$W_n \to U W_n$
$\xi_n \to U \xi_n$

High-energy scattering

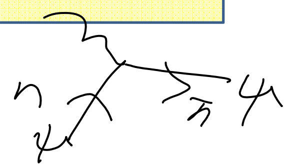
Collinear gauge-invariant combination $\chi_n = W_n^\dagger \xi_n,\ \chi_{\bar{n}} = W_{\bar{n}}^\dagger \xi_{\bar{n}}$

After the usoft factorization

$$\chi_n \to Y_n \chi_n, \overline{\chi}_n \to \overline{\chi}_n Y_n^\dagger$$

$$\chi_{\bar{n}} \to Y_{\bar{n}} \chi_{\bar{n}},\ \overline{\chi}_{\bar{n}} \to \overline{\chi}_{\bar{n}} Y_{\bar{n}}^\dagger$$

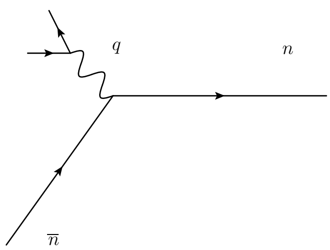
Full QCD back-to-back current $\overline{\psi}\Gamma\psi$



$$\begin{aligned}\overline{\psi}\Gamma\psi \quad &\to \quad \overline{\chi}_n Y_n^\dagger \Gamma Y_{\bar{n}} \chi_{\bar{n}} \\ &\to \quad \overline{\chi}_n Y_n^\dagger \Gamma C(\overline{\mathcal{P}}^\dagger, \mathcal{P}) Y_{\bar{n}} \chi_{\bar{n}} \to \int d\omega d\omega' C(\omega,\omega') O_{n\bar{n}}(\omega,\omega').\end{aligned}$$

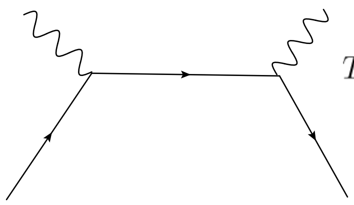
$$O_{n\bar{n}}(\omega,\omega') = \overline{\chi}_n \delta(\overline{\mathcal{P}}^\dagger - \omega) Y_n^\dagger \Gamma Y_{\bar{n}} \delta(\mathcal{P} - \omega') \chi_{\bar{n}}$$

1. Deep inelastic scattering $ep \to e + X$



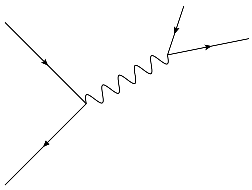
$$d\sigma = \frac{d^3\mathbf{k}'}{2|\mathbf{k}'|(2\pi)^3} \frac{\pi e^4}{sQ^4} L^{\mu\nu} W_{\mu\nu}(p,q)$$

$$W_{\mu\nu}(p,q) = \frac{1}{\pi} \mathrm{Im}\, T_{\mu\nu}(p,q)$$



$$T_{\mu\nu} = i \int d^4 z e^{iq\cdot z} \frac{1}{2} \sum_{\mathrm{spin}} \langle p | T[J_\mu(z) J_\nu(0)] | p \rangle$$

2. Drell-Yan process $p\bar{p} \to \ell^+ \ell^- X$



$$d\sigma = \frac{32\pi^2\alpha^2}{Q^4 s} L^{\mu\nu} W_{\mu\nu} \prod_{i=1,2} \frac{d^3 k_i}{(2\pi)^3 2k_i^0}$$

$$W_{\mu\nu} = \frac{1}{4}\sum_{\text{spins}} \int d^4 z e^{-iq\cdot z} \langle p\bar{p}| J_\mu(z) J_\nu(0) |p\bar{p}\rangle$$

$p^\mu = E n^\mu + k^\mu$

$\overline{\chi}_{\bar{n}} Y_{\bar{n}}^\dagger \gamma_\mu Y_n \chi_n(z) \overline{\chi}_n Y_n^\dagger \gamma_\nu Y_{\bar{n}} \chi_{\bar{n}}(0)$ Glauber gluon

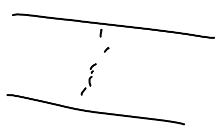
(1) DIS

$p^\mu = (\bar{n}\cdot p, n\cdot p, p_\perp) \sim (\lambda^2, \lambda^2, \lambda)$

$$T[\overline{\chi}_n(z)\chi_n(0)] = i\frac{\not{n}}{2}\int \frac{d^4k}{(2\pi)^4} \frac{J(n\cdot k)}{n\cdot k} e^{-ik\cdot z}$$

$$\int e^{-iq\cdot z} d^4x \langle p| \overline{\chi}_{\bar{n}}(z) \frac{\not{n}}{2} \chi_{\bar{n}}(0) |p\rangle$$

