Scattering Amplitudes in Gauge Theories

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Lecture 1

Prerequisites

- One semester course in Quantum Field Theory. (tree-level QED including 2-2 scattering process)

- Lie algebra of SU(N).

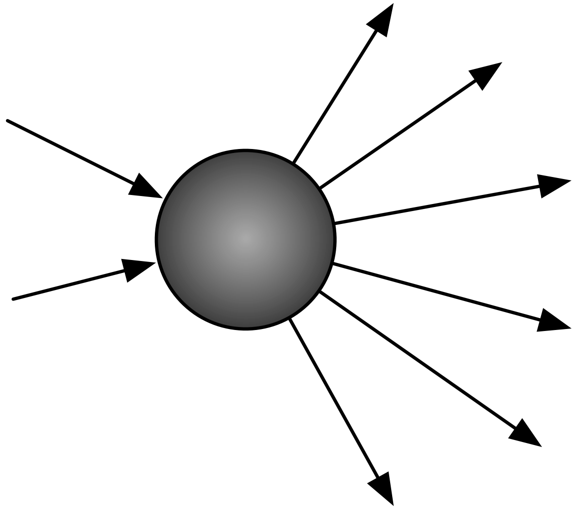
References

- Special volume of Journal of Physics A "Scattering Amplitudes in Gauge Theories"

- A good starting point is [L. Dixon 1105.0771]

Scattering amplitude

- What is scattering amplitude?



$$S_{\text{out,in}} = \langle \text{out} | \hat{S} | \text{in} \rangle$$

scattering operator

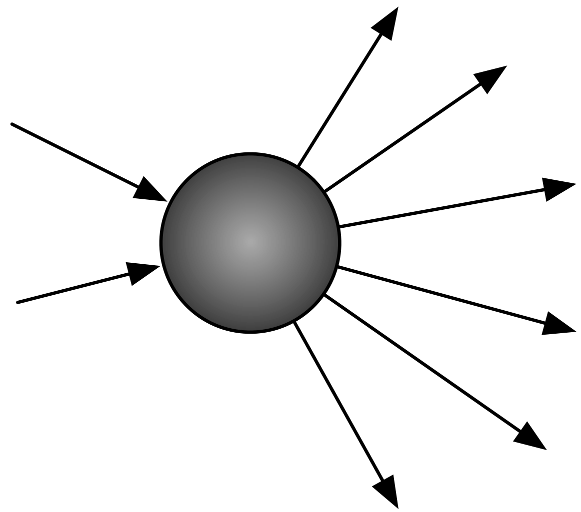
(determined by H_{int})

asymptotic (multi-particle) states

(determined by H_{free})

Use of scattering amplitudes

- Example: 2 to *m*-particle differential cross section



$$S = iT(2\pi)^4\delta^4(\sum_i p_i) \quad \Longrightarrow \quad d\sigma_{2\to m} = \frac{|T_{2\to m}|^2}{4|\vec{p}_1|_{\mathrm{CM}}\sqrt{s}} d\Omega_m$$

$$d\Omega_m \equiv (2\pi)^4\delta^4(p_1 + p_2 - \sum_{i=1}^{m} p_i) \prod_{j=1}^{m} \widetilde{dp}_i\,, \quad \widetilde{dp} = \frac{d^3\vec{p}}{(2\pi)^3 2p^0}$$

Scattering amplitudes in gauge theories - history

- QED (1940’s)

  Feynman, Schwinger, Tomonaga, Dyson

- Non-abelian gauge theory (1970’s)

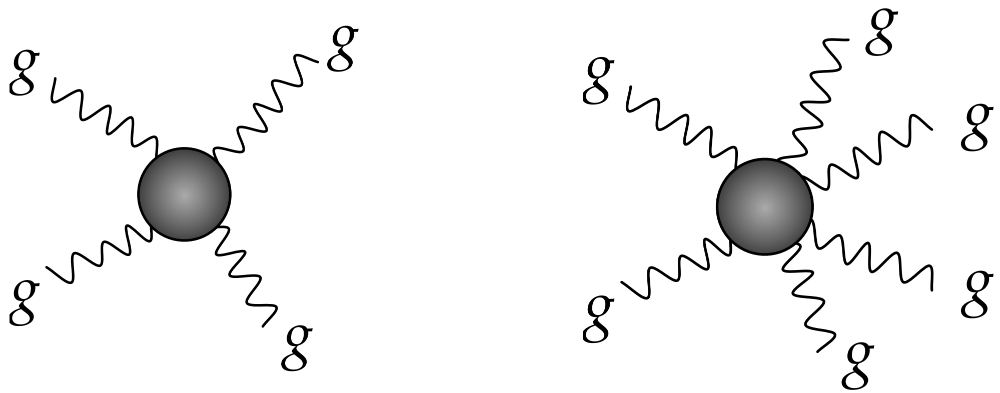
  Faddeev-Popov   ‘t Hooft-Veltman   Slavnov-Taylor

