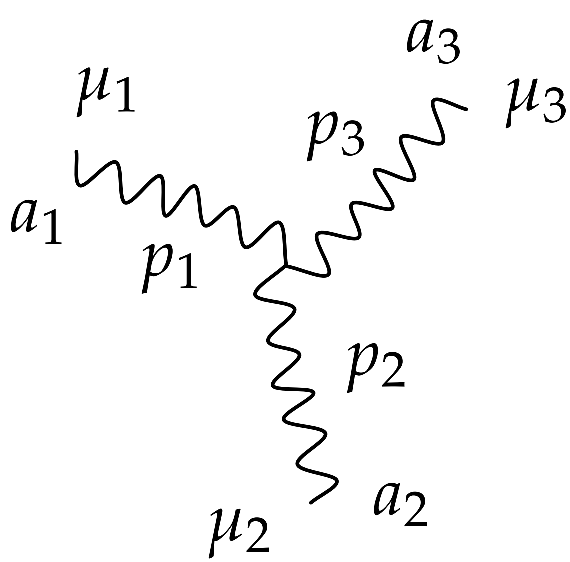
Lecture 2

3-gluon amplitude



$p_1 + p_2 + p_3 = 0$

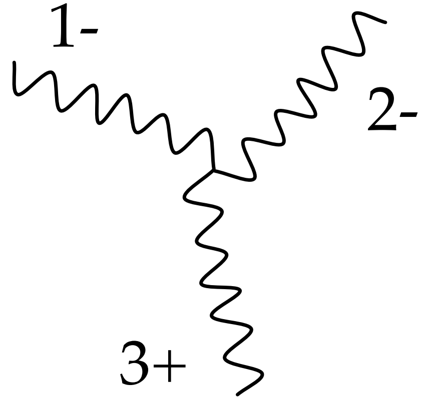
$$V^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3) = g f^{a_1 a_2 a_3} \left[(p_2 - p_3)_{\mu_1} \eta_{\mu_2 \mu_3} + (\text{cyclic}) \right]$$

$$f^{a_1 a_2 a_3} = \text{tr}(T^{a_1} T^{a_2} T^{a_3}) - \text{tr}(T^{a_2} T^{a_1} T^{a_3})$$

(color decomposition trivial)

$$A_3 = [\epsilon_1 \cdot (p_2 - p_3)](\epsilon_2 \cdot \epsilon_3) + (\text{cyclic})$$

3-gluon amplitude



$$p_1 + p_2 + p_3 = 0$$

$$A_3 = [\epsilon_1 \cdot (p_2 - p_3)](\epsilon_2 \cdot \epsilon_3) + (\text{cyclic})$$

$$\left(\epsilon_1^- = \frac{|1\rangle[\hat{1}|}{[\hat{1}1]}, \quad \epsilon_2^- = \frac{|2\rangle[\hat{2}|}{[\hat{2}2]}, \quad \epsilon_3^+ = \frac{|\hat{3}\rangle[3|}{\langle\hat{3}3\rangle}, \quad \epsilon_i \cdot p_i = 0 \right)$$

$$A_3 = \frac{\langle 12\rangle[\hat{1}2]\langle\hat{3}2\rangle[3\hat{2}] + \langle 23\rangle[\hat{2}3]\langle 1\hat{3}\rangle[\hat{1}3] + \langle 1\hat{3}\rangle[13]\langle 12\rangle[\hat{2}\hat{3}]}{[\hat{1}1][\hat{2}2]\langle\hat{3}3\rangle}$$