(2) Drell-Yan

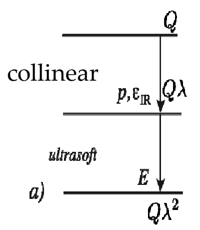


Double counting issue

In collinear loops, the divergence is of the form $I_{\rm col} = \frac{A}{\epsilon_{\rm UV}} + \frac{B}{\epsilon_{\rm IR}} + f(Q,\mu)$
As the momentum vanishes, it matches onto an usoft diagram with the structure

$$I_{\rm usoft} = -\frac{B}{\epsilon_{\rm UV}} + \quad + g(\bar{n}\cdot p,\mu).$$

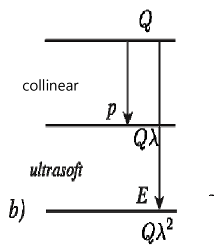
true IR divergence



$\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}} = 0$ in scaleless integral

$$I_{\rm col} + I_{\rm usoft} = \frac{A}{\epsilon_{\rm UV}} + \frac{C}{\epsilon_{\rm IR}} + f(\bar{n}\cdot p,\mu) + g(\bar{n}\cdot p,\mu)$$

Counterterms must be added for all $\frac{1}{\epsilon_{\rm UV}}$ divergences in the effective theory including those from scaleless integrals.

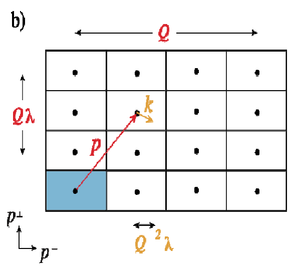


Pull up the usoft mode to the scale Q, then subtract the usoft contribution in the region $Q\lambda$ and Q.

$$I_{\rm col} = \frac{A}{\epsilon_{\rm UV}} + \frac{B}{\epsilon_{\rm IR}} + f(Q,\mu) \qquad I_0 = -\frac{B}{\epsilon_{\rm UV}} + \frac{B}{\epsilon_{\rm IR}}$$

$$\longrightarrow I_{\rm col} = \frac{A}{\epsilon_{\rm UV}} + \frac{B}{\epsilon_{\rm UV}} + f(Q,\mu)$$

Pullup mechanism used in NRQCD.



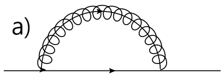
$$\sum_p \int dk \to \int dp \qquad \text{Naive integral}$$

$$\sum_{\vee\{p_i\neq 0\}} \int \prod_i dk_i F(\{p_i\},\{k_i\}) \to \int \prod_i dp_i \Big[F(\{p_i\}) - \sum_{j\in U} F_j^{\rm sub}(\{p_i\})\Big]$$

Jet function 

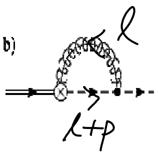
$$\bar{\xi}_n W_n \Gamma \qquad \exp -g\frac{1}{\bar{n}\cdot p}\bar{n}\cdot A_n$$

$$\bar{\xi}_n\left(1-g\frac{1}{\bar{n}\cdot q}\bar{n}^{\mu}T_a\right)$$

a) 

$$\bar{\xi}_n\, ig\frac{\bar{n}}{2}T_a n_\mu \left(\;\right)\frac{i\frac{n}{2}\bar{n}\cdot(l+p)}{(l+p)^2}(-ig)\frac{\bar{n}^{\mu}T_a}{\bar{n}\cdot l}\frac{-i}{l^2}$$

$$= g^2 C_F \tilde{I}_c$$

b) 

$$\begin{aligned}
\tilde{I}_C &= \int \frac{d^D}{(2\pi)^D}\frac{2\bar{n}\cdot(l+p)}{(\bar{n}\cdot l+i0^+)[(l+p)^2+i0^+](l^2+i0^-)} \\
&= -\frac{i}{16\pi^2}\left[-\frac{2}{\epsilon_{\rm IR}\epsilon_{\rm UV}}-\frac{2}{\epsilon_{\rm IR}}\ln\frac{\mu^2}{-p^2}-\ln^2\frac{\mu^2}{-p^2}+\left(\frac{2}{\epsilon_{\rm IR}}-\frac{2}{\epsilon_{\rm UV}}\right)\ln\frac{\mu}{\bar{n}\cdot p}\right]+\cdots \\
I_0 &= \int \frac{d^D l}{(2\pi)^D}\frac{2\bar{n}\cdot p}{(\bar{n}\cdot l+i0^+)(n\cdot l\bar{n}\cdot p+p^2+i0^+)(l^2+i0^+)} \\
&= -\frac{i}{16\pi^2}\left[\left(\frac{2}{\epsilon_{\rm UV}}-\frac{2}{\epsilon_{\rm IR}}\right)\left\{\frac{1}{\epsilon_{\rm UV}}+\ln\frac{\mu^2}{-p^2}-\ln\frac{\mu}{\bar{n}\cdot p}\right\}\right]
\end{aligned}$$