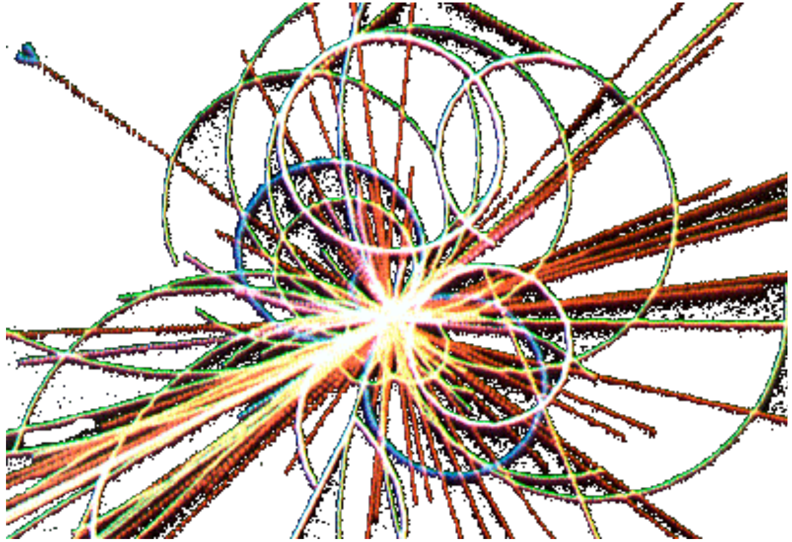
  Lee-ZinnJustin   Becchi-Rouet-Stora-Tyutin

- Scattering amplitudes of non-abelian gauge theory has been a serious research topic from mid-1980’s until ... now!

Amplitudes in non-abelian gauge theory

- Jet production at colliders





- Jets are produced at a rate many orders of magnitude higher than those of "interesting" events.

n	3	4	5	6	7	...
#(Feynman)	1	2	10	38	149	...

reduced counting after color decomposition

Scope of the lecture

We will mostly focus on ... (but brifely comment on)

- (1+3) dimension ((1+2)-dimension)
- Gauge theory (Gravity)
- Massless particles
- Tree amplitudes (Loop amplitudes)
- Planar (large N) sector