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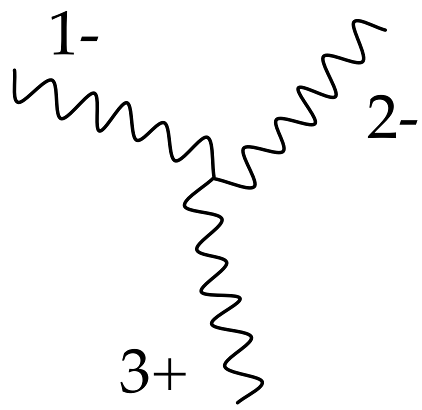

$$p_1 + p_2 + p_3 = 0$$

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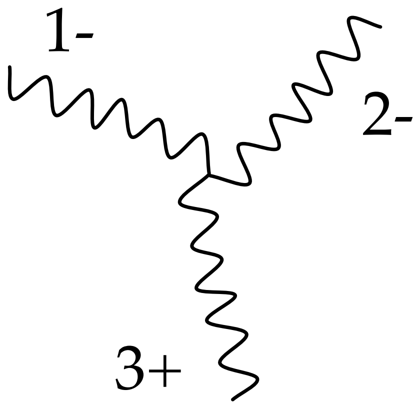
$$\lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2 + \lambda_3\bar{\lambda}_3 = 0 \quad \Longrightarrow \quad (\lambda_1 \propto \lambda_2 \propto \lambda_3) \text{ or } (\bar{\lambda}_1 \propto \bar{\lambda}_2 \propto \bar{\lambda}_3)$$

$$\bar{\lambda}_1 = \langle 23\rangle\bar{\zeta}\,, \;\; \bar{\lambda}_2 = \langle 31\rangle\bar{\zeta}\,, \;\; \bar{\lambda}_3 = \langle 12\rangle\bar{\zeta}\,.$$

$$\langle ab\rangle\langle cd\rangle + \langle bc\rangle\langle ad\rangle + \langle ca\rangle\langle bd\rangle = 0$$

$$A_3 = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,.$$

3-gluon amplitude



$$A_3 = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}.$$

$$p_1 + p_2 + p_3 = 0$$

- A_3 is the unique Lorentz invariant satisfying the following scaling properties:

$$A_n(t_i \lambda_i, t_i^{-1} \bar{\lambda}_i; h_i) = \left(\prod_i t_i^{-2h_i} \right) A_n(\lambda_i, \bar{\lambda}_i; h_i)$$

$$A_n(s\lambda_i, s\bar{\lambda}_i; h_i) = s^{-2(n-4)} A_n(\lambda_i, \bar{\lambda}_i; h_i)$$

- Same logic applied to gravity implies

$$\mathcal{A}_3(\text{gravity}) = [\mathcal{A}_3(\text{gauge})]^2$$

Maximally Helicity Violating (MHV) amplitudes

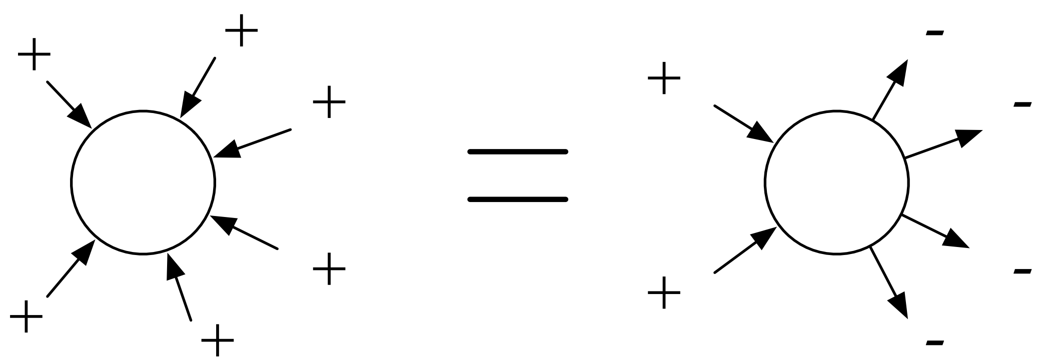
$$A_n(+,+,+,\cdots,+) = 0\,,$$

$$A_n(-,+,+,\cdots,+) = 0\,,$$

$$A_n(+,\cdots,\underset{j}{-},\cdots,\underset{k}{-},\cdots,+) = \frac{\langle jk\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,.$$

[Parke-Taylor '86]