Sample calculation

$$\frac{1}{(q^2)^n(q\cdot v))^m}=\frac{(n+m-1)!}{(n-1)!(m-1)!}\int_0^\infty\frac{2^m u^{m-1}du}{(q^2+2uq\cdot v)^{n+m}}$$

$$l^2+2xl\cdot p+xp^2+2u\bar{n}\cdot l=(l+xp+u\bar{n})^2-\left(-x(1-x)p^2+2ux\bar{n}\cdot p\right)$$

$$A^2$$

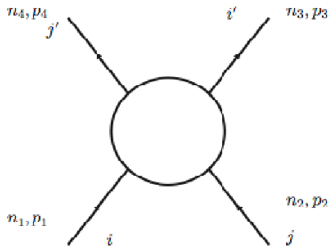
$$\begin{aligned}
\tilde{I}_c &= \int\frac{d^D l}{(2\pi)^D}\frac{2\bar{n}\cdot(l+p)}{l^2(l+p)^2\bar{n}\cdot l} \\
&= 8\bar{n}\cdot p\int_0^1 dx(1-x)\int_0^\infty du\int\frac{d^D l}{(2\pi)^D}\frac{1}{(l^2-A^2)^3} \\
&= -\frac{i}{16\pi^2}2\bar{n}\cdot p\int_0^1 dx(1-x)\int_0^\infty du\Big(ux\bar{n}\cdot p-x(1-x)p^2\Big)^{-1-\epsilon} \\
&= -\frac{i}{16\pi^2}2\int_0^1 dx x^{-1-\epsilon}(\bar{n}\cdot p)^{-\epsilon}\int_0^\infty\left(u-\frac{(1-x)p^2}{\bar{n}\cdot p}\right)^{-1-\epsilon} \\
&= -\frac{i}{16\pi^2}2\int_0^1 dx x^{-1-\epsilon}(1-x)^{-\epsilon}(\bar{n}\cdot p)^{-\epsilon}\frac{1}{\epsilon_{\rm UV}}\left(\frac{-p^2}{\bar{n}\cdot p}\right)^{-\epsilon_{\rm UV}} \\
&= -\frac{i}{16\pi^2}2\frac{-1}{\epsilon_{\rm IR}}(\bar{n}\cdot p)^{-\epsilon_{\rm IR}}\frac{1}{\epsilon_{\rm UV}}\left(\frac{-p^2}{\bar{n}\cdot p}\right)^{-\epsilon_{\rm UV}} \\
&= -\frac{i}{16\pi^2}2\left[-\frac{1}{\epsilon_{\rm IR}}+\ln\frac{\mu}{\bar{n}\cdot p}-\frac{\epsilon_{\rm IR}}{2}\ln^2\frac{\mu}{\bar{n}\cdot p}\right]\cdot\left[\frac{1}{\epsilon_{\rm UV}}+\ln\frac{\mu^2}{-p^2}-\ln\frac{\mu}{\bar{n}\cdot p}+\frac{\epsilon_{\rm UV}}{2}\ln^2\frac{-p^2}{\bar{n}\cdot p}\right] \\
&= -\frac{i}{16\pi^2}\left[-\frac{2}{\epsilon_{\rm IR}\epsilon_{\rm UV}}-\frac{2}{\epsilon_{\rm IR}}\ln\frac{\mu^2}{-p^2}-\ln^2\frac{\mu^2}{-p^2}+\left(\frac{2}{\epsilon_{\rm IR}}-\frac{2}{\epsilon_{\rm UV}}\right)\ln\frac{\mu}{\bar{n}\cdot p}\right]+\cdots.
\end{aligned}$$