Massless particle

- Lorentz group and its extension to include spinors

$$SO(1,3) \simeq SL(2,\mathbb{C})$$

$$SO(4) \simeq SU(2) \times SU(2)$$

$$SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$$

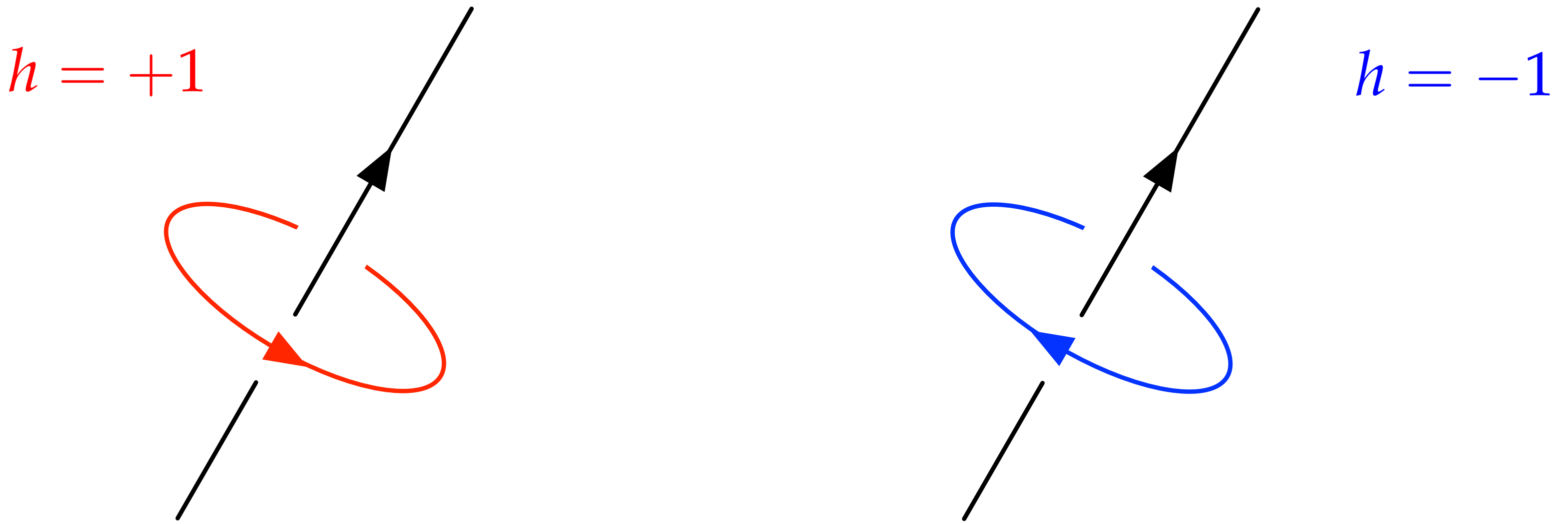
- Fields at a given point $(\phi, \psi_\alpha, A_\mu)$

finite dimensional, non-unitary representations

- Particle states $|\underbrace{p_\mu}_{\text{Momentum}}, \underbrace{h}_{\text{Helicity}}\rangle$

infinite dimensional, unitary representations

Momentum and Helicity



- Helicity = angular momentum around the momentum axis (spin under the SO(2) little group)

- Helicity is Lorentz invariant only for massless particles.

- h = +1 and h = -1 are related by discrete symmetry like parity but not by Lorentz transformation.

Gauge redundancy (problem with h = 1 or higher)

$$A_{\mu=0,1,2,3} \quad : \quad \text{(4 components)}$$

$$|p, h = \pm 1\rangle \quad : \quad \text{(2 states)}$$

$$A_\mu(x) = \epsilon_\mu(p) e^{ip\cdot x}$$

$$p^\mu \epsilon^\pm_\mu(p) = 0\,, \qquad \epsilon_\mu(p) \sim \epsilon_\mu(p) + \alpha p_\mu \qquad (A_\mu \sim A_\mu + \partial_\mu \Lambda)$$

$$\mathcal{A}_4^{++--}(p_1, p_2, p_3, p_4) = \epsilon^+_{\mu_1}(p_1)\epsilon^+_{\mu_2}(p_2)\epsilon^-_{\mu_3}(p_3)\epsilon^-_{\mu_4}(p_4) \times \mathcal{A}^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4)$$

"True" amplitude

What Feynman diagrams give you

"True" amplitudes are never written down in most QFT textbooks !!! (Exception: Srednicki)

General lessons

- Gauge redundancy is inevitable, if we want to keep locality and unitarity manifest at the same time.

- But, in Feynman diagram computations, gauge redundancy is the main source of computational complexity.
  (Individual diagrams are not gauge invariant.
  Gauge invariance is recovered only after summing over diagrams.)

- Explicit computations have shown that, quite often, a vast number of Feynman diagrams collapse into a simple 1-2 line expressions.

Could there be an alternative to Feynman calculus which reveals the secret behind the enormous simplification?

YES !!

