$\left[\frac{1}{n\cdot\mathcal{R}+i0}gn\cdot A_s\right]^m \rightarrow \frac{(-1)^m}{m!}\int_{-\infty}^{\bar{x}}d\bar{y}_1\cdots d\bar{y}_m P[gn\cdot A_s(\bar{y}_1)\cdots gn\cdot A_s(\bar{y}_m)]$$



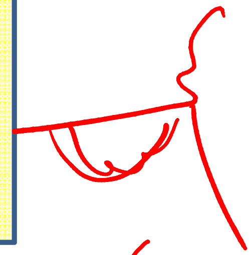
type	Wilson line	Fourier transform
(a):	$Y = \exp[\frac{1}{n\cdot\mathcal{P}+i\epsilon}(-gn\cdot A_s)]$	$Y(x) = P\exp\Big[ig\int_{-\infty}^{x} ds\, n\cdot A_s(ns)\Big]$
(b):	$\tilde{Y}^\dagger = \exp\Big[-gn\cdot A_s\frac{1}{n\cdot\mathcal{P}^\dagger+i\epsilon}\Big]$	$\tilde{Y}^\dagger(x) = P\exp\Big[ig\int_{x}^{\infty} ds\, n\cdot A_s(ns)\Big]$
(c):	$\tilde{Y} = \exp\Big[\frac{1}{n\cdot\mathcal{P}-i\epsilon}(-gn\cdot A_s)\Big]$	$\tilde{Y}(x) = \overline{P}\exp\Big[-ig\int_{x}^{\infty} ds\, n\cdot A_s(ns)\Big]$
(d):	$Y^\dagger = \exp\Big[-gn\cdot A_s\frac{1}{n\cdot\mathcal{P}^\dagger-i\epsilon}\Big]$	$Y^\dagger(x) = \overline{P}\exp\Big[-ig\int_{-\infty}^{x} ds\, n\cdot A_s(ns)\Big]$

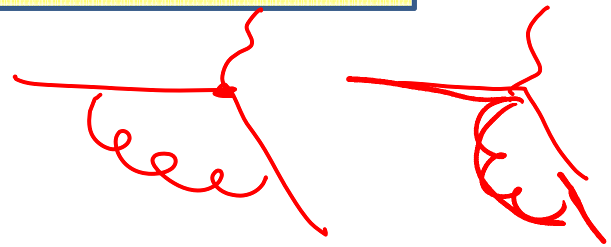
High-energy scattering

Collinear gauge-invariant combination $\chi_n = W_n^\dagger \xi_n, \;\; \chi_{\bar{n}} = W_{\bar{n}}^\dagger \xi_{\bar{n}}$

After the usoft factorization

$$\chi_n \to Y_n \chi_n, \overline{\chi}_n \to \overline{\chi}_n Y_n^\dagger$$
$$\chi_{\bar{n}} \to Y_{\bar{n}} \chi_{\bar{n}}, \;\; \overline{\chi}_{\bar{n}} \to \overline{\chi}_{\bar{n}} Y_{\bar{n}}^\dagger$$