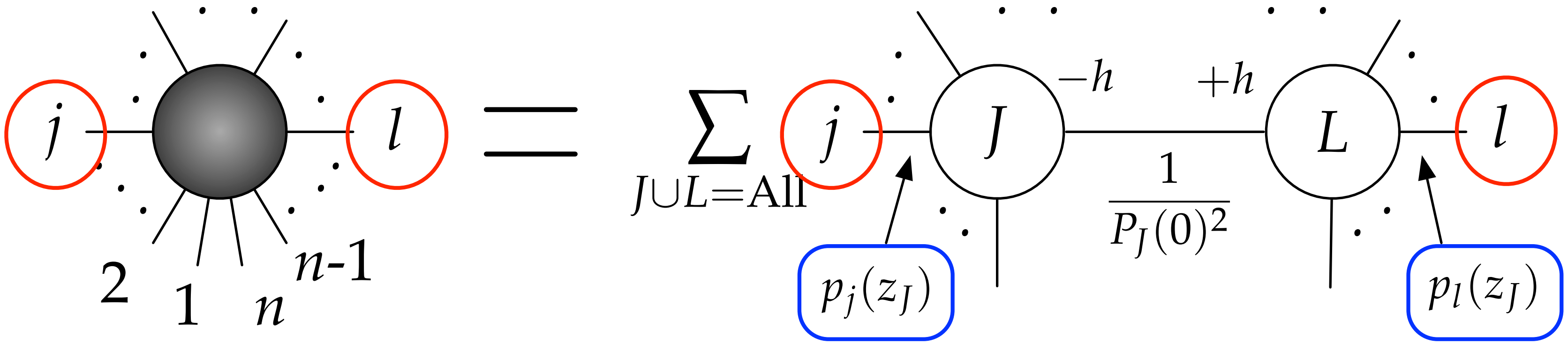
Watch out for the helicity flip in applying crossing symmetry.



BCFW recursion formula [Britto-Cachazo-Feng-Witten '05]

- Build up higher point amplitude from lower ones

$$A_n = \sum_{J,\pm} A_J^{\pm}(z_J)\frac{1}{P_J(0)^2}A_L^{\mp}(z_J)$$



- Deformation parameter z is determined by

$$P_J(z_J)^2 = 0$$

BCFW recursion formula - derivation

- On-shell deformation of momenta

$$\bar{\lambda}_j \to \bar{\lambda}_j - z\bar{\lambda}_l\,, \quad \lambda_l \to \lambda_l + z\lambda_j$$

$(p_j^2 = 0,\ p_l^2 = 0,\ p_1 + \cdots + p_n = 0 \ \text{unaffected})$

- Contour integral representation

$$A_n = A_n(z=0) = \int \frac{dz}{2\pi i} \frac{A_n(z)}{z}$$

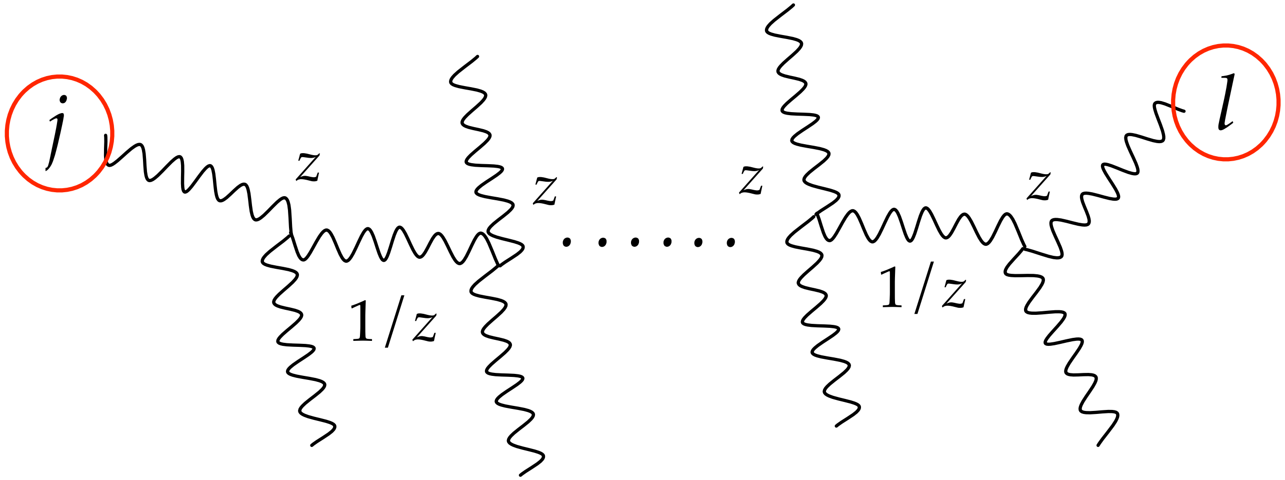
- Deformation of contour and residue theorem gives BCFW