Spinor-helicity variables

- Spinors for momenta

$$p^\mu = (p^0, \vec{p}) \qquad p_\mu p^\mu = (p^0)^2 - \vec{p}^2 = 0$$

$$p \cdot \sigma = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \equiv p_{\alpha\dot{\alpha}}$$

"bi-spinor"

$$\det(p \cdot \sigma) = (p^0)^2 - \vec{p}^2 = 0 \quad \Longrightarrow \quad p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$$

- Complexify all variables

$$SO(1,3) \;\to\; SL(2)_\lambda \times SL(2)_{\bar{\lambda}}$$

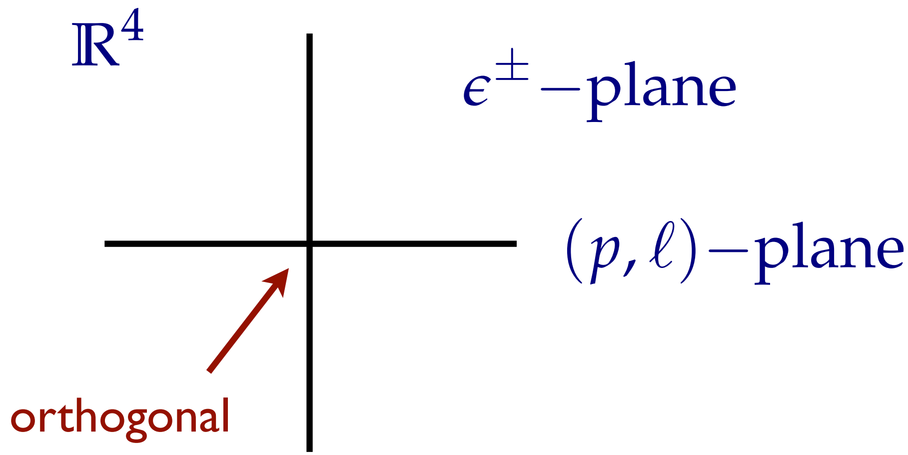
Complexified Lorentz group

$$\langle ij \rangle = \epsilon^{\alpha\beta}(\lambda_i)_\alpha (\lambda_j)_\beta\,, \quad [ij] = \epsilon^{\dot{\alpha}\dot{\beta}}(\bar{\lambda}_i)_{\dot{\alpha}}(\bar{\lambda}_j)_{\dot{\beta}}$$

Lorentz invariants

Polarization vectors revisited

- Momentum: $p_\mu \to p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$
- Introduce an additional reference vector: $\ell_\mu \to \ell_{\alpha\dot{\alpha}} = \rho_\alpha \bar{\rho}_{\dot{\alpha}}$



- Polarization vectors are now uniquely determined!

Polarization vectors - further remarks

- Gauge redundancy has been transferred to the ambiguity in the choice of $\ell_{\alpha\dot{\alpha}} = \rho_\alpha \bar{\rho}_{\dot{\alpha}}$.

- The final "true" amplitude should be gauge invariant: dependence on $\rho, \bar{\rho}$

- Bonus

  Under the "helicity scaling" $(\lambda, \bar{\lambda}) \to (t\lambda, t^{-1}\bar{\lambda})$,

$$\epsilon^+ \to t^{-2}\epsilon^+ , \quad \epsilon^- \to t^{+2}\epsilon^-$$

$$\Longrightarrow \quad A_n(t_i\lambda_i, t_i^{-1}\bar{\lambda}_i, h_i) = \left(\prod_{i=1}^{n} t_i^{-2h_i}\right) A_n(\lambda_i, \bar{\lambda}_i, h_i)$$

Color ordering

- SU(N$_c$) gauge group

  adjoint color indices: $a, b, c \in \{1, \ldots, N_c^2 - 1\}$

  fundamental indices: $i, j, \ldots \in \{1, \ldots, N_c\}$

  anti-fundamental indices: $\bar{\imath}, \bar{\jmath}, \ldots \in \{1, \ldots, N_c\}$