$$\begin{aligned}
I_0 &= \int \frac{d^D l}{(2\pi)^D} \frac{2\bar{n}\cdot p}{l^2 \bar{n}\cdot l(\bar{n}\cdot p n\cdot l + p^2)} = 2\int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 \bar{n}\cdot l(n\cdot l + p^2/\bar{n}\cdot p)} \\
&= 16\int_0^\infty du \int_0^\infty dv \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 - B^2)^3} \\
&= -\frac{i}{16\pi^2} 4\int du dv \Big(2uv - \frac{vp^2}{\bar{n}\cdot p}\Big)^{-1-\epsilon} \\
&= -\frac{i}{16\pi^2}\frac{2}{\epsilon_{\rm UV}}\int \frac{dv}{v}\Big(\frac{-vp^2}{\bar{n}\cdot p}\Big)^{-\epsilon} = -\frac{i}{16\pi^2}\frac{2}{\epsilon_{\rm UV}}\Big(\frac{-p^2}{\bar{n}\cdot p}\Big)^{-\epsilon_{\rm UV}}\int dv v^{-1-\epsilon} \\
&= -\frac{i}{16\pi^2}\Big(\frac{2}{\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm IR}}\Big)\Big(\frac{1}{\epsilon_{\rm UV}} + \ln\frac{\mu^2}{-p^2} - \ln\frac{\mu}{\bar{n}\cdot p}\Big).
\end{aligned}$$

$$l^2 + 2u\bar{n}\cdot l + 2v(n\cdot l + p^2/\bar{n}\cdot p) = (l + u\bar{n} + vn)^2 - \Big(-2v\frac{p^2}{\bar{n}\cdot p} + 4uv\Big)$$

$$I_C = \tilde{I}_C - I_0 = -\frac{i}{16\pi^2}\Big[-\frac{2}{\epsilon_{\rm UV}^2} - \frac{2}{\epsilon_{\rm UV}}\ln\frac{\mu^2}{-p^2} - \ln^2\frac{\mu^2}{-p^2}\Big] + \cdots$$

Consider 2-2 scattering.



Current operators

$$J_n(x) = e^{i(\bar{n}\cdot p_1' - \bar{n}\cdot p_1) n\cdot x/2} \overline{\chi}_n(x) \frac{\not{\bar{n}}}{2} \chi_n(x),$$

$$J_{\bar{n}}(x) = e^{i(n\cdot p_2' - n\cdot p_2)\bar{n}\cdot x/2} \overline{\chi}_{\bar{n}}(x) \frac{\not{n}}{2} \chi_{\bar{n}}(x).$$

Now we consider

$$W = \int d^4x e^{iq\cdot x} \overline{\chi}_{n,i'}(x) \frac{\not{\bar{n}}}{2} \chi_{n,i}(x) \overline{\chi}_{\bar{n},j'} \frac{\not{n}}{2} \chi_{\bar{n},j}(0) G(x)_{ij}^{i'j'},$$

In qq scattering, the relevant Wilson lines are given as

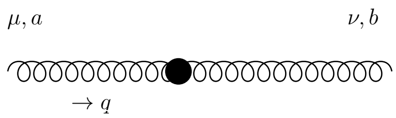
$$(O_2)^{i'i}_{j'j} = (\tilde{Y}^\dagger_{n_3} Y_{n_1})_{i'i} (\tilde{Y}^\dagger_{n_4} Y_{n_2})_{j'j}$$

$$(O_1)^{i'j}_{j'i} = (\tilde{Y}^\dagger_{n_3} Y_{n_2})_{i'j} (\tilde{Y}^\dagger_{n_4} Y_{n_1})_{j'i}$$

$$\begin{aligned}(O_2)^{i'i}_{j'j} = (\tilde{Y}^\dagger_{n_3} Y_{n_1})_{i'i} (\tilde{Y}^\dagger_{n_4} Y_{n_2})_{j'j} &= \left[\left(1 - g n_3 \cdot A \frac{1}{n_3 \cdot R^\dagger + i0}\right)\left(1 - \frac{1}{n_1 \cdot R + i0} g n_1 \cdot A\right)\right]_{i'i} \\ &\times \left[\left(1 - g n_4 \cdot A \frac{1}{n_4 \cdot R^\dagger + i0}\right)\left(1 - \frac{1}{n_2 \cdot R + i0} g n_2 \cdot A\right)\right]_{j'j}\end{aligned}$$

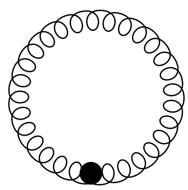
$$\left[\left(1 - gn_3 \cdot A \frac{1}{n_3 \cdot R^\dagger + i0}\right)\left(1 - \frac{1}{n_1 \cdot R + i0} gn_1 \cdot A\right)\right]_{i'i}$$
$$\times \left[\left(1 - gn_4 \cdot A \frac{1}{n_4 \cdot R^\dagger + i0}\right)\left(1 - \frac{1}{n_2 \cdot R + i0} gn_2 \cdot A\right)\right]_{j'j}$$