$$\frac{1}{P_J(z)^2} = \frac{1}{P_J(0)^2 - 2zP_J(0)\cdot q}\,, \quad q \equiv \lambda_j \bar{\lambda}_l$$

... provided that A falls off at $z =$ infinity.

Large z behavior

- Naive power counting



$$\frac{1}{P_J(z)^2} = \frac{1}{P_J(0)^2 - 2zP_J(0)\cdot q}, \quad q \equiv \lambda_j \bar{\lambda}_l$$

ϕ^4 theory: $A(z) \to z^0$

Gauge theory: $A^{-+}_{\text{naive}}(z) \to z\,, \quad A^{--/++}_{\text{naive}}(z) \to z^2\,, \quad A^{+-}_{\text{naive}}(z) \to z^3\,.$

Large z behavior

- Correct counting

$$A^{-+}(z)\,,\,A^{--}(z)\,,\,A^{++}(z)\to 1/z\,,\quad A^{+-}(z)\to z^3\,.$$

Gauge symmetry plays a crucial role.

- Simple example: scalar QED [ArkaniHamed-Kaplan '08]

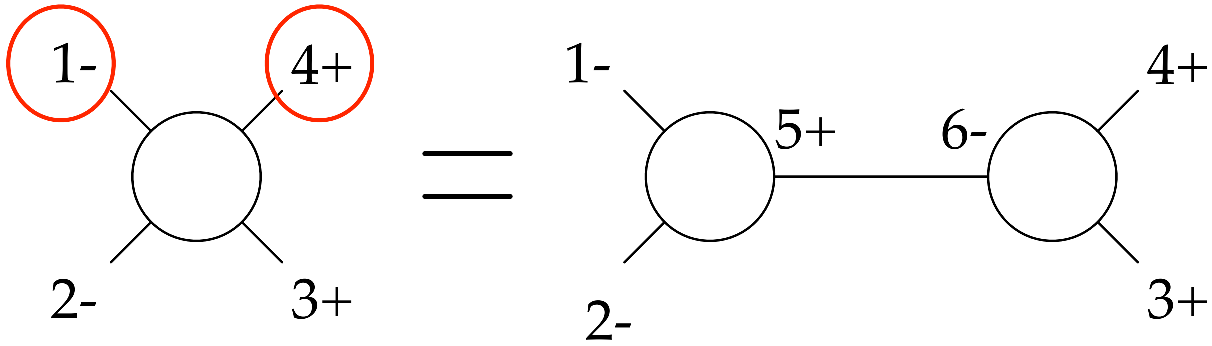
$$p(z)=p\pm zq$$

$$\mathcal{L}=(D^\mu\phi)^* D_\mu\phi\,,\quad D_\mu\phi\equiv\partial_\mu\phi-iA_\mu\phi$$

O(z) vertex disappear if we choose the gauge $q^\mu A_\mu=0\,.$

Example: 4-gluon via BCFW

- $4 = 3 * 3$



$$\bar{\lambda}_1 \to \bar{\lambda}_1 - z\bar{\lambda}_4\,, \quad \lambda_4 = \lambda_4 + z\lambda_1$$

$$A^{--++}_{4,\mathrm{BCFW}} = \frac{\langle 12\rangle^3}{\langle 15\rangle\langle 25\rangle} \times \frac{1}{\langle 12\rangle[12]} \times \frac{[34]^3}{[36][46]}$$

$$\left([36] = \frac{\langle 41\rangle[34]}{\langle 16\rangle}, \quad [46] = \frac{\langle 23\rangle[34]}{\langle 26\rangle}, \quad p_5 + p_6 = 0 \right)$$