- Generators of SU(N$_c$), $(T^a)_i{}^{\bar{\jmath}}$, satisfy

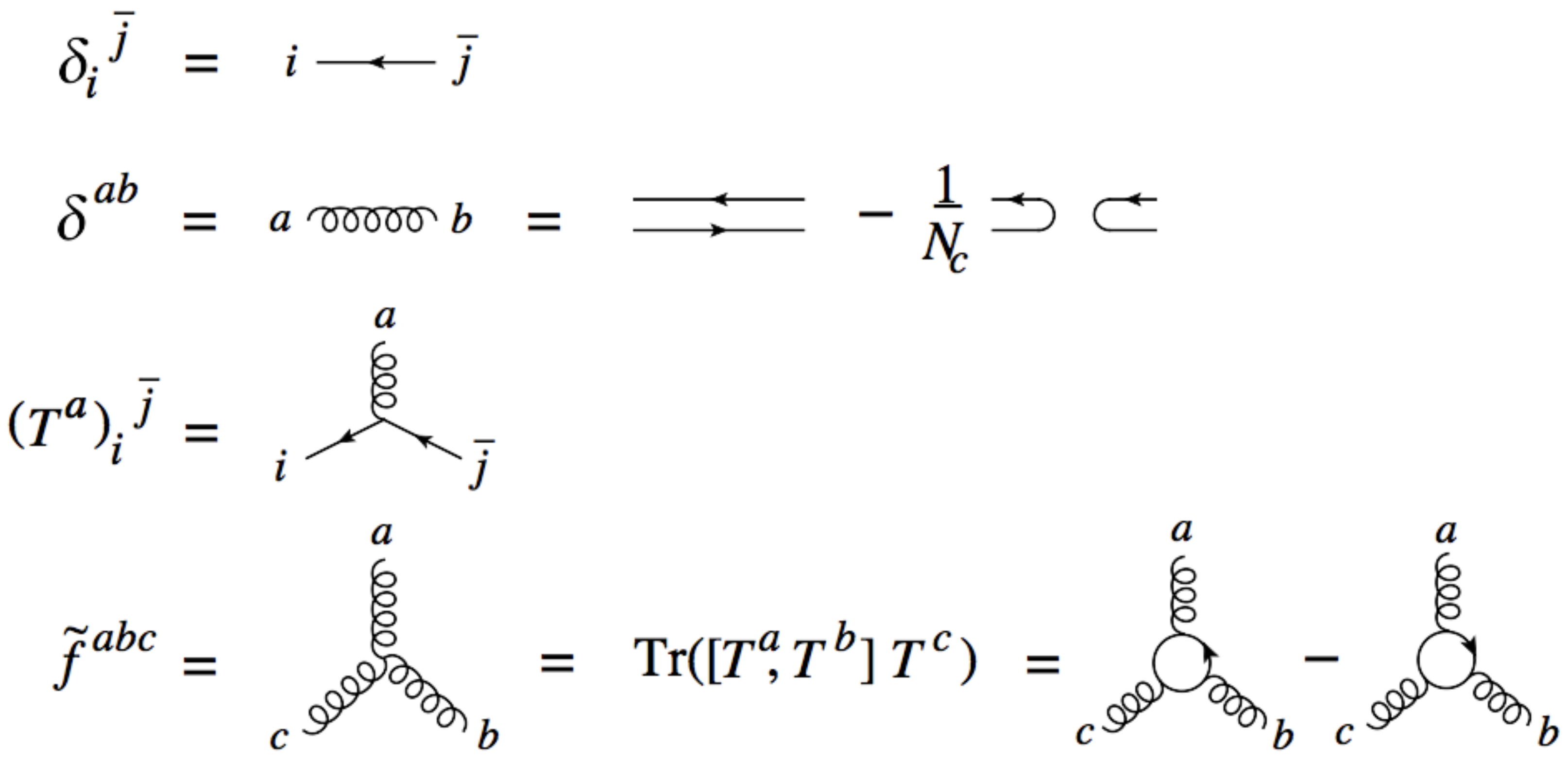
$$[T^a, T^b] = f^{ab}{}_c T^c$$

$$\mathrm{tr}(T^a T^b) = \delta^{ab}$$

$$(T^a)_{i_1}{}^{\bar{\jmath}_1} (T^a)_{i_2}{}^{\bar{\jmath}_2} = \delta_{i_1}{}^{\bar{\jmath}_2} \delta_{i_2}{}^{\bar{\jmath}_1} - \frac{1}{N_c} \delta_{i_1}{}^{\bar{\jmath}_1} \delta_{i_2}{}^{\bar{\jmath}_2}$$

Color ordering

- Double-line notation ['t Hooft '74]



figures copied from [Dixon 1105.0771]

Color ordering

- Trace-based color decomposition

“Color-ordered” amplitude
(cyclic symmetry only)

Full amplitude (permutation symm)

$$A_n^{\text{tree}}(p_i, h_i, a_i) = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_n(\sigma(1^{h_1}), \cdots, \sigma(n^{h_n}))$$

- Heuristic argument [Dixon 1105.0771]