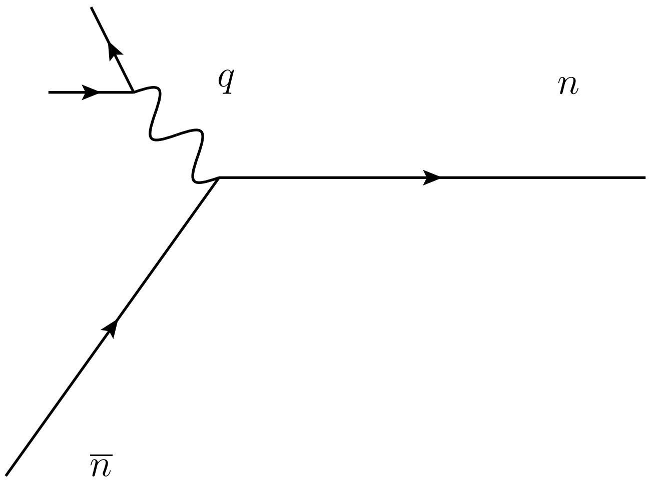


Full QCD back-to-back current $\overline{\psi}\Gamma\psi$

$$\begin{aligned} \overline{\psi}\Gamma\psi \;\; &\to \;\; \overline{\chi}_n Y_n^\dagger \Gamma Y_{\bar{n}} \chi_{\bar{n}} \\ &\to \;\; \overline{\chi}_n Y_n^\dagger \Gamma C(\overline{\mathcal{P}}^\dagger, \mathcal{P}) Y_{\bar{n}} \chi_{\bar{n}} \to \int d\omega d\omega' C(\omega, \omega') O_{n\bar{n}}(\omega, \omega'). \end{aligned}$$

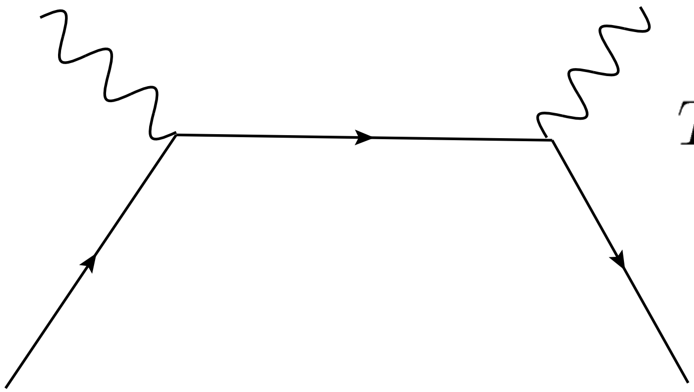
$$O_{n\bar{n}}(\omega, \omega') = \overline{\chi}_n \delta(\overline{\mathcal{P}}^\dagger - \omega) Y_n^\dagger \Gamma Y_{\bar{n}} \delta(\mathcal{P} - \omega') \chi_{\bar{n}}$$

1. Deep inelastic scattering $ep \to e + X$



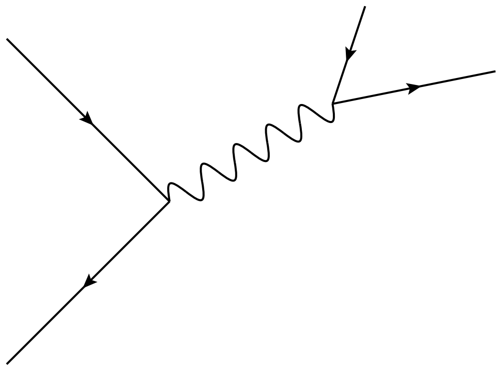
$$d\sigma = \frac{d^3\mathbf{k}'}{2|\mathbf{k}'|(2\pi)^3}\frac{\pi e^4}{sQ^4}L^{\mu\nu}W_{\mu\nu}(p,q)$$

$$W_{\mu\nu}(p,q) = \frac{1}{\pi}\mathrm{Im}\,T_{\mu\nu}(p,q)$$



$$T_{\mu\nu} = i\int d^4z e^{iq\cdot z}\frac{1}{2}\sum_{\mathrm{spin}}\langle p|T[J_\mu(z)J_\nu(0)]|p\rangle$$

2. Drell-Yan process $p\overline{p} \to \ell^+\ell^- X$



$$d\sigma = \frac{32\pi^2\alpha^2}{Q^4 s} L^{\mu\nu} W_{\mu\nu} \prod_{i=1,2} \frac{d^3 k_i}{(2\pi)^3 2k_i^0}$$

$$W_{\mu\nu} = \frac{1}{4}\sum_{\text{spins}} \int d^4 z e^{-iq\cdot z} \langle p\overline{p}| J_\mu(z) J_\nu(0) |p\overline{p}\rangle$$

$$\overline{\chi}_{\bar{n}} Y_{\bar{n}}^{\dagger} \gamma_{\mu} Y_{n}^{\dagger} \chi_{n}(z) \overline{\chi}_{n} Y_{n}^{\dagger} \gamma_{\nu} Y_{\bar{n}} \chi_{\bar{n}}$$