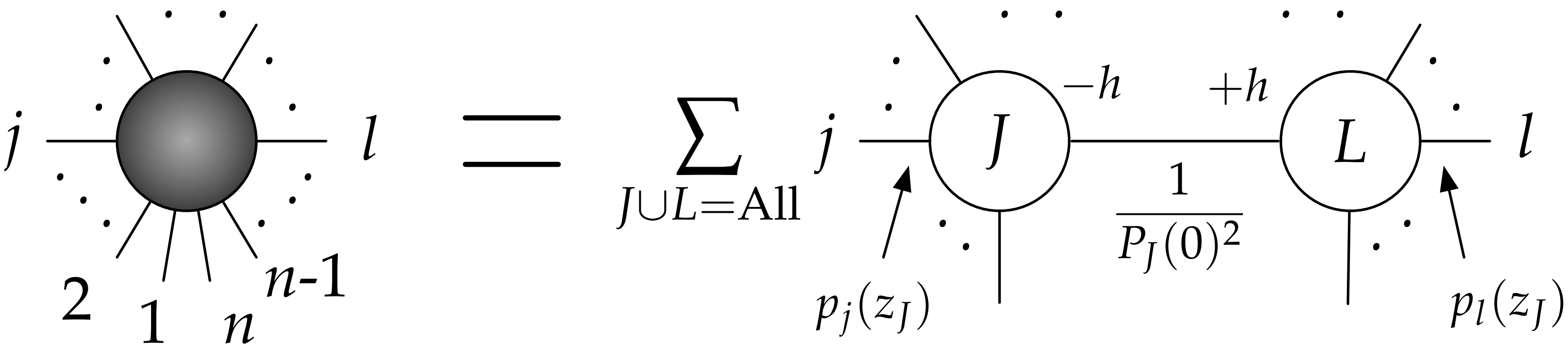
$$A^{--++}_{4,\mathrm{BCFW}} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

in agreement with Parke-Talyor

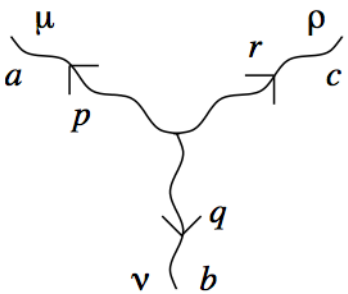
BCFW and gauge symmetry

$$A_n = \sum_{J,\pm} A_J^{\pm}(z_J) \frac{1}{P_J(0)^2} A_L^{\mp}(z_J)$$

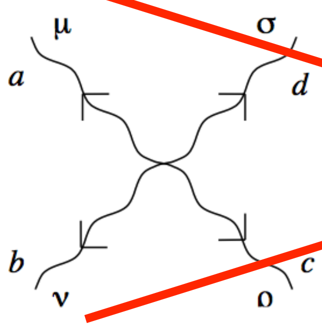
- Build up higher point amplitude from lower ones



- No need for 4-point vertex!



$$= g f^{abc}[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}]$$



$$= -ig^2 [f^{abe} f^{cde} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + f^{ace} f^{dbe} (g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma}) + f^{ade} f^{bce} (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\sigma\nu})]$$

Simplest non-trivial non-MHV amplitude

- 6-point next-to-MHV (nMHV) amplitude

$$\{4\} = \frac{\langle 46\rangle^4 [13]^4}{[12][23]\langle 45\rangle\langle 56\rangle (p_4+p_5+p_6)^2 \langle 6|5+4|3]\langle 4|5+6|1]},$$

$$\{6\} = g^2\{4\}, \{2\} = g^4\{4\}$$

$$A_{6,\mathrm{BCFW}}^{+-+-+-} = \{2\} + \{4\} + \{6\}\,.$$