Feynman rule



$$\begin{aligned} &= g^2 (T_a)_{i'i} (T_b)_{j'j} \left(\frac{n_3^\mu}{n_3 \cdot q + i0} + \frac{n_1^\mu}{-n_1 \cdot q + i0}\right)\left(\frac{n_4^\nu}{-n_4 \cdot q + i0} + \frac{n_2^\nu}{n_2 \cdot q + i0}\right) \\ &+ g^2 (T_a T_b)_{i'i} \delta_{j'j} \frac{n_3^\mu}{n_3 \cdot q + i0} \frac{n_1^\nu}{n_1 \cdot q + i0} + g^2 \delta_{i'i} (T_a T_b)_{j'j} \frac{n_4^\mu}{n_4 \cdot q + i0} + \frac{n_2^\nu}{n_2 \cdot q + i0} \\ &+ (a \leftrightarrow b, \mu \leftrightarrow \nu, q \to -q) \end{aligned}$$

Radiative corrections



$$\begin{aligned} (K_2)^{i'i}_{j'j} &= -\frac{ig^2}{2} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2} \\ &\times \left[2(T_a)_{i'i}(T_a)_{j'j}\Big((1,4) + (2,3) - (3,4) - (1,2)\Big) + C_F \delta_{i'i} \delta_{j'j}\Big((1,3) + (2,4) + (1,3) + (2,4)\Big)\right], \end{aligned}$$

where $(i,j) = \dfrac{n_i \cdot n_j}{n_i \cdot l n_j \cdot l}$.

The basic integral is

$$
\begin{aligned}
J_{ij} &= \int \frac{d^D l}{(2\pi)^D} \frac{n_i \cdot n_j}{l^2(n_i \cdot l - \lambda_i)(n_j \cdot l - \lambda_j)} \quad (\lambda_i = p_i^2/\overline{n}_i \cdot p_i) \\
&= -\frac{i}{16\pi^2} 4 \int_0^\infty du\, dv \frac{n_i \cdot n_j}{(2uv n_i \cdot n_j + 2u\lambda_i + 2v\lambda_j)^{1+\epsilon}} \\
&= -\frac{i}{16\pi^2} \Big(\frac{n_i \cdot n_j}{2}\Big)^{-\epsilon} 2 \int_0^\infty du\, dv \Big(uv + \frac{2u\lambda_i}{n_i \cdot n_j} + \frac{2v\lambda_j}{n_i \cdot n_j}\Big)^{-1-\epsilon} \\
&= -\frac{i}{16\pi^2} \Big(\frac{n_i \cdot n_j}{2}\Big)^{-\epsilon} \frac{2}{\epsilon} \Big(\frac{2\lambda_j}{n_i \cdot n_j}\Big)^{-\epsilon} \int dv \frac{v^{-\epsilon}}{v + \frac{2\lambda_i}{n_i \cdot n_j}} \\
&= -\frac{i}{16\pi^2} \frac{2}{\epsilon^2} \Big(\frac{2\lambda_i \lambda_j}{n_i \cdot n_j}\Big)^{-\epsilon} = -\frac{i}{16\pi^2} \Big(\frac{2}{\epsilon^2} + \frac{2\beta_{ij}}{\epsilon}\Big).
\end{aligned}
$$

where $\beta_{ij} = \ln \frac{-2\sigma_{ij} p_i \cdot p_j \mu^2}{(-p_i^2)(-p_j^2)}$

Generalization to n-parton scattering

Jet function

$$J_q = 1 + \frac{\alpha_s}{4\pi} C_F \Big(\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \ln \frac{\mu^2}{-p^2} + \frac{3}{2\epsilon}\Big)$$

$$S = 1 + \frac{\alpha_s}{4\pi} \sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \Big(\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \ln \frac{-\sigma_{ij} n_i \cdot n_j \overline{n}_i \cdot p_i \overline{n}_j \cdot p_j \mu^2}{2(-p_i^2)(-p_j)^2}\Big)$$

$$\sum_i \mathbf{T}_i = 0$$

$$S \prod_i J_i = 1 - \frac{\alpha_s}{4\pi} \Big[\sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \Big(\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \ln \frac{\mu^2}{-s_{ij}}\Big) + \sum_i \frac{\gamma_0^i}{2\epsilon}\Big]$$

$$\gamma_0^i = -3C_F$$